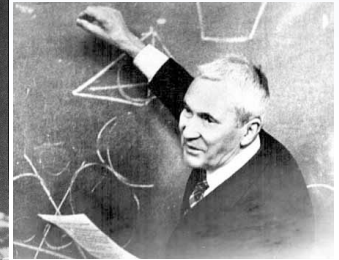
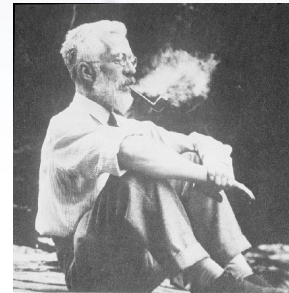
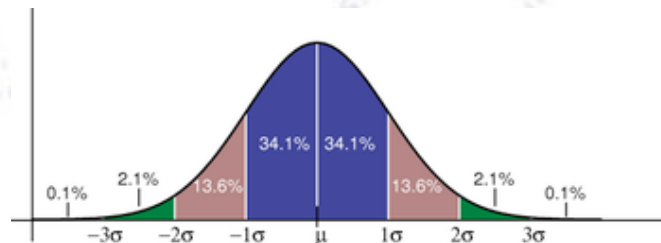


Applied Statistics

Hypothesis Testing



Troels C. Petersen (NBI)



"Statistics is merely a quantisation of common sense"

Hypothesis testing

Suppose in a beer tasting, that someone gets 49 out of 50 right.

Does that prove that the person can taste difference between beers?

Hypothesis testing

Suppose in a beer tasting, that someone gets 49 out of 50 right.

Does that prove that the person can taste difference between beers?

NO!

What we can say is that the result is **inconsistent** (at some significance level) with the hypothesis that the person chooses at random.

This leaves us with the alternative hypotheses, that the person can taste the difference or have cheated (consciously or unconsciously).

In statistics one can never prove a hypothesis directly. However, one can set up alternative hypotheses and disprove these. That is how one works in statistics...

Hypothesis testing

Hypothesis testing is like a criminal trial. The basic “null” hypothesis is **Innocent** (called H_0) and this is the hypothesis we want to test, compared to an “alternative” hypothesis, **Guilty** (called H_1).

Innocence is initially assumed, and this hypothesis is only rejected, if enough evidence proves otherwise, i.e. that the probability of innocence is very small (“beyond a reasonable doubt”), and the hypothesis can be rejected.

	Truly innocent (H_0 is true)	Truly guilty (H_1 is true)
Acquittal (Accept H_0)	Right decision	Wrong decision Type II error
Conviction (Reject H_0)	Wrong decision Type I error	Right decision

The rate of type I/II errors are correlated, and one can only choose one of these!

Hypothesis terminology

$H_0 = \text{Null Hypothesis:}$

Definition: The initial / simplest hypothesis.

Examples: Data is background, data follows simple model, particle is a pion.

$H_1 = \text{Alternative Hypothesis:}$

Definition: The alternative to the null hypothesis, possibly more advanced.

Examples: Data is background + signal, data does not follow simple model, particle is an electron.

$\alpha = \text{Significance:}$

Definition: Probability to **reject H_0** , even if it is **true**.

Example: Concluding no signal, even if there; deciding electron when pion.

Note: The selection efficiency = $1 - \alpha$

$\beta = 1 - \text{Power:}$

Definition: Probability to **accept H_0** , even if it is **false**.

Example: Concluding signal, even if not there; deciding pion when electron.

Note: The misidentification probability = β

Taking decisions

You are asked to take a decision or give judgement - it is yes-or-no.

Given data - how to do that best?

