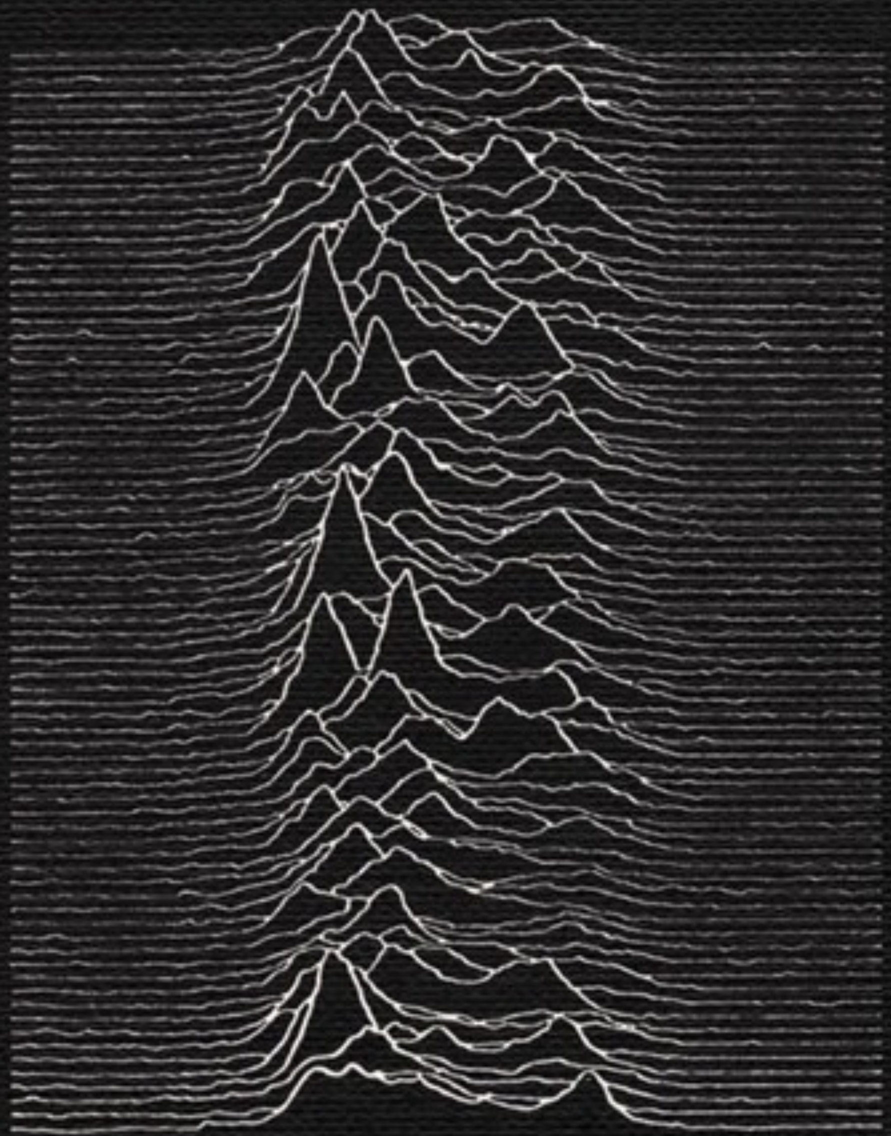
Signal Processing With Wavelets



JAMES MONK

Niels Bohr Institute, University of Copenhagen.

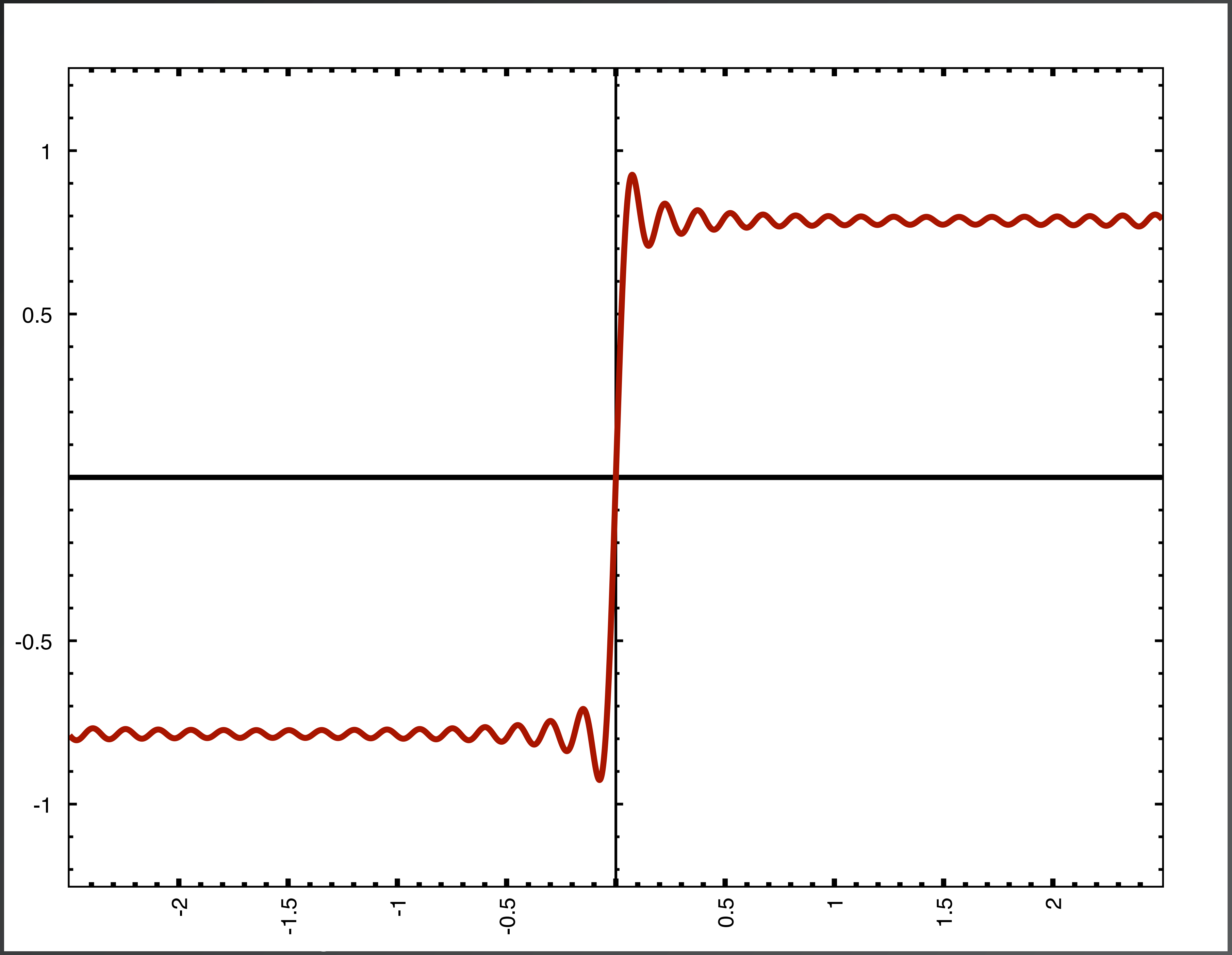
Reminder of the Fourier Transform

$$g(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt$$

- Tells you the frequency components in a signal
- One method of encoding a signal (e.g. a piece of music): take the Fourier transform, keep only those contributions in the frequency domain that are large - this is a (bad!) lossy compression technique.
- Remember the uncertainty principle - you need an infinite number of Fourier terms in order to make a sharp spike.
- Put another way, if you have a spike in your data with a width approaching zero, you start getting a very large number of populated frequencies.
- The Fourier basis functions, sine and cosine, have infinite extent.
- The Fourier transform does not tell you *when* (or *where*) in your data a particular frequency is occurring. It just tells you what contribution a given frequency makes.
- The result of all this is *ringing*. Although using a Fourier basis can be a good way of encoding some signals, in some situations you get artefacts due to the finite number of terms.

