Statistical Hypothesis Tests

Markus Ahlers
markus.ahlers@nbi.ku.dk

Niels Bohr International Academy

March 3, 2020

The Niels Bohr International Academy

UNIVERSITY OF COPENHAGEN

Statistical Hypothesis Tests

- Typical problem in physics and astronomy:

You have collected data with your experiment or observatory and want to test a theory (signal hypothesis H_1)?

➜ How can you judge if the hypothesis is correct/wrong?
➜ How does the alternative hypothesis (null hypothesis H_0) look like?
➜ How confident can you be that your conclusions are correct?

- In most cases there is a chance that your decision is wrong:
  - ✘ You **decided** that H_1 is correct, but it is actually wrong? **(type I error)**
  - ✘ You **decided** that H_1 is wrong, but it is actually correct? **(type II error)**

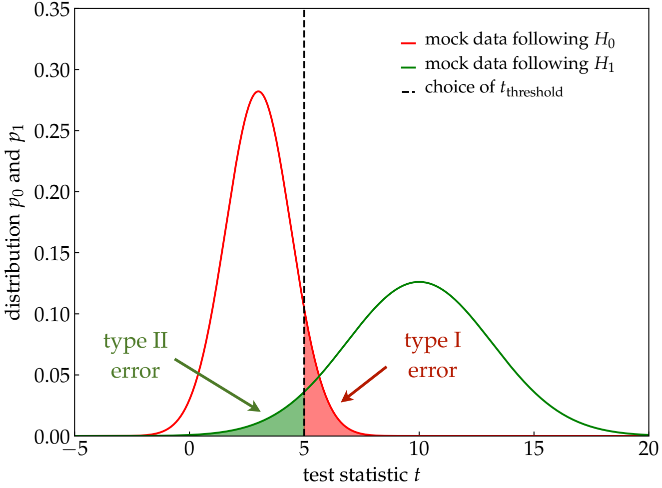
Statistical Hypothesis Tests

- A **statistical hypothesis test** is based on a quantity called **test statistic** that allows to quantify the degree of confidence that your decision was right or wrong.

- A useful test statistic:
  - is **sensitive** to the signal hypothesis H_1 (that's a must!)
  - is **efficiently calculable** (e.g. fast calculation on your computer)
  - has a **well-known behaviour** for data following the null hypothesis H_0 (more on this later)

- If we apply the statistical test to the observed data we can quantify the Type I ("false positive") and Type II ("false negative") errors by comparing to the **expected** test statistic distribution, p_0 and p_1, of data following background (H_0) and signal (H_1) hypothesis, respectively.

Test Statistic Distribution



In a hypothesis test we have to choose a **critical** t-value to either reject or accept the hypothesis.

Test Statistic Distribution

- **significance** (α) :
  Probability that background would have created outcome with same t or larger (**type I error**):

$$\alpha = \int_{t_{\rm obs}}^{\infty} {\rm d}t\, p_0(t) = \text{“}p\text{-value”}$$

- **Note:** It is a **convention** that t *increases* for a more “signal-like” outcome. If not, just define a new test statistic $t' = -t$.
- **power of test** ($1 - \beta$) :
  Probability that signal would have created outcome with same t or less (**type II error**):

$$\beta = \int_{-\infty}^{t_{\rm obs}} {\rm d}t\, p_1(t)$$

Statistical Hypothesis Tests

➜ A good statistical test will have good "separation" of p_0 and p_1 to allow a minimize type I/II errors. Separation from background allows to quantify **significance** of event excesses:

- **discovery** (in particle physics) :

$$\alpha \simeq 5.7 \times 10^{-7} (\text{“}5\sigma\text{”})$$

- **evidence** (in particle physics) :

$$\alpha \simeq 2.7 \times 10^{-4} (\text{“}3\sigma\text{”})$$

- Often, we want to estimate the **performance** of a statistical test prior to a measurement by simulations. We can determine this by tuning the signal strength, *e.g.* the IceCube experiment uses:
  - **discovery potential**:

$$\alpha \simeq 5.7 \times 10^{-7} (\text{“}5\sigma\text{”}) \qquad \text{and} \qquad \beta = 0.5$$

  - **90% sensitivity level**:

$$\alpha = 0.5 \qquad \text{and} \qquad \beta = 0.1$$

Today's Program

- **Today**, we will explore various examples of hypothesis tests and test statistics:
- Maximum likelihood ratio test
  - This is the most powerful test statistic (Neyman-Pearson theorem).
  - Allows to quantify background distributions p_0 (Wilks theorem).
  - We will study the applicability of Wilks theorem by a **numerical example** (exercise 1).
  - Discussion of trials factor corrections.
- Kolmogorov-Smirnov test
  - We will introduce this test by the cumulative auto-correlation function of event distributions on a sphere.
  - This test allows to study hidden structure in event distributions, *e.g.* deviations from an isotropic distribution.
  - We will **generate mock data** following isotropic and simple anisotropic distributions and study the performance of the test (exercise 2).

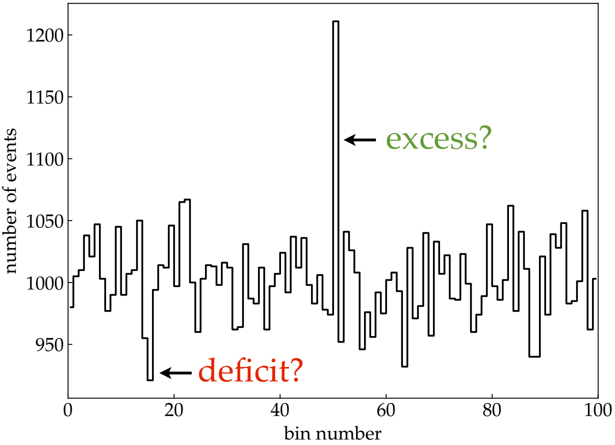
Today's Program (cont.)

- Angular power spectrum (optional, depending on time)
  - The power spectrum C_ℓ can be used as a test statistic that allows to study distributions of data (large number of events, temperature flucuations (CMB),...) on a sphere.
  - Brief introduction of spherical harmonics $Y_{\ell m}$ as basis functions on a sphere (exercise 3).
  - Introduction of the two-point angular correlation function and its relation to the power spectrum.
  - Introduction of the power spectrum.
  - Extraction of power spectra from mock data and background (exercise 4).

Part I
Maximum Likelihood Ratio

Recap: Maximum Likelihood Ratio

- Consider data ($N_{\rm tot}$ "events") distributed in $N_{\rm bins}$ bins.
- **Question:** Is there an excess or deficit in the data?