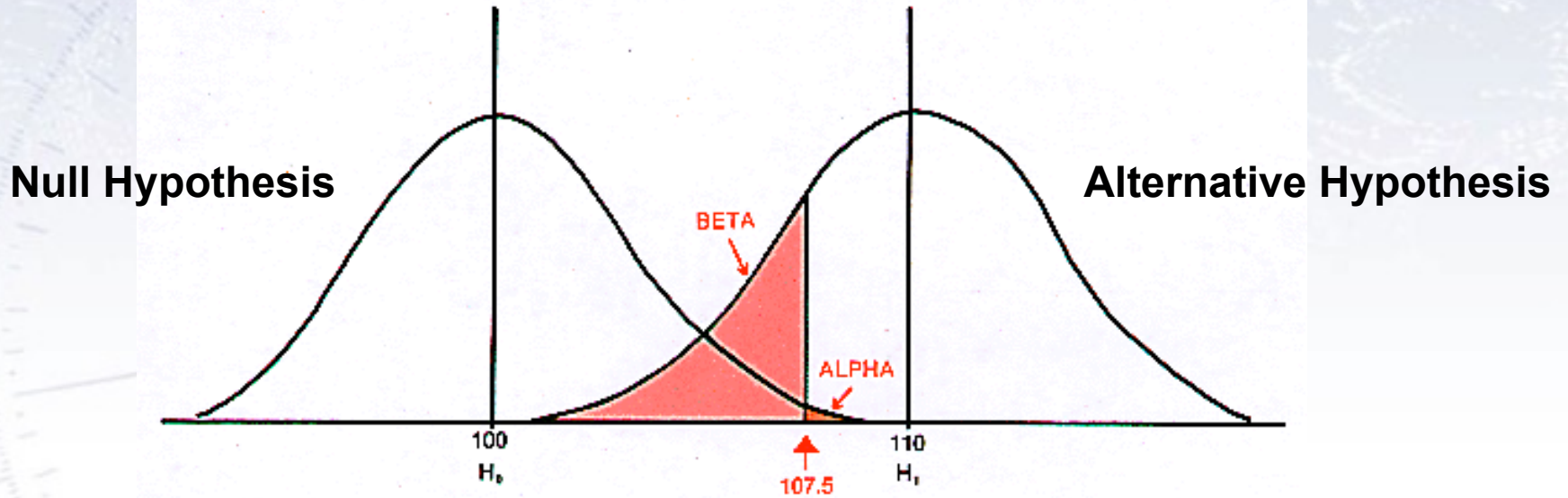
That is the basic question in hypothesis testing.

Trouble is, you may take the wrong decision, and there are TWO errors:

- The hypothesis is **true**, but you **reject** it (Type I).
- The hypothesis is **wrong**, but you **accept** it (Type II).

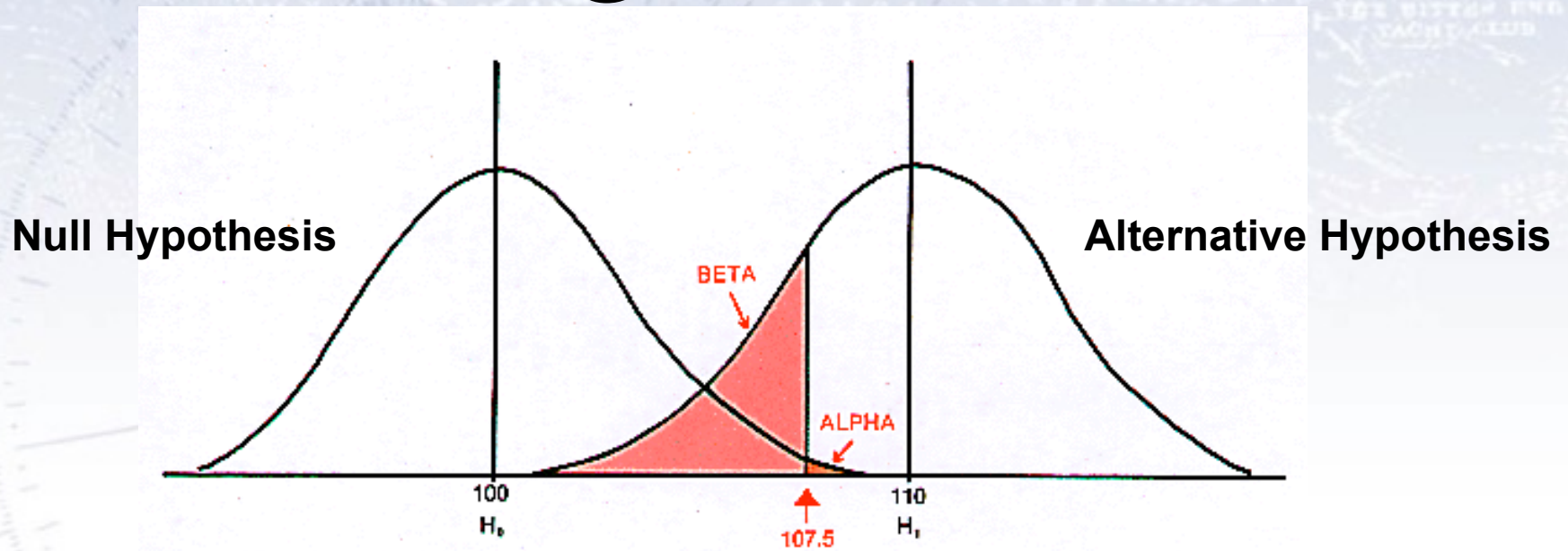
		REALITY	
		Null is True	Null is False
STATISTICAL DECISION:	Do Not Reject Null	$1 - \alpha$ Correct	β Type II error
	Reject Null	α Type I error	$1 - \beta$ Correct