Example of the Square Wave

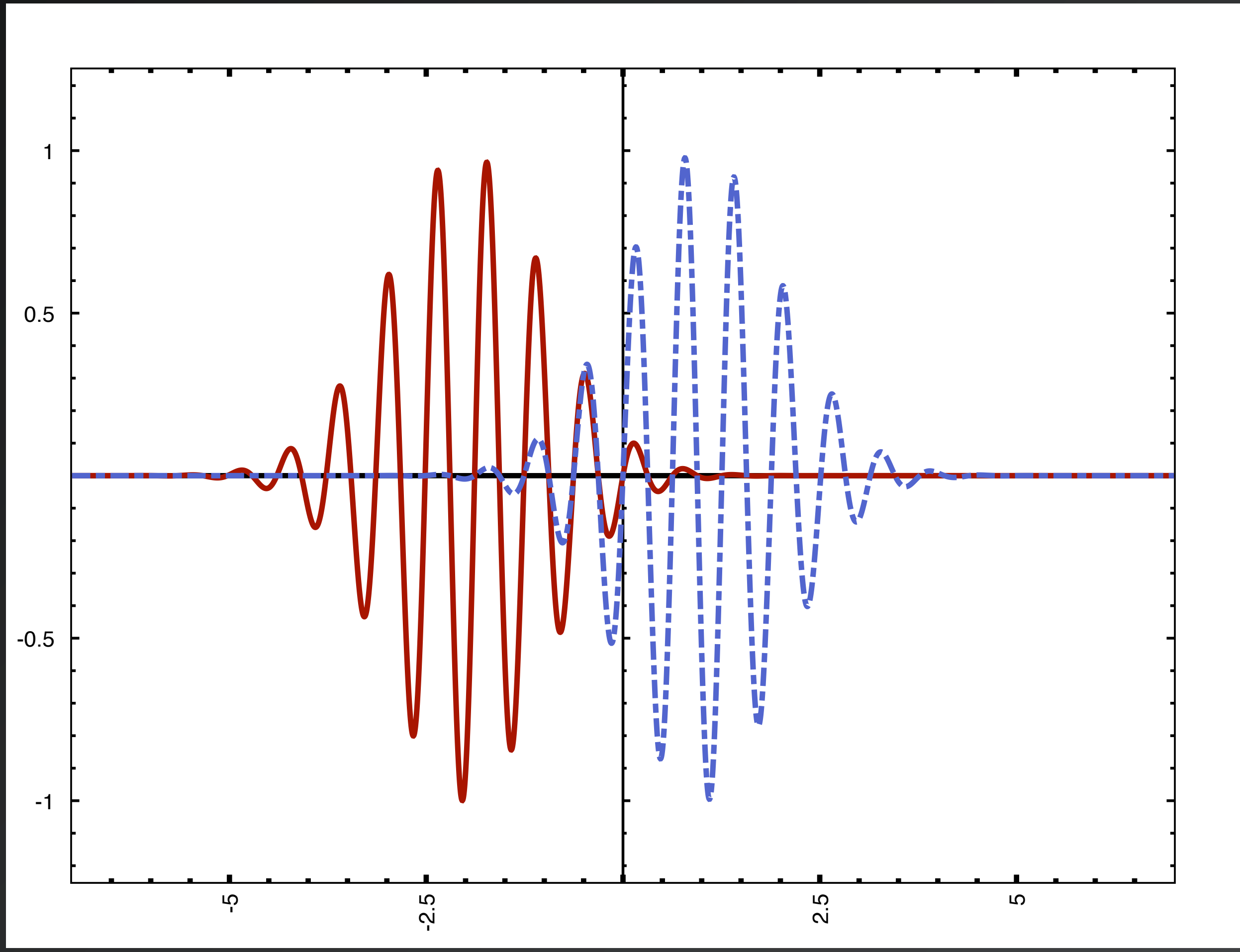
- Using the first ~20 Fourier components of a step function.
- Note the wiggles - ringing



Short-Time Fourier Transform

- To get around these limitations, people tried modifying the Fourier basis functions by a moveable Gaussian window

$$g(\omega, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\tau) e^{-i\omega\tau} e^{-\frac{(t-\tau)^2}{\sigma^2}} d\tau$$



Note how the transformed function now depends on frequency *and* time

But the width of the Gaussian window is fixed

Enter Wavelets

Mathematical tool developed in the 1980s and 90s

Grew out of short-time Fourier Transforms, i.e. windowed by a Gaussian (Morlet & Grossman,1980)

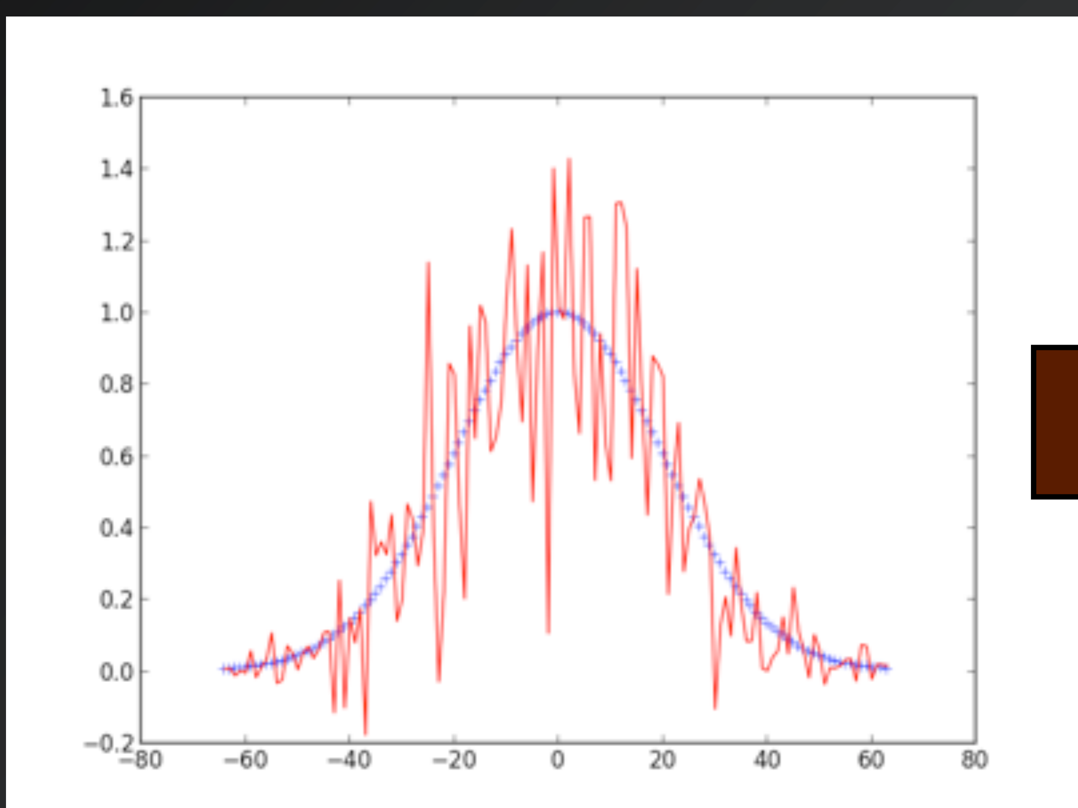
Modern, discrete and orthogonal wavelet basis developed in large part by Ingrid Debauchies (~1988)

Many applications in a wide range of subjects. **Deep** relevance to the way the natural world *appears* to work.