Recap: Maximum Likelihood Ratio

- Likelihood for data vector $\mathbf{x}$ and parameter vector $\boldsymbol{\mu}$:

$$\mathcal{L}(\boldsymbol{\mu}|\mathbf{x}) = \underbrace{\prod_{i=1}^{N_{\rm bins}} \frac{\mu_i^{x_i}}{x_i!} e^{-\mu_i}}_{\text{Poisson distributions}}$$

- Null hypothesis ("no signal")

$$\mu_i = \mu_{\rm bg} = {\rm const}$$

- Signal hypothesis ("signal (excess or deficit) in bin 1")

$$\mu_i = \begin{cases} \mu_{\rm sig} + \mu^*_{\rm bg} & i = 1 \\ \mu^*_{\rm bg} & 2 \leq i \leq N_{\rm bins} \end{cases}$$

! **Important note:** $\mu^*_{\rm bg} \neq \mu_{\rm bg}$

Maximum of Null Hypothesis

- **for convenience** : likelihood $\rightarrow$ log-likelihood (LLH)

$$\ln \mathcal{L}(\boldsymbol{\mu}|\mathbf{x}) = \sum_{i=1}^{N_{\rm bins}} (x_i \ln \mu_i - \mu_i) + \underbrace{\rm const}_{\text{independent of } \boldsymbol{\mu}}$$

- In general, maximum of LH (or LLH) can be derived numerically. **This example is easy enough to solve analytically:**
- maximum LH value determined by:

$$\frac{d \ln \mathcal{L}}{d\mu_{\rm bg}} = 0 = \sum_{i=1}^{N_{\rm bins}} \left(\frac{x_i}{\mu_{\rm bg}} - 1 \right)$$

- maximum $\hat{\mu}_{\rm bg}$ obeys:

$$\hat{\mu}_{\rm bg} = \frac{N_{\rm tot}}{N_{\rm bins}}$$

Maximum of Signal Hypothesis

- For the signal hypothesis we have to find the maximum w.r.t. signal and background strength:

$$\frac{d\ln\mathcal{L}}{d\mu^*_{\rm bg}} = 0 \quad \text{and} \quad \frac{d\ln\mathcal{L}}{d\mu_{\rm sig}} = 0$$

- Signal term $\mu_{\rm sig}$ is (by construction) only present in bin 1.
- maximum $\{\hat\mu^*_{\rm bg}, \hat\mu_{\rm sig}\}$ obeys:

$$\hat\mu^*_{\rm bg} = \frac{N_{\rm tot} - x_1}{N_{\rm bins} - 1}$$

$$\hat\mu_{\rm sig} = x_1 - \hat\mu^*_{\rm bg} = \frac{x_1 N_{\rm bins} - N_{\rm tot}}{N_{\rm bins} - 1}$$

Maximum LH Ratio

- test statistic λ is defined as maximum likelihood ratio:

$$\lambda(\mathbf{x}) = -2\ln\frac{\mathcal{L}(\mathbf{x}|\hat{\mu}_{\rm bg},0)}{\mathcal{L}(\mathbf{x}|\hat{\mu}^*_{\rm bg},\hat{\mu}_{\rm sig})}$$

- after some algebra using the solutions of $\hat{\mu}_{\rm bg}$, $\hat{\mu}^*_{\rm bg}$, and $\hat{\mu}_{\rm sig}$:

$$\lambda(\mathbf{x}) = 2x_1\ln\left(\frac{N_{\rm bins}}{N_{\rm tot}}x_1\right) + 2(N_{\rm tot}-x_1)\ln\left(\frac{N_{\rm bins}}{N_{\rm tot}}\frac{N_{\rm tot}-x_1}{N_{\rm bins}-1}\right) \quad (1)$$

- **Note:** The first (or second) term in Eq.(1) vanishes in the special case $x_1 = 0$ (or $N_{\rm tot} - x_1 = 0$).

- **bonus exercise**: Derive $\hat{\mu}_{\rm bg}$, $\hat{\mu}^*_{\rm bg}$, $\hat{\mu}_{\rm sig}$, and Eq.(1).

➜ **exercise 1** : Let's explore the behaviour of Eq.(1).

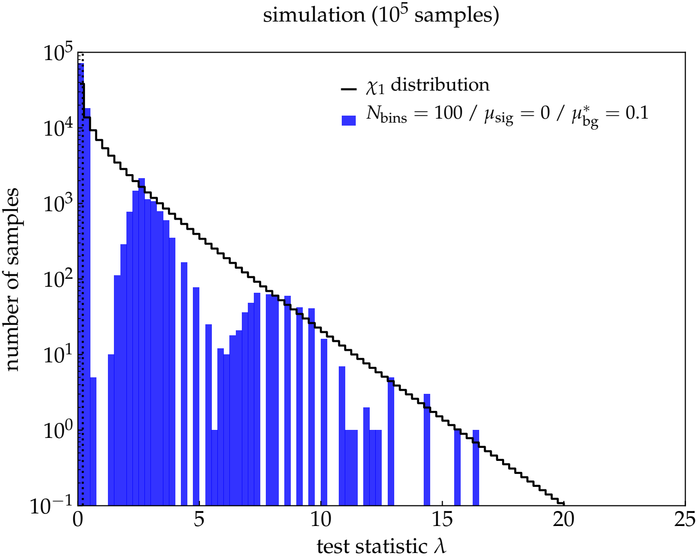
Exercise 1

- Generate mock data assuming $N_{\rm bins} = 100$ bins.
- Consider two categories:
  - **three background cases:**
    choose $\mu_{\rm sig} = 0$ and $\mu_{\rm bg} = 0.1$, 10, or 1000.
  - **two signal cases:**
    choose $\mu^*_{\rm bg} = 1000$ and signal in first bin ($i = 1$) with $\mu_{\rm sig} = 100$ and 200.
- For each case generate many (10^5) samples $\mathbf{x} = \{x_1, \ldots, x_{N_{\rm bins}}\}$ of mock data and calculate $\lambda(x_1, N_{\rm tot} = \sum_{i=1}^{N_{\rm bins}} x_i)$:

$$\lambda = 2x_1 \ln\left(\frac{N_{\rm bins}}{N_{\rm tot}} x_1\right) + 2(N_{\rm tot} - x_1) \ln\left(\frac{N_{\rm bins}}{N_{\rm tot}} \frac{N_{\rm tot} - x_1}{N_{\rm bins} - 1}\right)$$

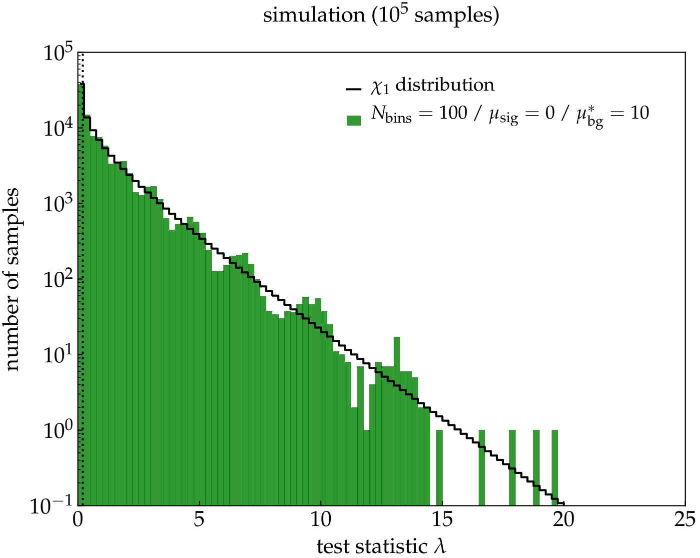
- Make histograms of the λ values to estimate the null and signal distributions.

Exercise 1: Background Cases



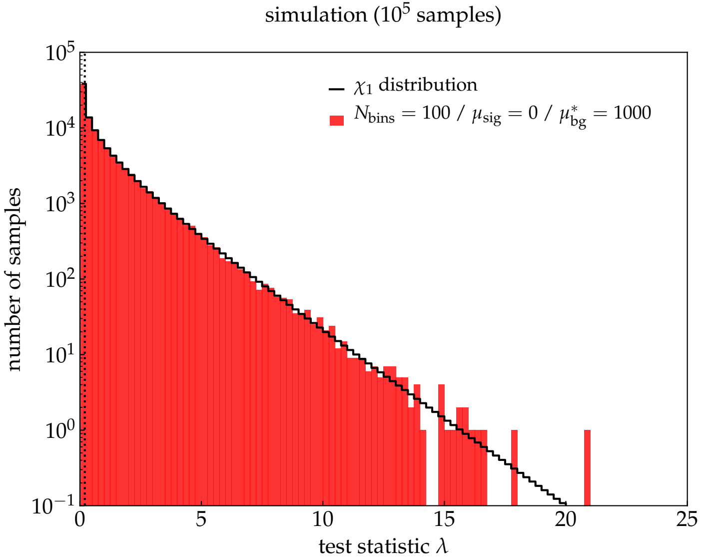
for python code see : `maxLH_produce.py` & `maxLH_show.py`

Exercise 1: Background Cases



for python code see : `maxLH_produce.py` & `maxLH_show.py`

Exercise 1: Background Cases



for python code see : `maxLH_produce.py` & `maxLH_show.py`

Wilks Theorem (1938)

THE LARGE-SAMPLE DISTRIBUTION OF THE LIKELIHOOD RATIO FOR TESTING COMPOSITE HYPOTHESES[1]

By S. S. Wilks

(...)