Taking decisions



		REALITY	
		Null is True	Null is False
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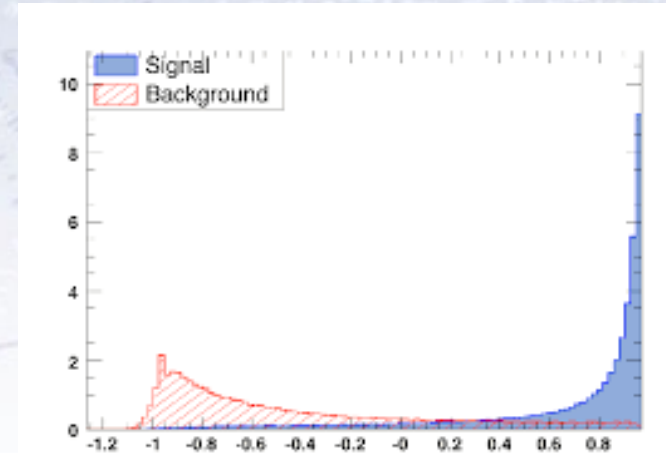
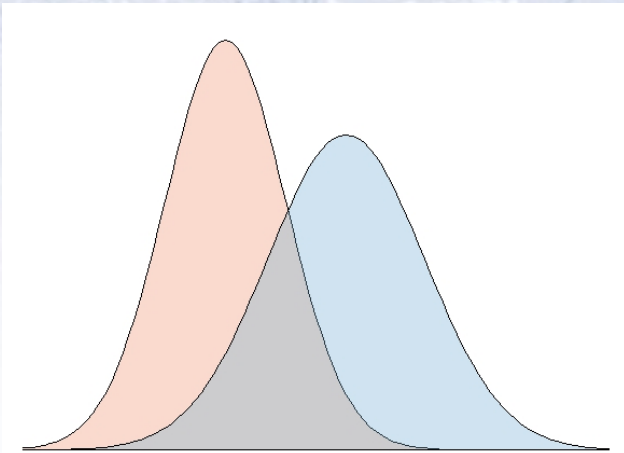
Taking decisions