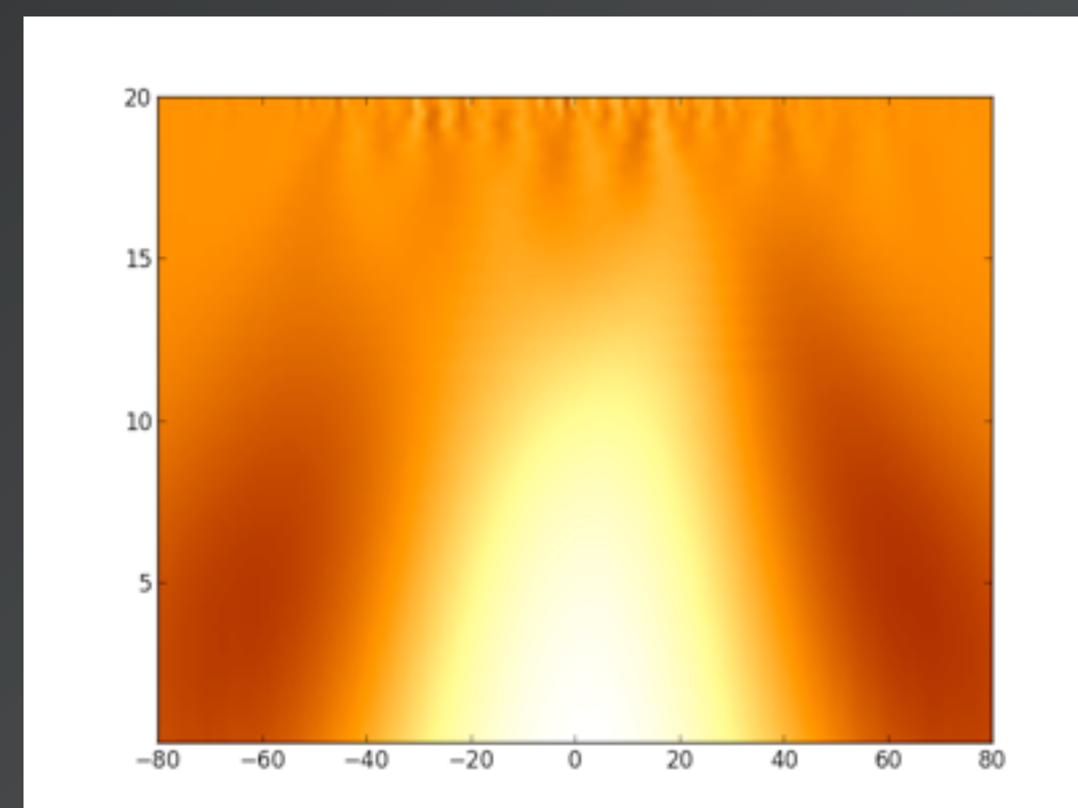
Continuous Wavelet Transform

$$W(S,t) = \frac{1}{\sqrt{S}} \int_{-\infty}^{\infty} f(\tau)\psi\left(\frac{t-\tau}{S}\right) d\tau$$

Yields time-frequency information on a signal



Frequency



Discrete Wavelet Transform

Coefficient

Wavelet function

Signal

$$C_{mn} = \sqrt{\frac{2^{(m-M)}}{S_0}} \int \bar{\psi}\left(2^{(m-M)}\frac{\phi}{S_0} - n\right) P\left(\phi\right) d\phi$$

Arbitrary scale (limit of resolution is a good choice)

Index m identifies the physical scale of the coefficient (c.f. wavelength for Fourier)

Index n identifies the location (translation) of the contribution

Wavelet coefficients have both scale, **and** translation (FT has only scale)

$$\psi_{mn}\left(\phi\right) = \sqrt{2^m}\psi\left(2^m\phi - n\right)$$

The wavelet bases are re-scalings and translations of a (scale-less) mother wavelet

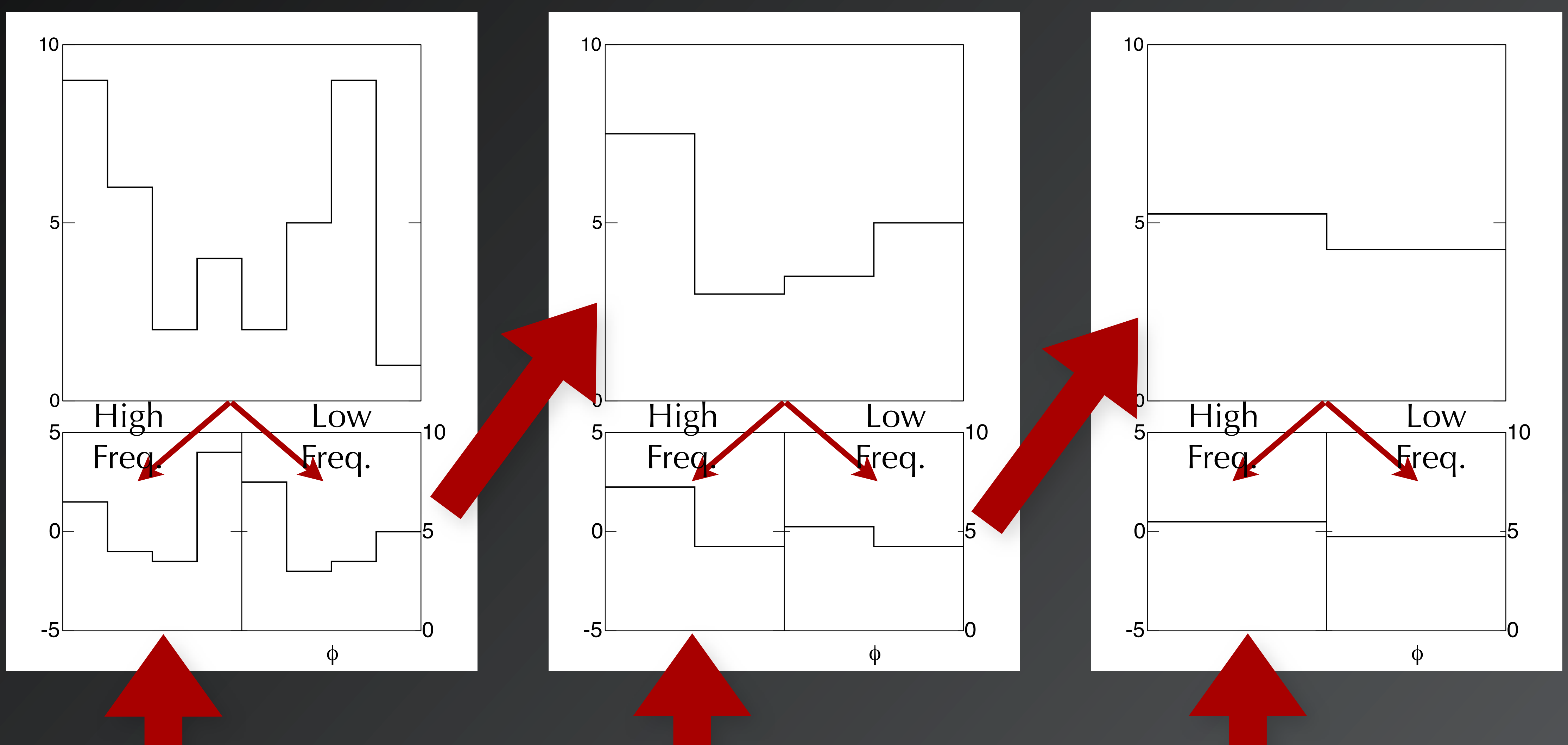
Demonstration of the Haar Wavelet

(More generally, a high-pass and low-pass filter)

The Haar wavelet is the simplest wavelet, consisting of a step function that takes the difference between adjacent points

After taking the difference, the two points are **averaged**, and the output is a re-scaled version of the signal