Theorem: If a population with a variate x is distributed according to the probability function $f(x, \theta_1, \theta_2 \cdots \theta_h)$, such that optimum estimates $\bar{\theta}_i$ of the θ_i exist which are distributed in large samples according to (3), then when the hypothesis H is true that $\theta_i = \theta_{0i}$, $i = m + 1, m + 2, \cdots h$, the distribution of $-2 \log \lambda$, where λ is given by (2) is, except for terms of order $1/\sqrt{n}$, distributed like χ^2 with $h - m$ degrees of freedom.

bonus exercise: Try to find this publication online.

Wilks Theorem

- **Prerequisites**:
  - Let $\mathbf{x}$ be data that follows a probability function $f(\mathbf{x}|\theta_1, \ldots, \theta_n)$.
  - The corresponding likelihood function $\mathcal{L}(\theta_1, \ldots, \theta_n|\mathbf{x})$ has a maximum at $\hat{\theta}_1$, ..., $\hat{\theta}_n$.
  - Let the true hypothesis have $\theta_1 = \theta_1^{(0)}$, ..., $\theta_m = \theta_m^{(0)}$ with $m < n$.
  - The *constrained* likelihood function $\mathcal{L}(\theta_1^{(0)}, \ldots, \theta_m^{(0)}, \theta_{m+1}, \ldots, \theta_n|\mathbf{x})$ has a maximum at $\hat{\hat{\theta}}_{m+1}$,...,$\hat{\hat{\theta}}_n$.
- **Wilks theorem:**

  For a large number of samples $\mathbf{x}$, the distribution of the test statistic

$$-2\ln\frac{\mathcal{L}(\theta_1^{(0)}, \ldots, \theta_m^{(0)}, \hat{\hat{\theta}}_{m+1}, \ldots, \hat{\hat{\theta}}_n|\mathbf{x})}{\mathcal{L}(\hat{\theta}_1, \ldots, \hat{\theta}_n|\mathbf{x})}$$

  approaches a χ^2_k distribution with $k = n - m$ in the limit of a large number of events, N_{tot}.

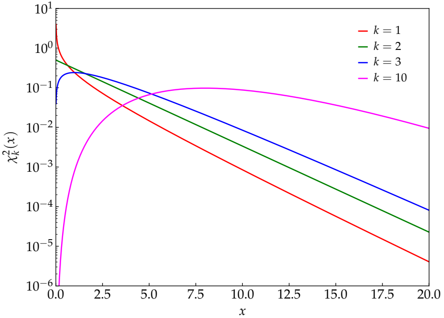
χ^2_k Distributions

- Definition of χ^2_k distributions:

$$\chi^2_k(x) = \frac{x^{k/2-1}e^{-x/2}}{2^{k/2}\Gamma(k/2)}$$

➜ **our example**:

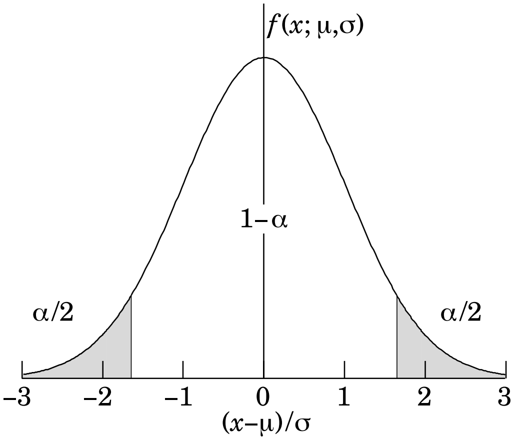
$$k = 2\,(\hat{\mu}^*_{\rm bg},\, \hat{\mu}_{\rm sig}) - 1\,(\hat{\mu}_{\rm bg}) = 1$$



➜ $\chi^2_k(x)$ is related to the integrated probability of a **k-variate normal distribution** (s : units of "sigma") :

$$\int\limits_{s^2} {\rm d}x \chi^2_k(x) = \int\limits_{\mathbf{r}^T\mathbf{\Sigma}^{-1}\mathbf{r}>s^2} {\rm d}r_1 \ldots {\rm d}r_k \frac{1}{\sqrt{(2\pi)^k det\mathbf{\Sigma}}} \exp(-\mathbf{r}^T\mathbf{\Sigma}^{-1}\mathbf{r}/2)$$

Example: χ^2_1 Distributions



$$\alpha = \int_{s^2} \mathrm{d}x \chi^2_1(x) = \int_{r^2/\sigma^2 > s^2} \mathrm{d}r \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{r^2}{2\sigma^2}\right) \qquad (\boldsymbol{\Sigma} \to \sigma^2)$$

Quick Example

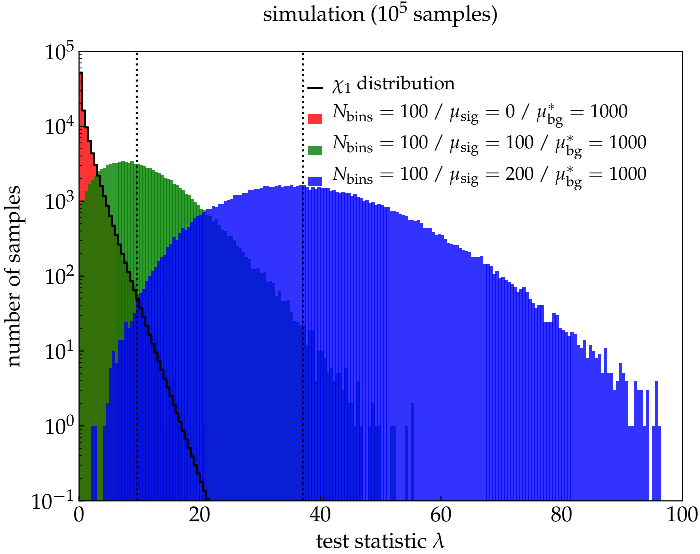
- For large N_{tot} we can apply Wilks theorem and assume that the background distribution follows a χ^2_1 distribution.

$$p-\text{value} = \int\limits_{\lambda_{\text{obs}}}^{\infty} \mathrm{d}x\chi^2_k(x) = 1 - \text{erf}(\sqrt{\lambda_{\text{obs}}/2})$$

- Assume $N_{\text{tot}} = 10^5$, $N_{\text{bins}} = 100$ and first bin contains:
  - 1100 events : maximum likelihood value $\lambda_{\text{obs}} \simeq 9.8$
    Wilks theorem: $p \simeq 0.0017$
  - 1150 events : maximum likelihood value $\lambda_{\text{obs}} \simeq 21.7$
    Wilks theorem: $p \simeq 3.2 \times 10^{-6}$
  - 1200 events : maximum likelihood value $\lambda_{\text{obs}} \simeq 38.0$
    Wilks theorem: $p \simeq 7.1 \times 10^{-10}$

➜ the 5σ discovery threshold corresponds to $x_1 \simeq 1162$ events

Exercise 1, cont.: Signal vs. Background



for python code see : `maxLH_produce.py` & `maxLH_show.py`

Sensitivity and Discovery Potential

- performance of the test
  - **sensitivity level:**
    defined as the level of $\mu_{\rm sig}$ such that 90% of the signal distribution is above 50% of the background distribution
  - **discovery potential:**
    defined as the level of $\mu_{\rm sig}$ such that 50% of samples have a chance probability of 5.7×10^{-7} to be generated by background only

➜ This is a **challenge for brute-force background simulation** – you need $N_{\rm samples} \gg 10^7$ for accuracy!