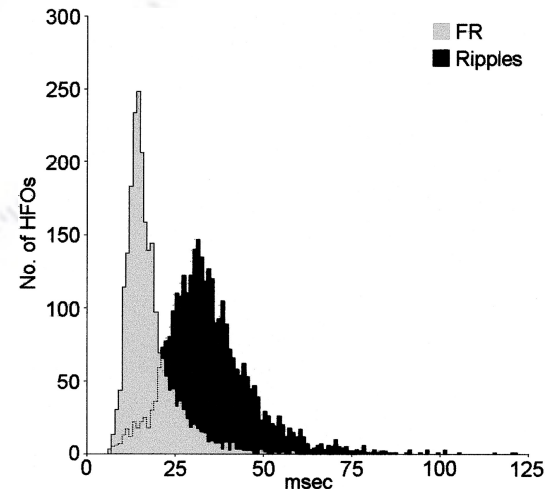
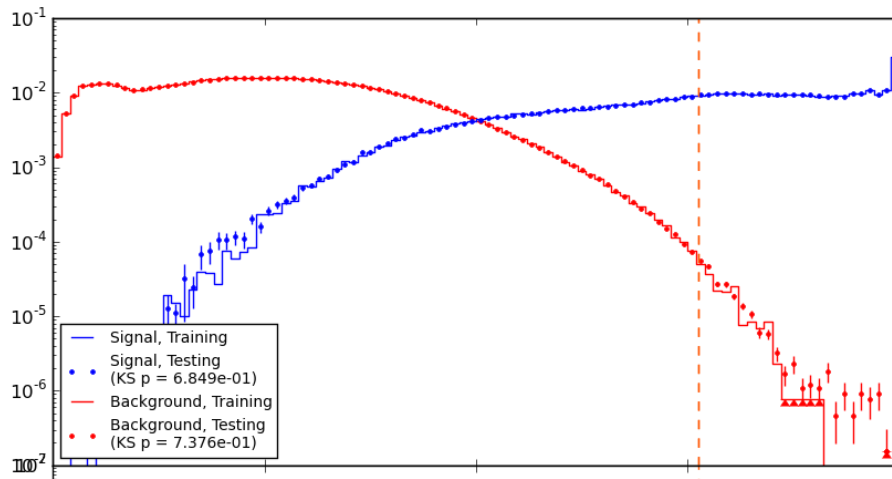
The purpose of a **test** is to yield (calculable/predictable) distributions for the **Null** and **Alternative hypotheses**, which are *as separated from each other as possible*, to minimise α and β .

The likelihood ratio test can be the best such test.

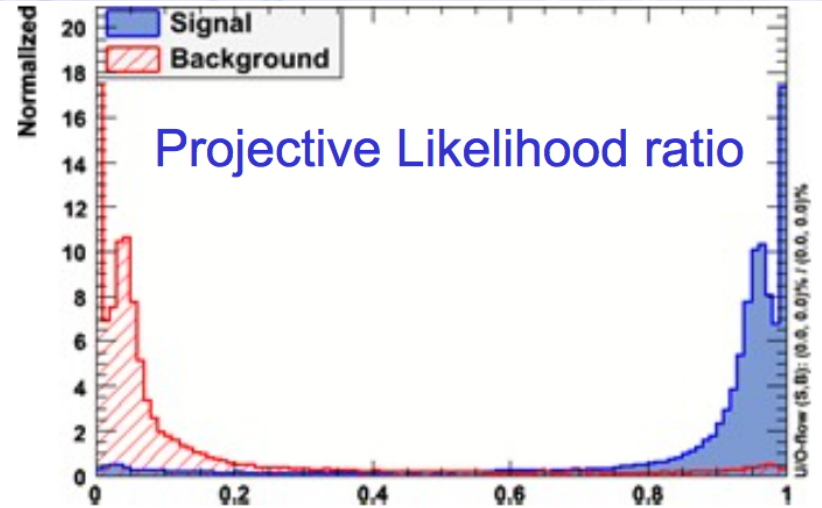
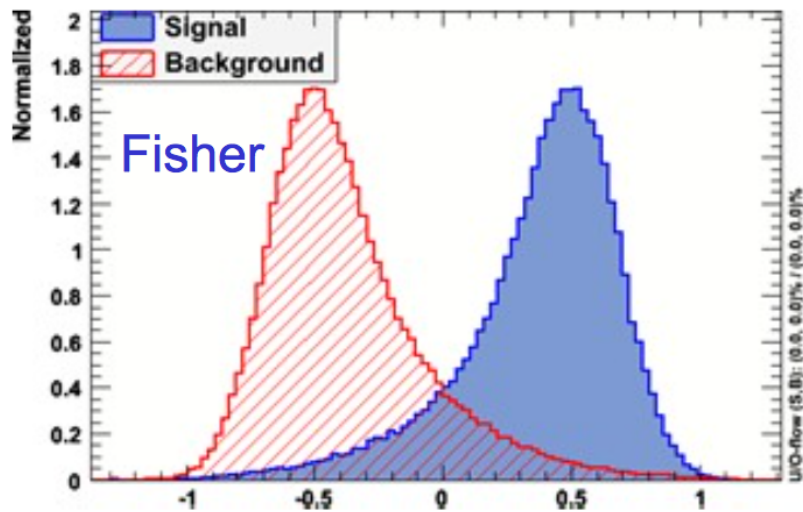
Measuring separation