Re-apply the wavelet to the re-scaled signal



Wavelet Coefficients

Wavelet Coefficients

Wavelet Coefficients

Iterative Harr

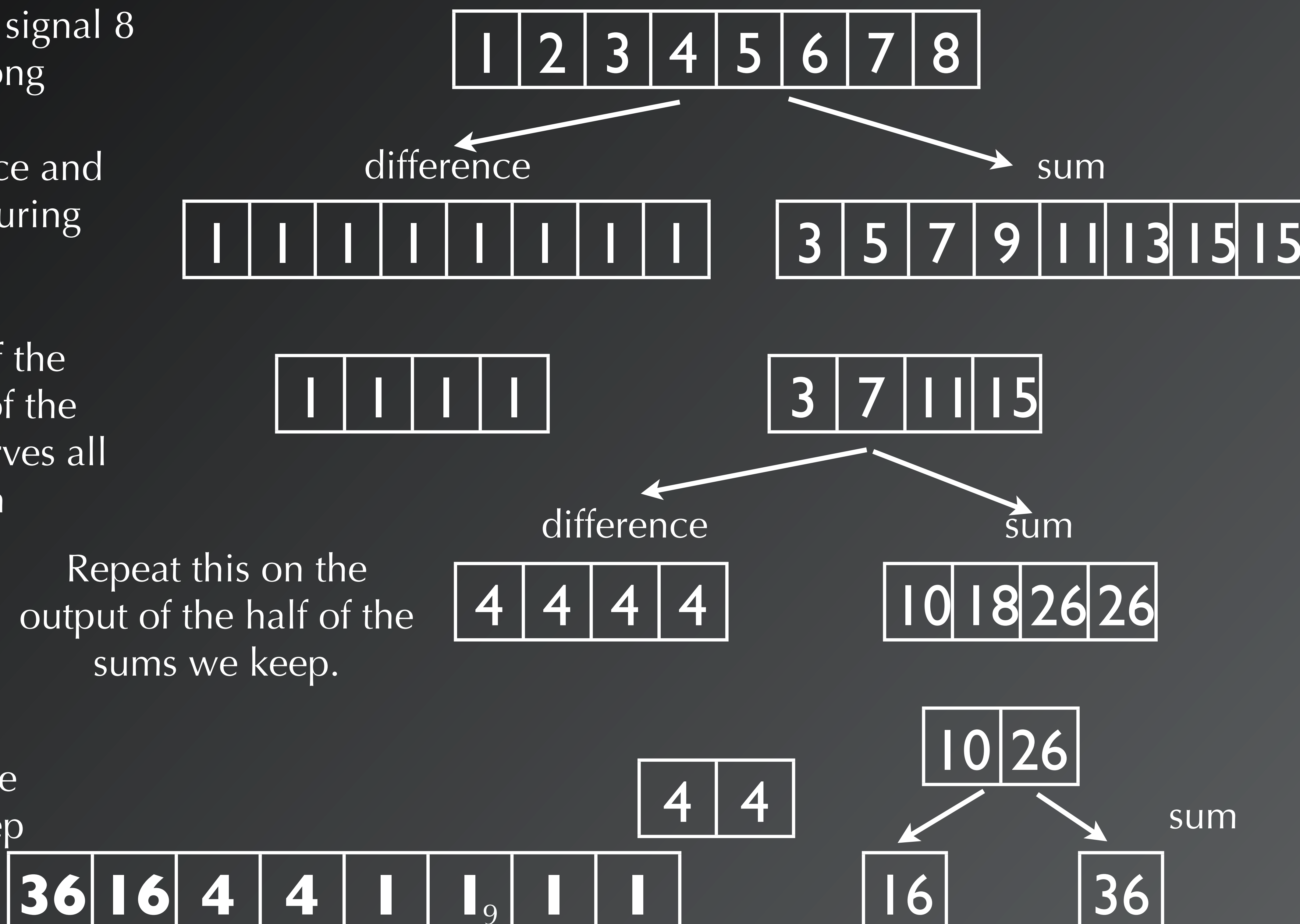
Imagine a short signal 8 samples long

Take the difference and sum of neighbouring cells

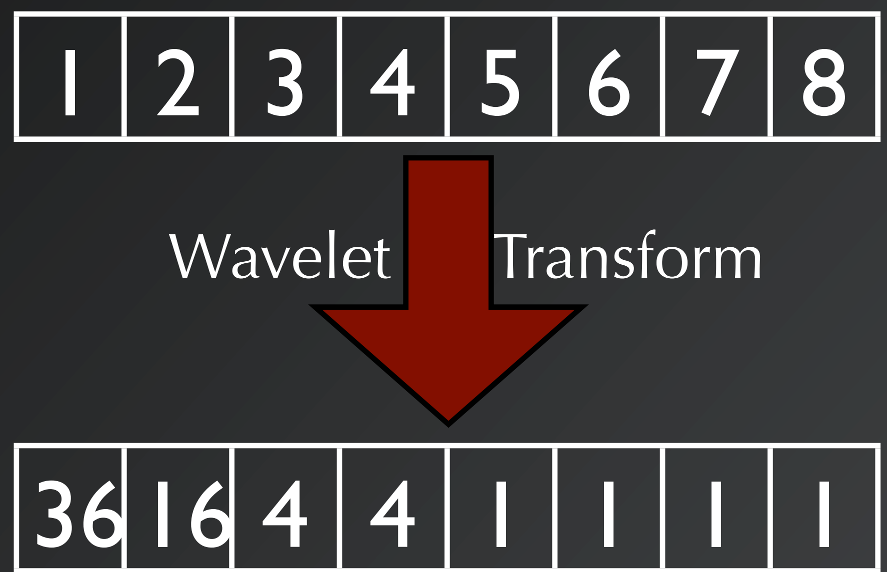
Keeping half of the sums and half of the differences preserves all information

Repeat this on the output of the half of the sums we keep.

The Wavelet Coefficients are the differences we Keep



Interpretation

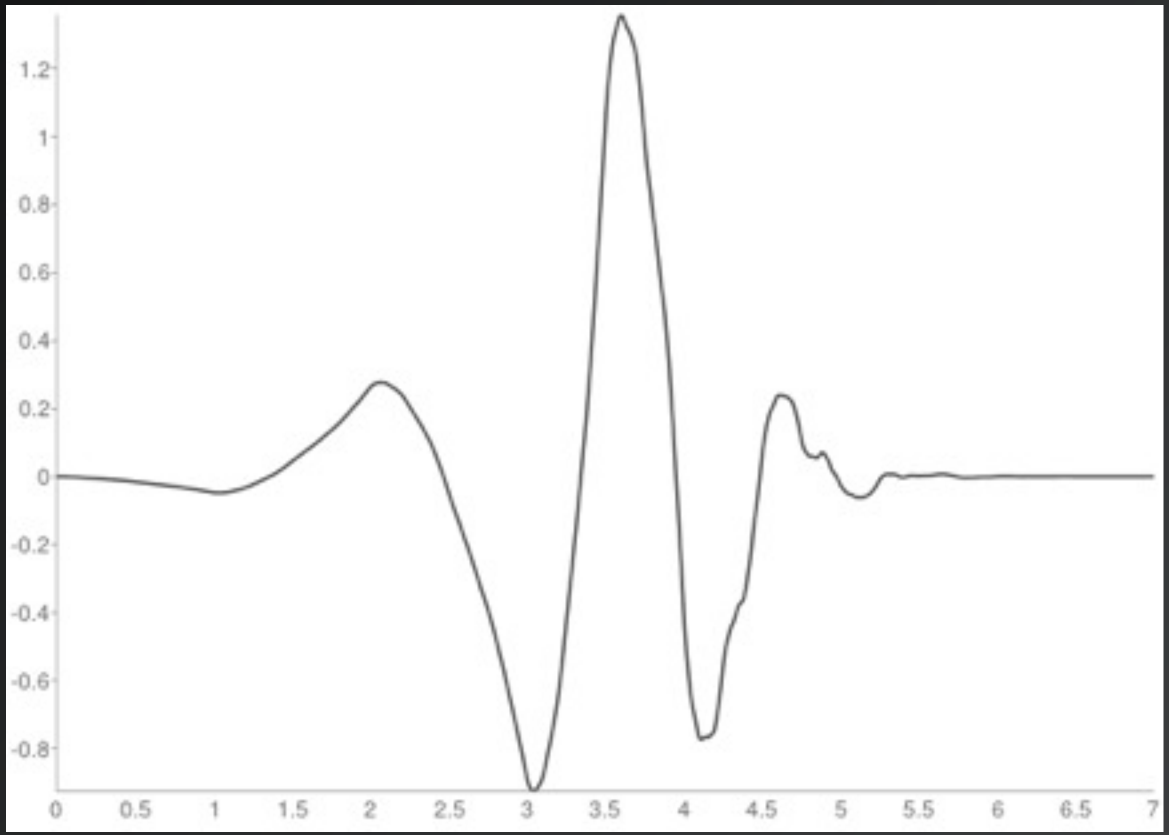


- The **high-frequency**, small scale cell-by-cell changes appear on the **right**. There is a uniform change of 1 at this scale.
- **Moderate** scale changes over scales of **two** cells are in the **middle** coefficients
- The **total** sum is on the **leftover** coefficient (36)
- Note the **high-frequency** component occupies fully **half** of the information!
- This is how a **compression** algorithm works