- However, **Wilks theorem** allows to extrapolate the background distribution very easily:

➜ For χ_1 distribution we know that the "5σ" level corresponds to:

$$\lambda_{\rm threshold} = 5^2 = 25$$

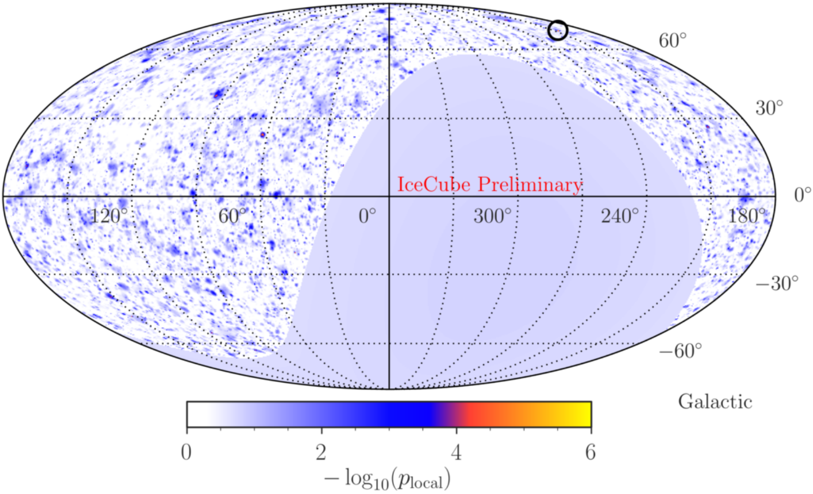
Trial Correction

- What happens if we want to find a signal not just in bin 1 but in *any* of the $N_{\rm bins}$ bins?
- We can simply repeat the test over all bins and identify the bin with minimum p-value p_*.
- **Problem:** There are many bins ("hypothesis") and we have to account for the fact that there can be a chance fluctuation in the local p-values.
- If $N_{\rm bins}$ are independent of each other (as in our example) then we can define a post-trial p-value as

$$p_{\rm post} = 1 - \underbrace{(1-p_*)^{N_{\rm bins}}}_{\text{background probability}} \simeq N_{\rm bins} p_*$$

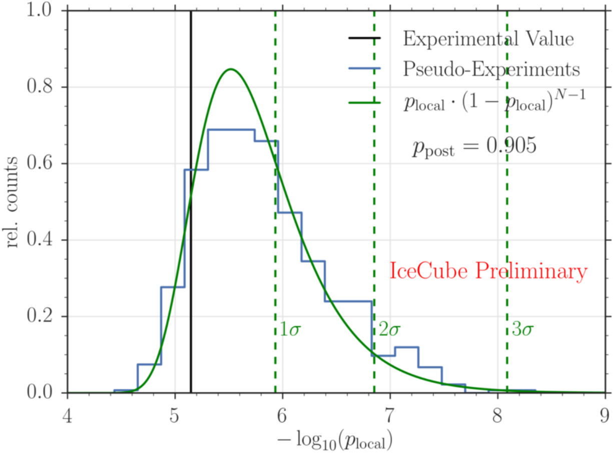
- Number of independent "trials", $N_{\rm trials}$, is often difficult to estimate.

Example: IceCube Neutrino Data



"All-sky" point-like source search:
each location tested for an excess!

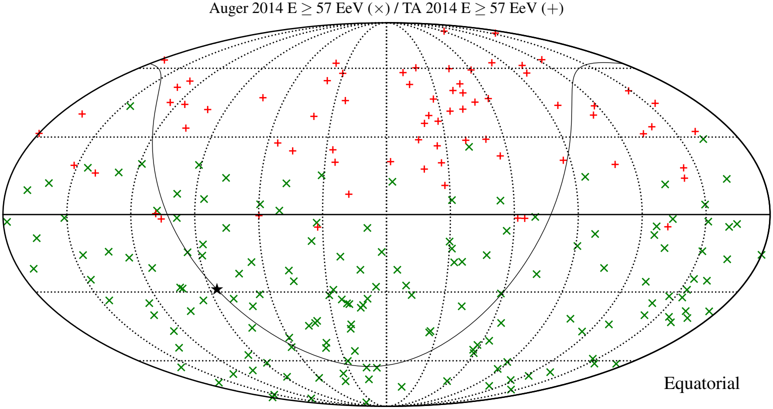
Example: IceCube Neutrino Data



- Trial factor: $N_{\rm trials} \sim N_{\rm bins} \sim \mathcal{O}(1000)$
- **IceCube procedure:** choose maximal $p_{\rm local}$ in sky map as a new **test statistic** and compare against maximal $p_{\rm local}$ of randomly generated sky maps

Part II
Kolmogorov Smirnov Test

Example: Arrival Direction of Cosmic Rays



Anisotropies in the arrival directions of ultra-high energy cosmic rays (data from the observatories Telescope Array (TA) and Auger).

Auto-Correlation

- So far, we have only looked into local excesses in individual bins.
- This method was not sensitive to the correlation between events, *e.g.* in neighbouring bins or in small clusters.
- Consider $N_{\rm tot}$ events distributed on a sphere with position $\mathbf{n}_i$ (unit vector).
- For two events with label i and j ($i \neq j$) we can define an angular distance:

$$\cos\varphi_{ij} = \mathbf{n}_i \cdot \mathbf{n}_j$$

- The **cumulative two-point auto-correlation function** is defined as

$$\mathcal{C}(\{\mathbf{n}_i\},\varphi) = \frac{2}{N_{\rm tot}(N_{\rm tot}-1)}\sum_{i=1}^{N_{\rm tot}}\sum_{j=1}^{i-1}\Theta(\cos\varphi_{ij} - \cos\varphi) \qquad (2)$$

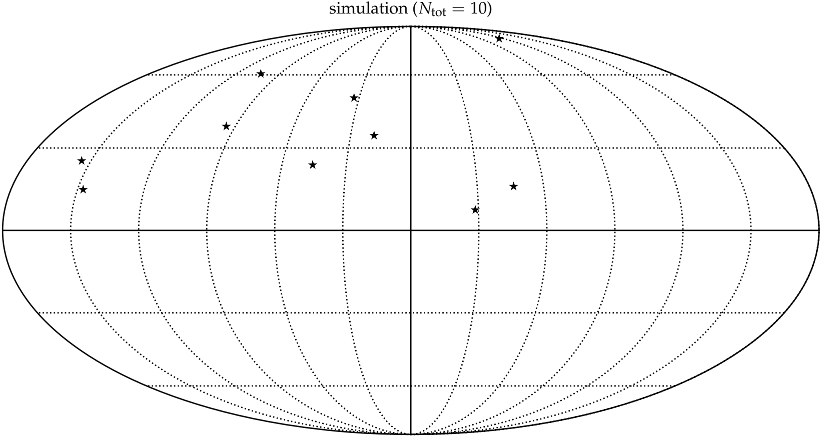
  with **step function** $\Theta(x) = 1$ for $x \geq 0$ and $\Theta(x) = 0$ for $x < 0$.