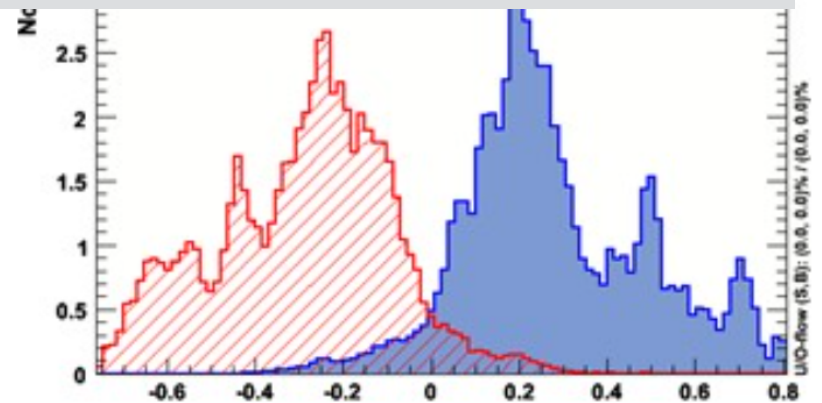
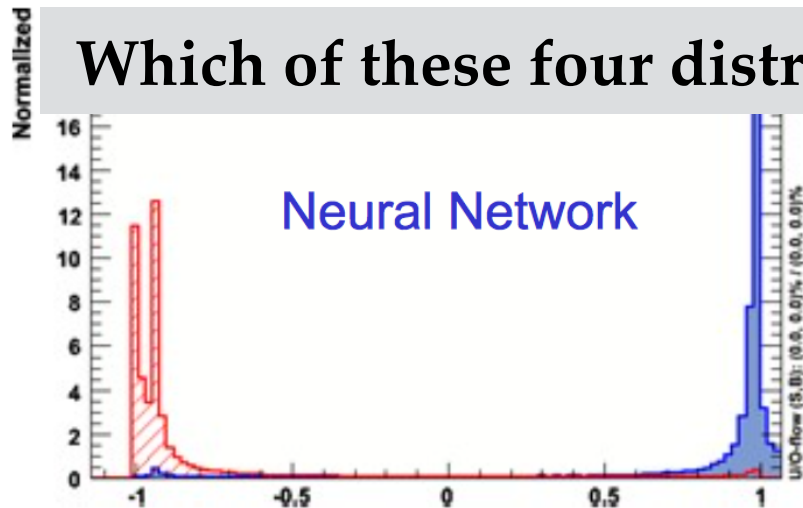
Which of these three distributions are most separated?
How do you “measure” this?



Measuring separation



Which of these four distributions are most separated?