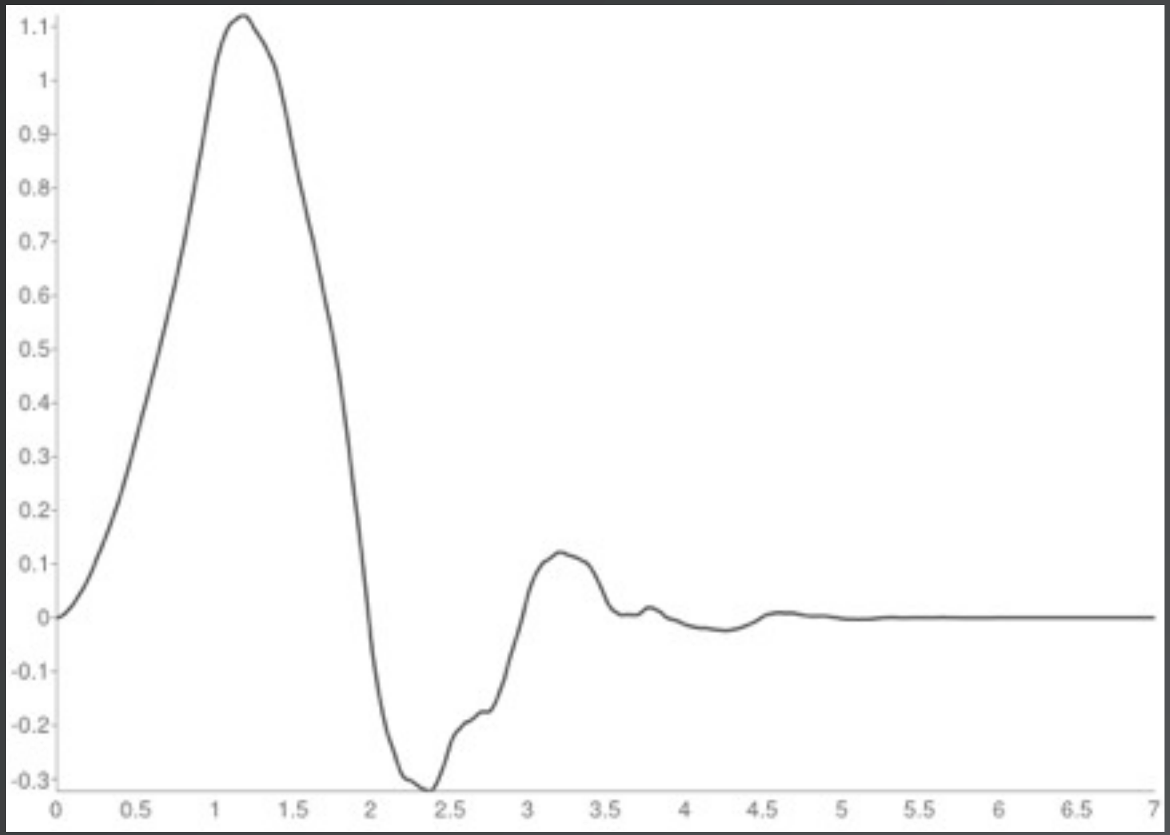
Daubechies Wavelet Basis

The Daubechies family of wavelets are usually more useful - encodes high frequency features better

Derived recursively by inverse transforming {1,0,0,...N}



The “well known” Daubechies 4 wavelet

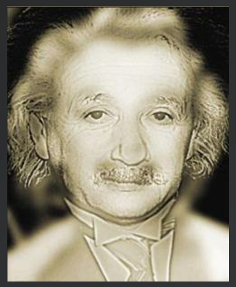


And the re-scaling function

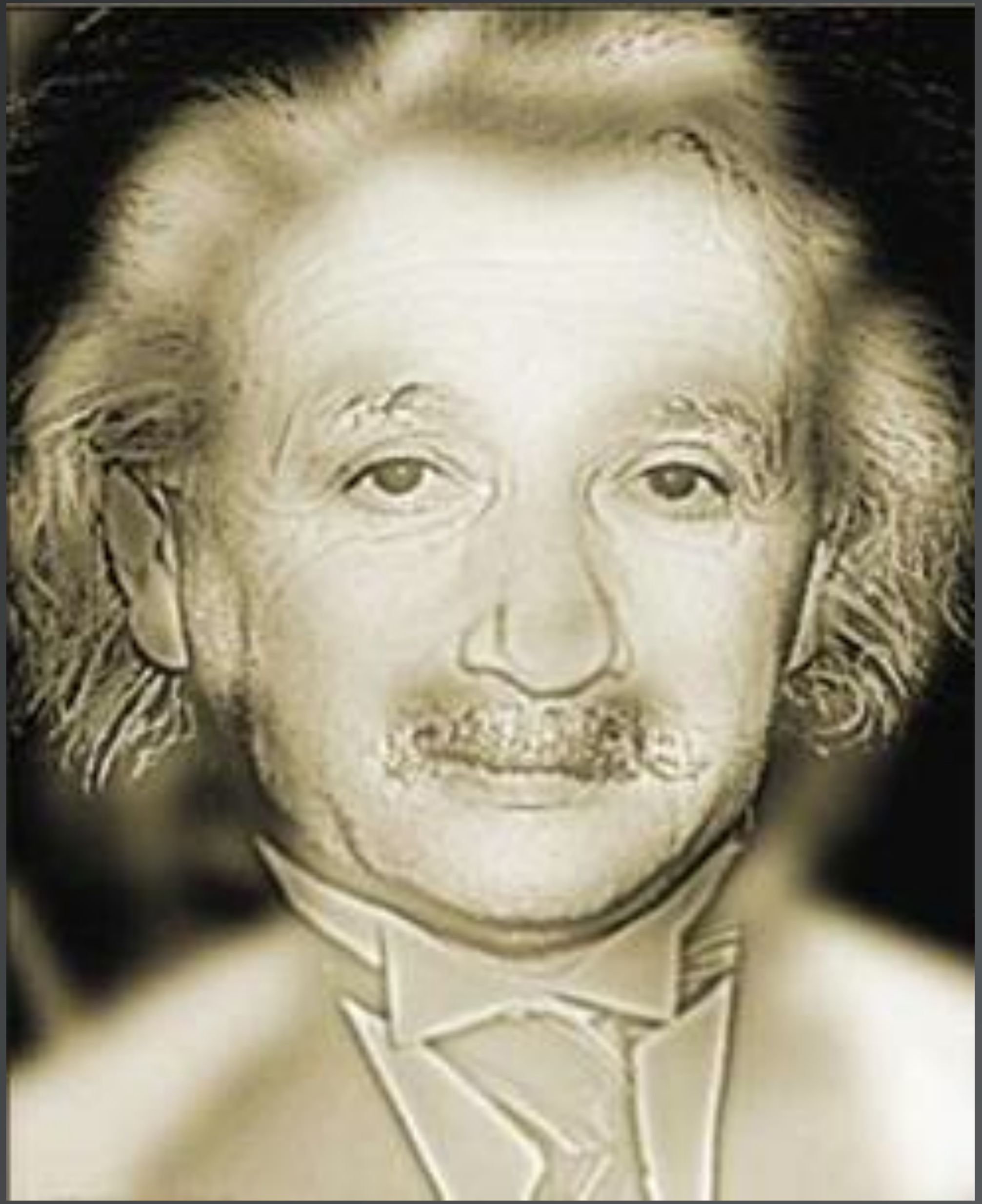
Multi-Resolution Analysis

Wavelets are an example of **multi-resolution analysis**

Your brain processes vision like this - analysing contrast changes over the local background