➜ This expression counts the pairs of events within angular distance φ.

Exercise 2: Event Distributions

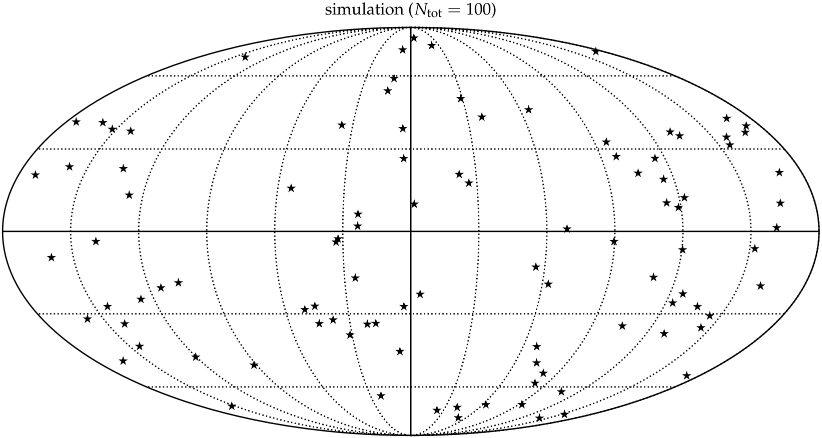
- Generate mock data of events on a sphere for two categories:
- **isotropic distribution**:
  - generate $N_{\rm tot}$ events randomly distributed on a sphere
  - e.g. `python` module `healpy` allows for pixelised sky maps with equal pixel sizes
  - In general: How would you sample from an azimuth angle φ and zenith angle θ to obtain a random distribution?
  - Derive the two-point auto-correlation function for the distribution.
  - What distribution do you expect for a large number of events?
- **biased distribution** (bonus exercise):
  - generate $N_{\rm tot}$ events following a non-isotropic distribution
  - *e.g.* only sample events within a limited azimuth or zenith range, or events following a dipole distribution
  - How does the auto-correlation function compare to that of the isotropic distribution?

Exercise 2: Isotropic Distribution



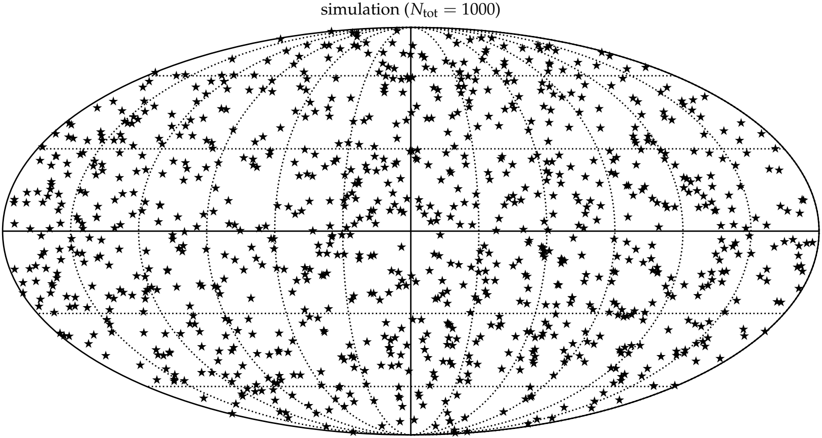
for python code see : `twopoint.py`

Exercise 2: Isotropic Distribution



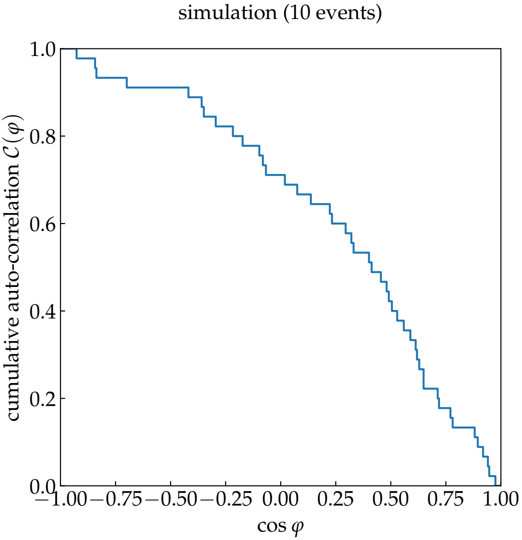
for python code see : `twopoint.py`

Exercise 2: Isotropic Distribution



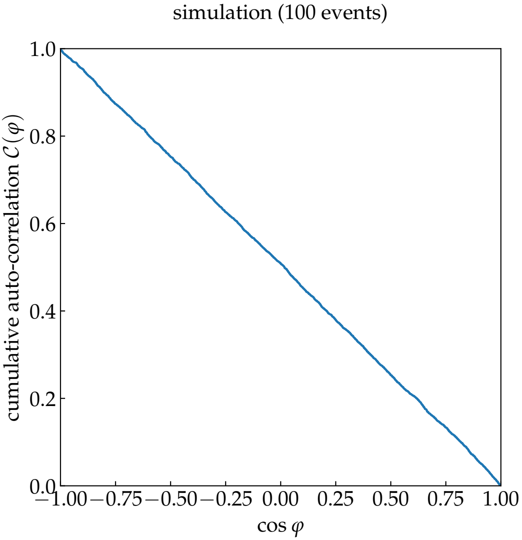
for python code see : `twopoint.py`

Exercise 2: Isotropic Distribution



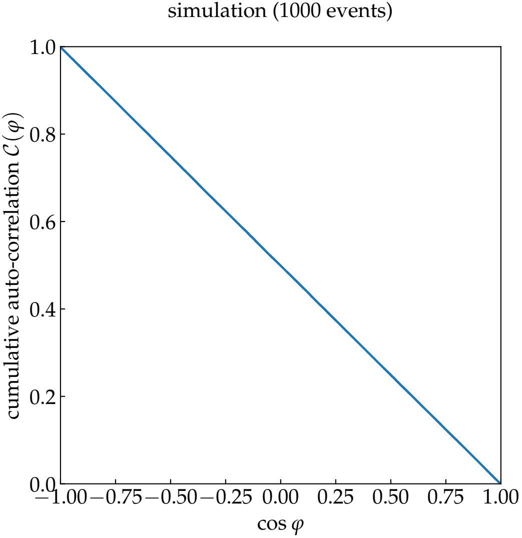
for python code see : `twopoint.py`

Exercise 2: Isotropic Distribution



for python code see : `twopoint.py`

Exercise 2: Isotropic Distribution



for python code see : `twopoint.py`

Exercise 2: Large-N limit

- In the limit of a large number of events, $N_{\rm tot}$ the cumulative distribution is just given by the relative size of the solid angle $\Delta\Omega$ with half-opening angle φ

$$\lim_{N_{\rm tot}\to\infty} \mathcal{C}(\{\mathbf{n}_i\},\varphi) \to \mathcal{C}_{\rm iso}(\varphi) = \frac{\Delta\Omega}{4\pi}$$

- solid angle

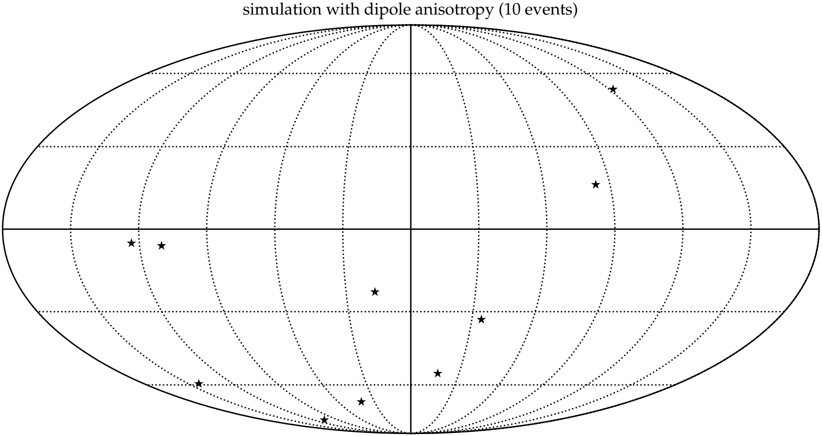
$$\Delta\Omega = 2\pi(1-\cos\varphi)$$

- isotropic distribution:

$$\mathcal{C}_{\rm iso}(\varphi) = \frac{1}{2}(1-\cos\varphi)$$

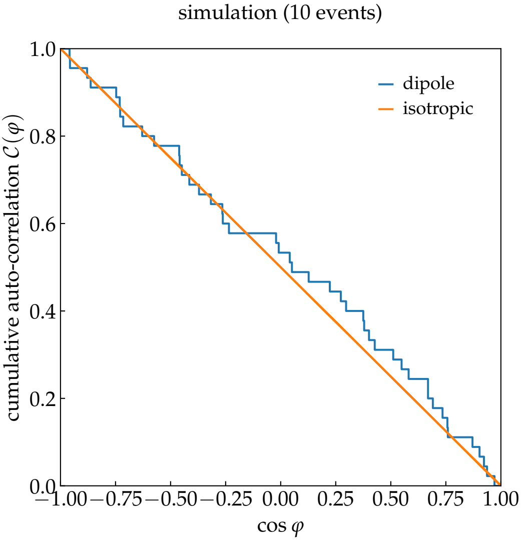
! **Note:** an isotropic distribution of a **finite** number of events will always show deviations from $\mathcal{C}_{\rm iso}$.

Exercise 2: Anisotropic Distribution



for `python` code see : `twopoint.py`

Exercise 2: Anisotropic Distribution



for python code see : `twopoint.py`

Kolmogorov-Smirnov (KS) Test

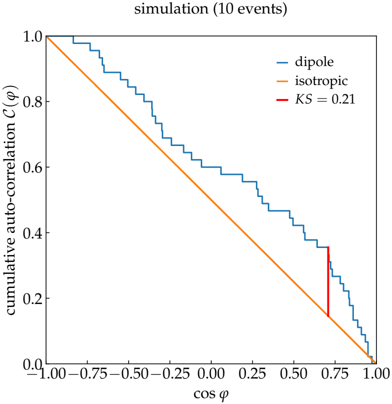
- We want to define a quantity that is a statistical measure for the difference between the empirical distribution and background distribution.
- Area between two curves?

$$\int \mathrm{d}\cos\varphi |\mathcal{C}(\{\mathbf{n}_i\},\varphi) - \mathcal{C}_{\rm iso}(\varphi)|$$

- Or, more general (L^p norm)?

$$\left[\int \mathrm{d}\cos\varphi\, |\mathcal{C}(\{\mathbf{n}_i\},\varphi) - \mathcal{C}_{\rm iso}(\varphi))\,|^p\right]^{\frac{1}{p}}$$

- **Kolmogrov-Smirnov:** $p \to \infty$.



Kolmogorov-Smirnov (KS) Test

- In general, given two cumulative probability distributions, $0 \leq A(x) \leq 1$ and $0 \leq B(x) \leq 1$, we can define the **Kolmogorov-Smirnov test** as:

$$KS = \mathbf{sup}_x |A(x) - B(x)|$$

- Cumulative auto-correlation function $\mathcal{C}(\{\mathbf{n}_i\}, \varphi)$ follows the probability distributions to find a pair of events within an angular distance φ.

- We will use this in the following to define a test statistic, that describes **deviation from an isotropic background distribution**:

$$KS(\{\mathbf{n}_i\}) = \mathbf{sup}_\varphi |\mathcal{C}(\{\mathbf{n}_i\}, \varphi) - \mathcal{C}_{\rm iso}(\varphi)|$$

Kolmogorov-Smirnov (KS) Test

- **Plan:** For a fixed number of events $N_{\rm tot}$ we can simulate isotropic event distributions (null hypothesis) and their KS values (test statistic).

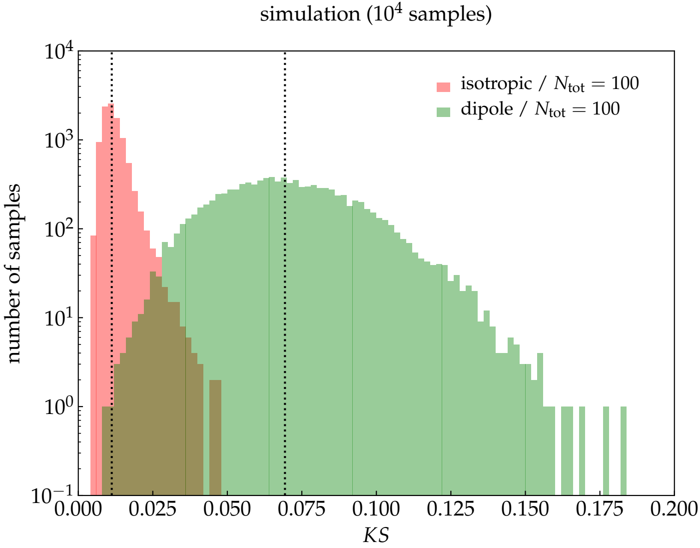
➜ Separation of KS for observed data from background distribution allows to **estimate significance of an excess**.

- Similar to Wilks theorem the background distribution approaches a **predictive asymptotic behaviour** for large number of events, but we will not cover this here.
- number of event pairs increases as

$$N_{\rm pair} = \frac{1}{2}N_{\rm tot}(N_{\rm tot} - 1) \propto N_{\rm tot}^2$$

✘ Cumulative auto-correlation function in Eq. (2) becomes numerically inefficient.

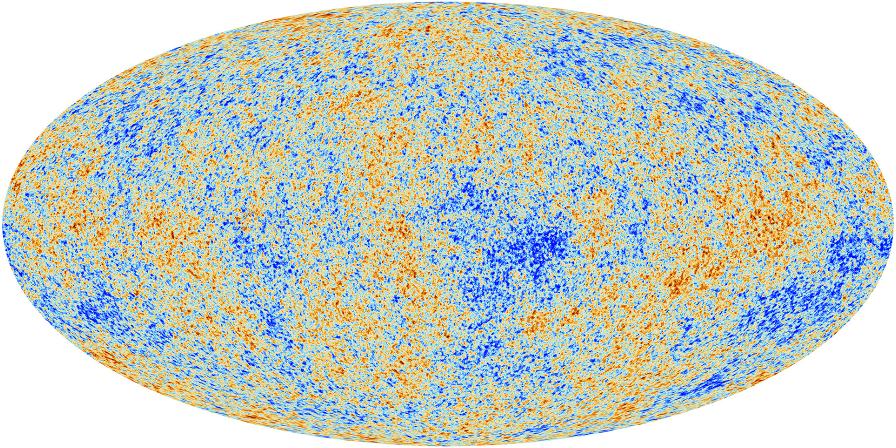
Kolmogorov-Smirnov (KS) Test



for python code see : `KS_produce.py` & `KS_show.py`

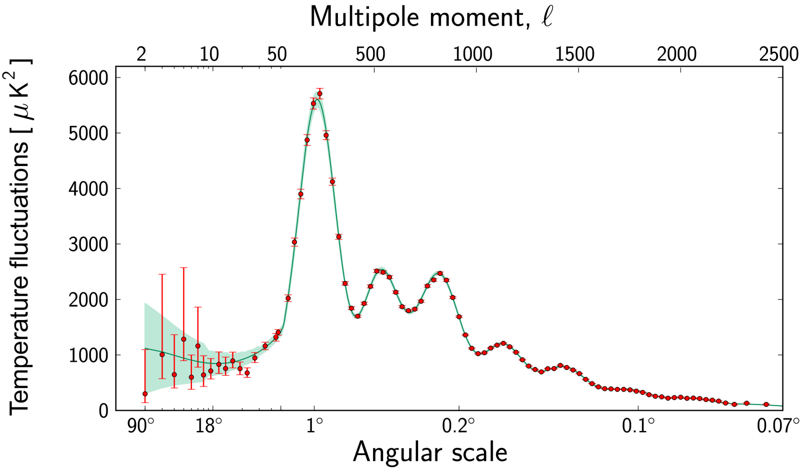
Part III
Angular Power Spectrum
(optional, depending on time)

Example: Temperature Fluctuation in CMB



Temperature anisotropies of the cosmic microwave background (CMB) observed by the Planck satellite.