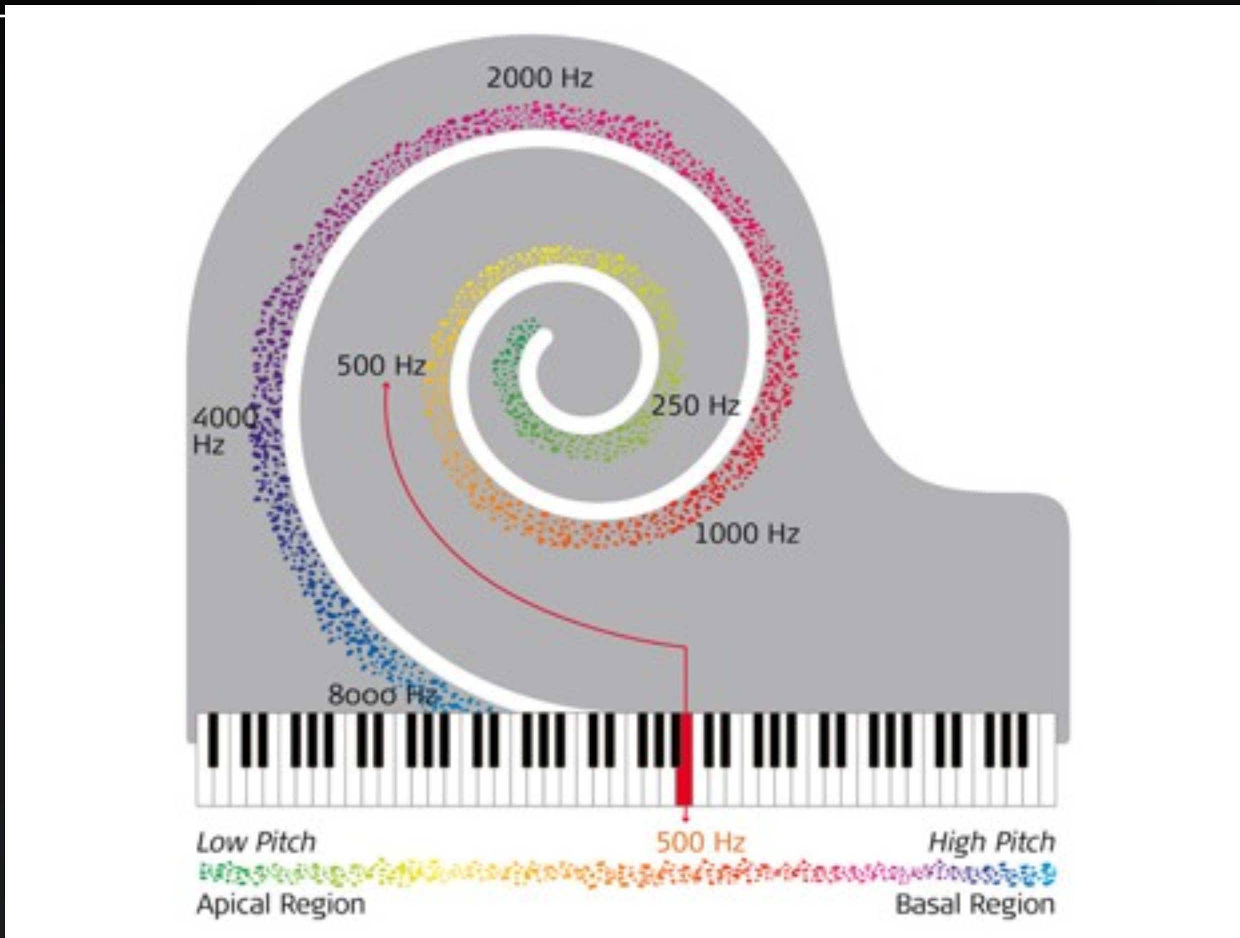


(This is the same image, but when seen from afar your brain uses the large-scale structure, up close it prioritises the fine detail)



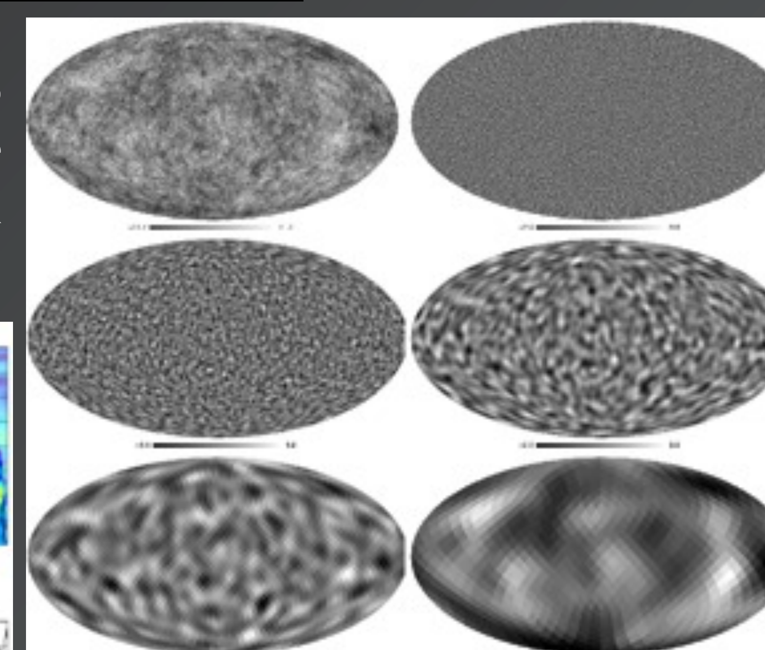
Already we can see how this is useful in physics to separate fine details from broad structures

Use of Wavelets



Astronomers use these techniques for image analysis, extraction of fine details like Einstein rings.

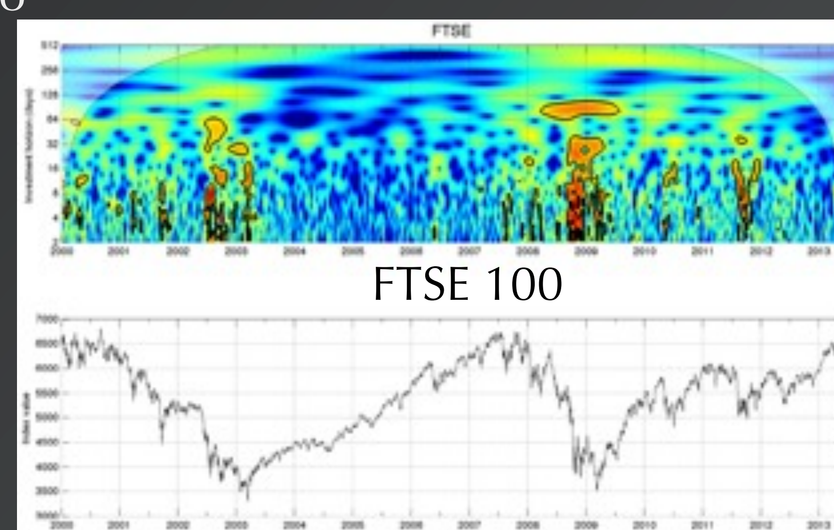
Wavelets used to decompose the CMBR



The cochlea inside your ear is arranged such that it performs a wavelet transform on sound. Attempts to "sonify" LHC data were doing wavelet analysis!

Wavelets can be used as the basis of a compression algorithm, including JPEG 2000

The Stock market is (allegedly) fractal, and subject to wavelet analysis

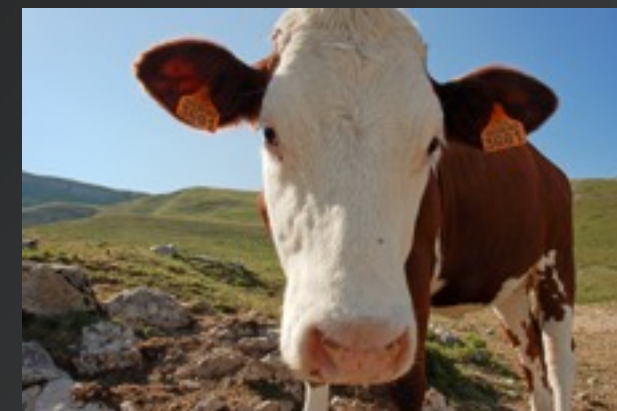


Self-Similarity

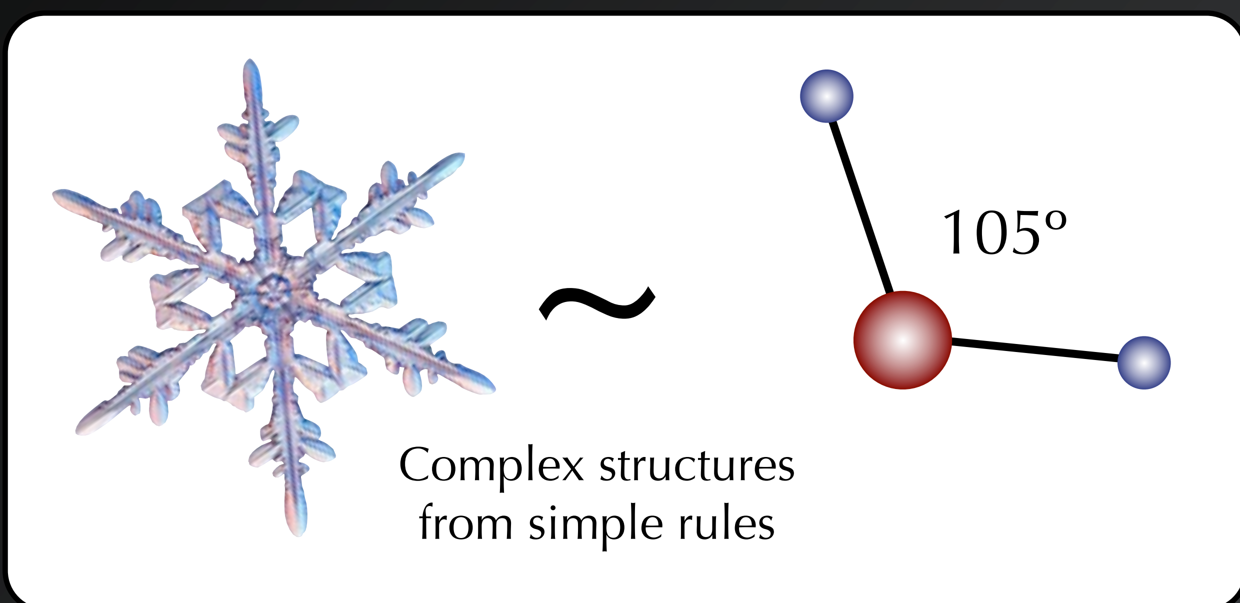
Benoît B.* Mandlebrot:

"Clouds are not spheres, mountains are not cones, coastlines are not circles, and bark is not smooth, nor does lightning travel in a straight line."