Example Temperature Fluctuation in CMB



The angular power spectrum C_ℓ of the temperature fluctuations.

Auto-Correlation for Large $N_{\rm tot}$

- In the Kolmogorov-Smirnov test we observed that for large $N_{\rm tot}$ the number of pairs increase as $N_{\rm tot}^2$ and the calculation can become very inefficient.
- In **large-$N_{\rm tot}$ limit** we can approximate the event distribution by a smooth function

$$g(\Omega) = \lim_{N_{\rm bins}\to\infty}\frac{\Delta n(\Omega)}{N_{\rm tot}\Delta\Omega}$$

- On a smooth distribution we can define the **two-point auto-correlation function** as

$$\xi(\varphi) = \int {\rm d}\Omega_1 \int {\rm d}\Omega_2 \delta(\mathbf{n}(\Omega_1)\mathbf{n}(\Omega_2) - \cos\varphi) g(\Omega_1) g(\Omega_2)$$

- **Note:** This is the differential version of cumulative auto-correlation function.

Auto-Correlation for Large $N_{\rm tot}$

- **comment 1** : *cumulative* two-point auto-correlation function:

$$\mathcal{C}(\varphi) = \int\limits_{\cos\varphi}^{1} \mathrm{d}\cos\varphi' \xi(\varphi')$$

- **comment 2** : isotropic distribution $g(\Omega) = 1/(4\pi)$

$$\xi(\varphi) \stackrel{\dagger}{=} \frac{1}{2} \quad \to \quad \mathcal{C}_{\rm iso}(\varphi) = \int\limits_{\cos\varphi}^{1} \mathrm{d}\cos\varphi' \frac{1}{2} = \frac{1}{2}(1-\cos\varphi) \qquad (\checkmark)$$

† follows from:

$$\delta(\mathbf{n}(\Omega_1)\mathbf{n}(\Omega_2) - \cos\varphi) = 2\pi \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} P_\ell(\cos\varphi) Y^*_{\ell m}(\Omega_1) Y_{\ell m}(\Omega_2)$$

Spherical Harmonics

- Every smooth function $g(\theta,\phi)$ on a sphere can be decomposed in terms of spherical harmonics $Y_{\ell m}$:

$$g(\theta,\phi) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} a_{\ell m} Y_{\ell m}(\theta,\phi)$$

- coefficients given by:

$$a_{\ell m} = \int \mathrm{d}\Omega Y^*_{\ell m}(\theta,\phi) g(\theta,\phi)$$

➜ for real-valued functions:

$$a^*_{\ell m} = (-1)^m a_{\ell -m}$$

Spherical Harmonics

- The low-ℓ components are
  - $\ell = 0$: **monopole** $Y_{00} = 1/\sqrt{4\pi}$
  - $\ell = 1$: **dipole**

$$Y_{10} = \sqrt{\frac{3}{4\pi}}\cos\theta \quad Y_{1-1} = \sqrt{\frac{3}{8\pi}}\sin\theta e^{-i\varphi} \quad Y_{11} = -\sqrt{\frac{3}{8\pi}}\sin\theta e^{i\varphi}$$

  - $\ell = 2$: **quadrupole**, $\ell = 3$: **octupole**, etc.
- **angular power spectrum**:

$$C_\ell = \frac{1}{2\ell+1}\sum_{m=-\ell}^{\ell}|a_{\ell m}|^2$$

- simple relation to ξ via Legendre polynomials P_ℓ:

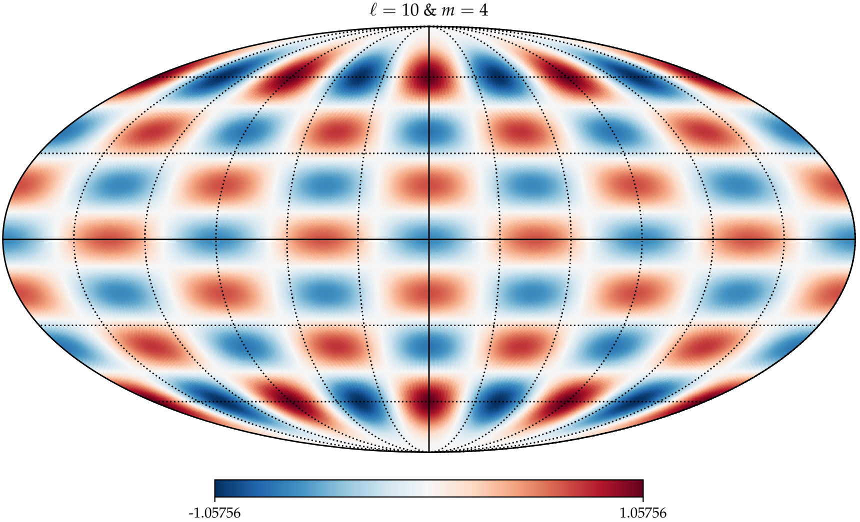
$$\xi(\varphi) = 2\pi\sum_\ell(2\ell+1)C_\ell P_\ell(\cos\varphi)$$

Exercise 3

- visualize spherical harmonics for various combinations of ℓ and m
- for example, in `python` use `healpy`:

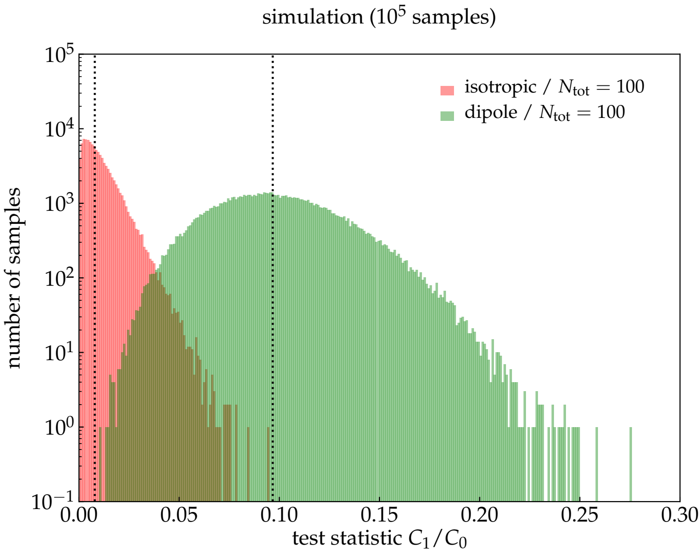
```
nside = 128
npix = H.nside2npix(nside)

LMAX = 4*nside
almsize = np.int(((LMAX+2)*(LMAX+1))/2)
alm = np.zeros(almsize,dtype=np.complex)

l = 10
m = 4

index = H.sphtfunc.Alm.getidx(LMAX,l,m)
alm[index] = 1.0

map = H.alm2map(alm,nside,lmax=LMAX)
mapmax = max(max(map),max(-map))
maptitle = r'$\ell= ' + str(l) + '$ \& $m= ' + str(m) + '$'

H.mollview(map,cmap=cm.RdBu_r,max=mapmax,min=-mapmax,title=maptitle)
H.graticule()
show()
```