*The B. stands for "Benoît B. Mandlebrot"



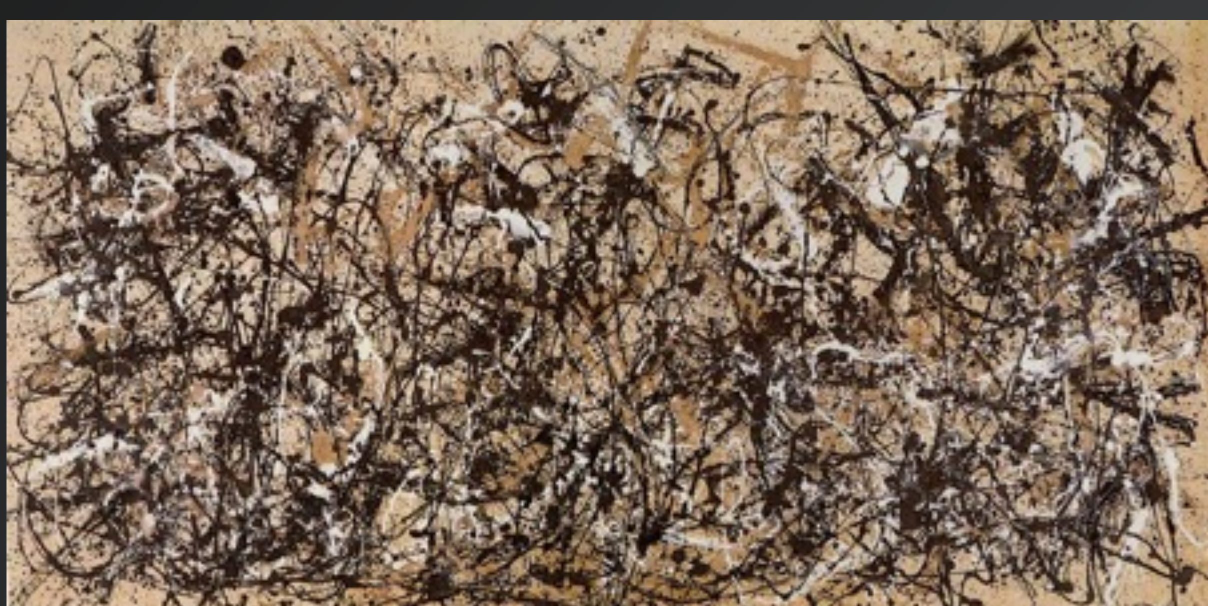
$\neq$



Building up a structure from repeated re-scalings of the same basic shape is very common in nature - fractal structures.

Wavelets can be a good way of understanding and looking for this self-similarity

If you have come across the scale-dependence of e.g. coupling constants in particle physics (α_s), then you might see this is a similar idea.



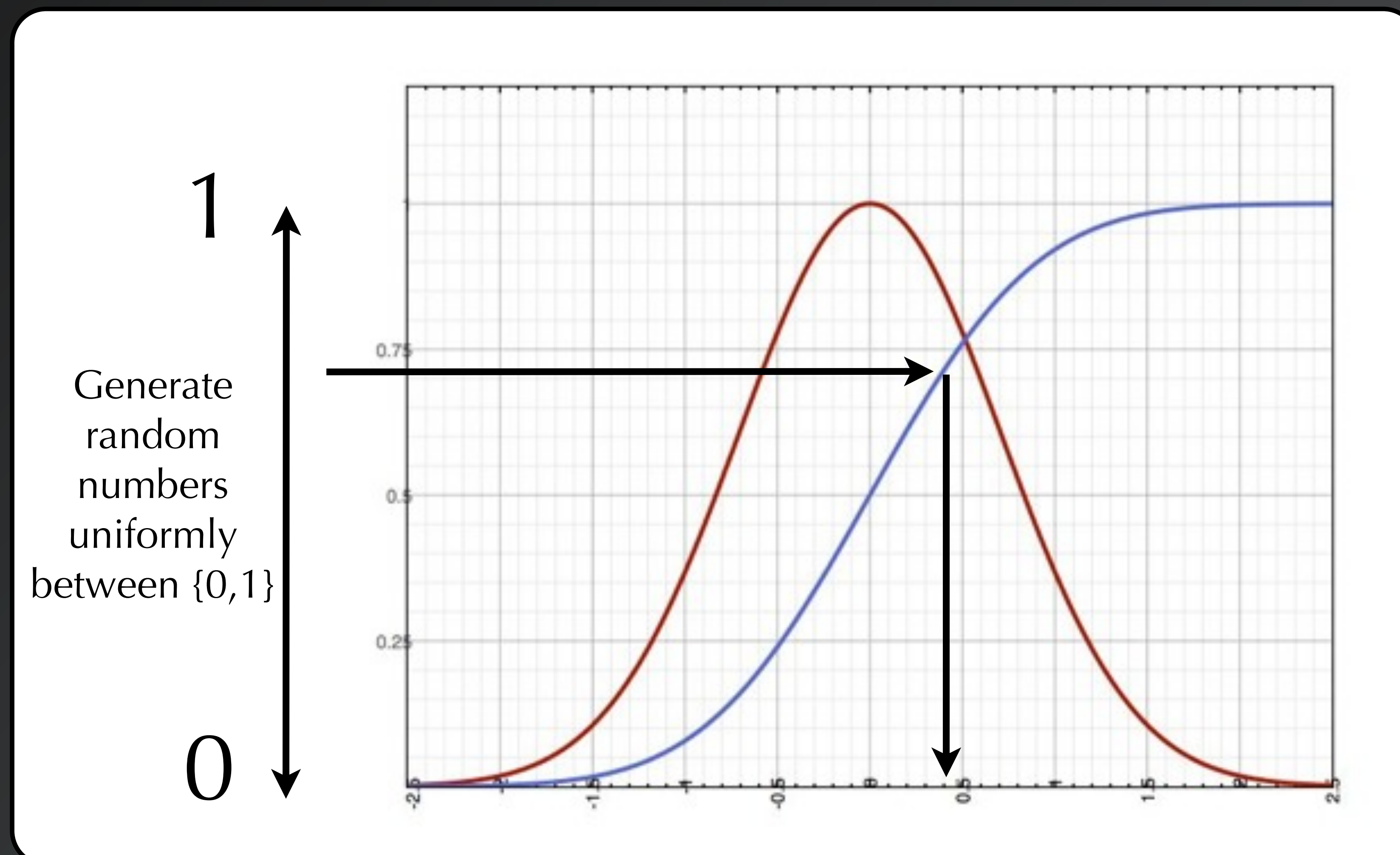
Jackson Pollock

Gaussian Noise example: `wavelet_gaussian.py`

- Uses Python and the PyWavelets package: http://pywavelets.readthedocs.org/en/latest/
- Produces a simple Gaussian and then adds noise
- You can vary the amount of noise by increasing or decreasing the “stats” parameter in the code, and also by changing the random seed
- First performs a continuous wavelet transform - note the separation in frequency between the noise and the signal
- Then perform a discrete transform, filter the coefficients to reduce the noise, and reverse the transform to re-obtain the signal
- At its simplest, the threshold sets to zero any coefficient whose value is below the RMS for its “level”
- Wavelet level = all coefficients having the same scaling parameter
- Try and play around with the filtering, see the effect of different thresholds, removing high and low frequency contributions, using different basis functions.