Exercise 3 : Example Map of Spherical Harmonic



for `python` code see : `Ylm.py`

Power Spectrum



for python code see : `C1_produce.py` & `C1_show.py`

Power Spectrum

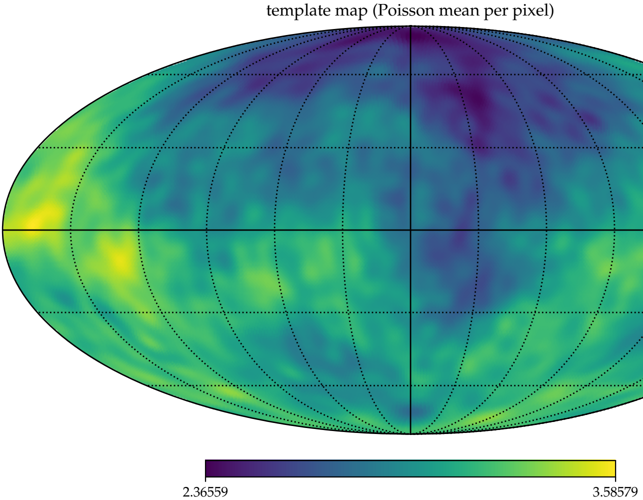
- In general, we want to judge if a distribution of events shows evidence for an excess in the power spectrum compared to background expectations.
- **Strategy**: Generate background maps from data via scrambling:
  a) choose two random bins i and j
  b) interchange the events in the two bins
  c) repeat from a) until $N_{\rm scramble} \gg N_{\rm bins}$
- The distribution of the power spectrum of these maps gives an estimate of the median and variance of the background power.
- Expected median noise level:

$$\mathcal{N} = \frac{1}{N_{\rm tot}}$$

Exercise 4

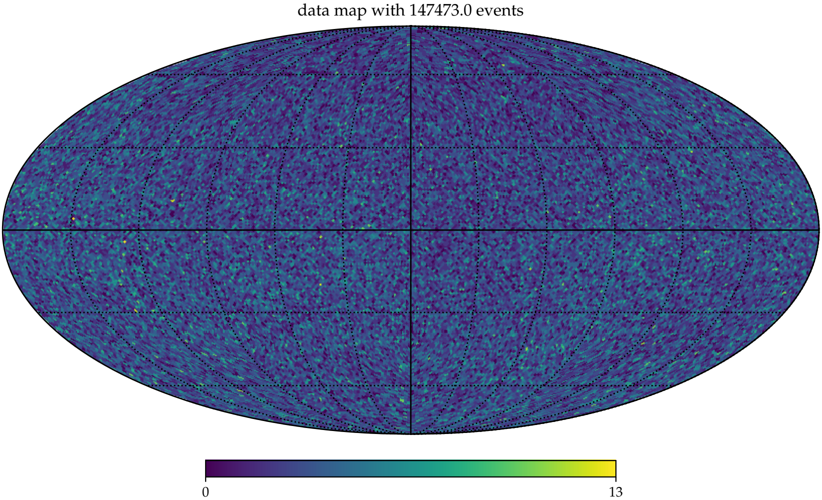
- Load the two data files `truemap1.fits` and `eventmap1.fits` (the second file is a bin-wise Poisson sample with mean given in the first map)
- Display the maps
- Determine and compare the power spectra C_ℓ/C_0 of the two maps, e.g. with `HealPix` or `healpy`
- Generate a background map via data scrambling, as described on the previous slide.
- Compare the power spectrum of the event map to the expected noise level $1/N_{\rm tot}$.

Exercise 4 : Template vs. Event Map



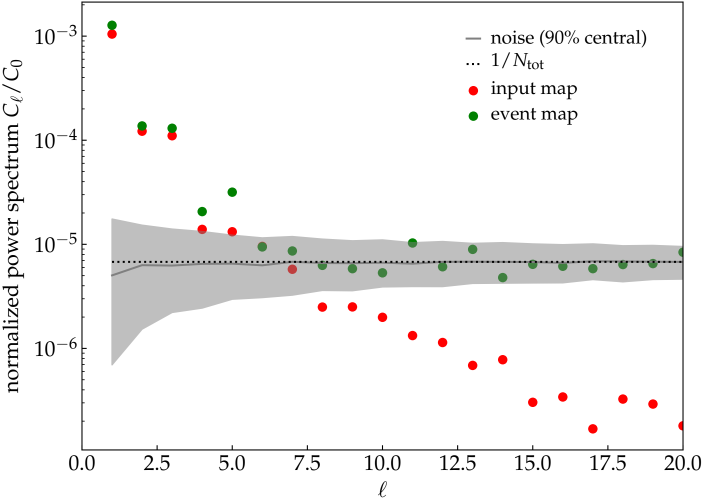
for `python` code see : `powerspectrum.py`

Exercise 4 : Template vs. Event Map



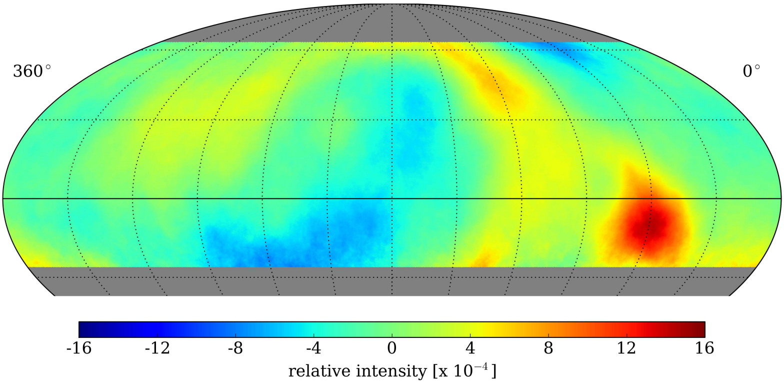
for python code see : `powerspectrum.py`

Exercise 4 : Power Spectra



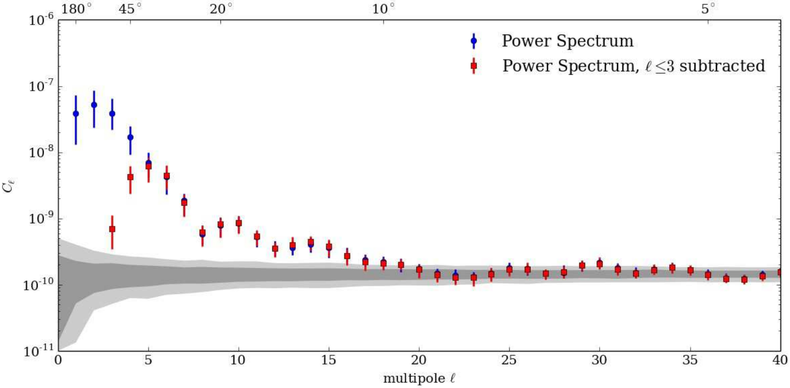
for python code see : `powerspectrum.py`

Example: HAWC Anisotropies



Study of cosmic ray arrival directions with the
High Altitude Water Cherenkov (HAWC) detector.

Example: HAWC Anisotropies



Study of cosmic ray arrival directions with the High Altitude Water Cherenkov (HAWC) detector.