Note on Random Noise

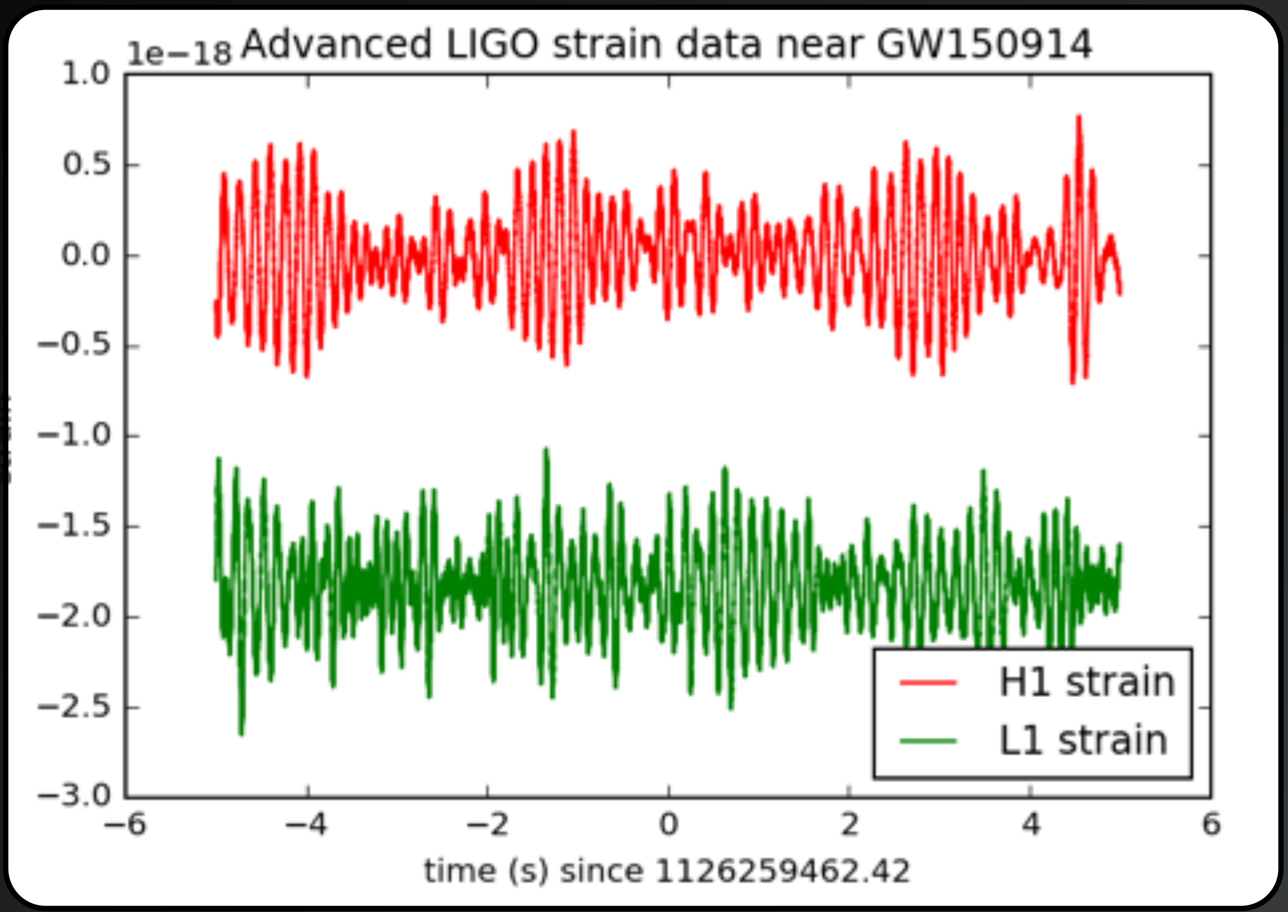
- To generate Gaussian-distributed random numbers for the noise, we need the inverse of the ERF function
- The ERF function is the *cumulative distribution function* of the Gaussian function
- The output of ERF is between 0 and 1
- So when you take the inverse ERF of a uniform random number between 0 and 1, you get a Gaussian distribution



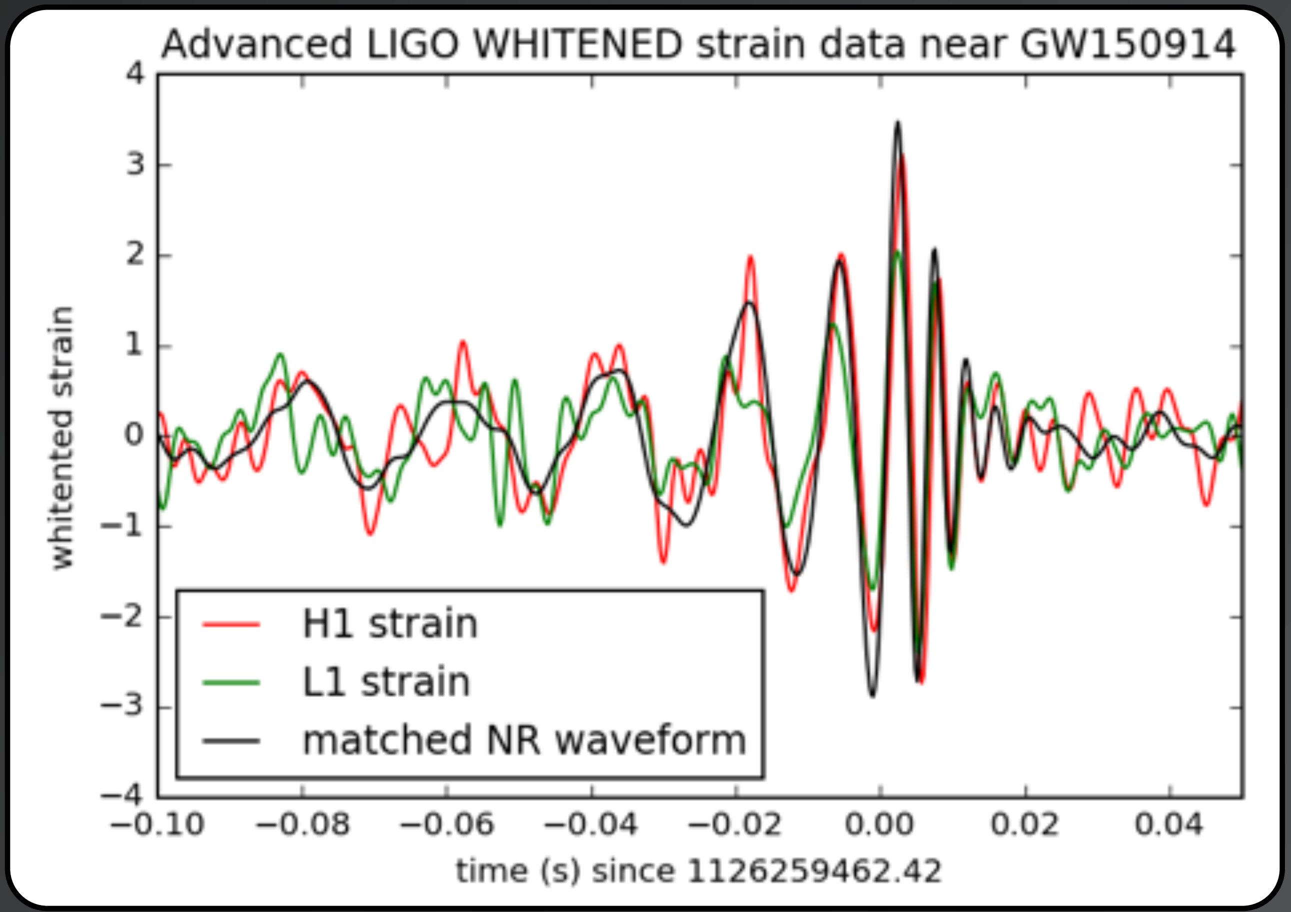
LIGO data

- The second example uses the LIGO data available from here: https://losc.ligo.org/events/GW150914/
- LIGO provide their own tutorial using Fourier analysis. You can follow it here: https://losc.ligo.org/s/events/GW150914/GW150914_tutorial.html
- We will use the 4096 Hz samples from the Livingston and Hanford detectors. Note they are 6.9 ms apart.
- Without signal processing, they do not look anything like the famous chirp sound formed when two black holes collide.
- Our goal will be to filter using wavelets to see if we can extract the chirp!

LIGO data



Advanced LIGO strain data near GW150914
1e−18
1.0
0.5
0.0
−0.5
−1.0
−1.5
−2.0
−2.5
−3.0
strain
−6
−4
−2
0
2
4
6
time (s) since 1126259462.42
H1 strain
L1 strain



Advanced LIGO WHITENED strain data near GW150914
4
3
2
1
0
−1
−2
−3
−4
whitented strain
−0.10
−0.08
−0.06
−0.04
−0.02
0.00
0.02
0.04
time (s) since 1126259462.42
H1 strain
L1 strain
matched NR waveform