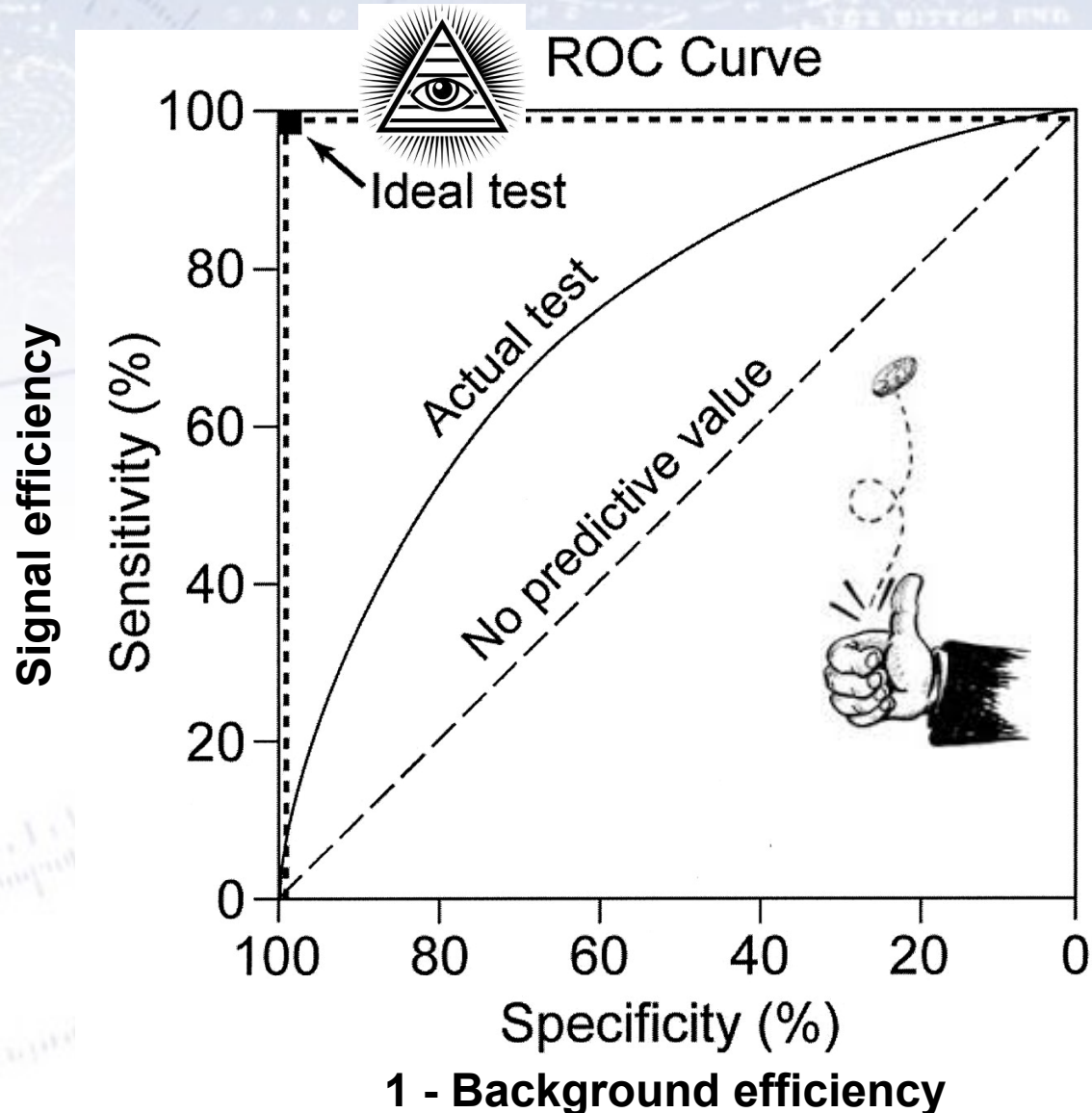
ROC-curves

The **Receiver Operating Characteristic** or just ROC-curve is a graphical plot of the sensitivity, or true positive rate, vs. false positive rate.

It is calculated as the integral of the two hypothesis distributions, and is used to evaluate the power of a test.

Often, it requires a testing data set to actually see how well a test is performing.

Dividing data, it can also detect overtraining!



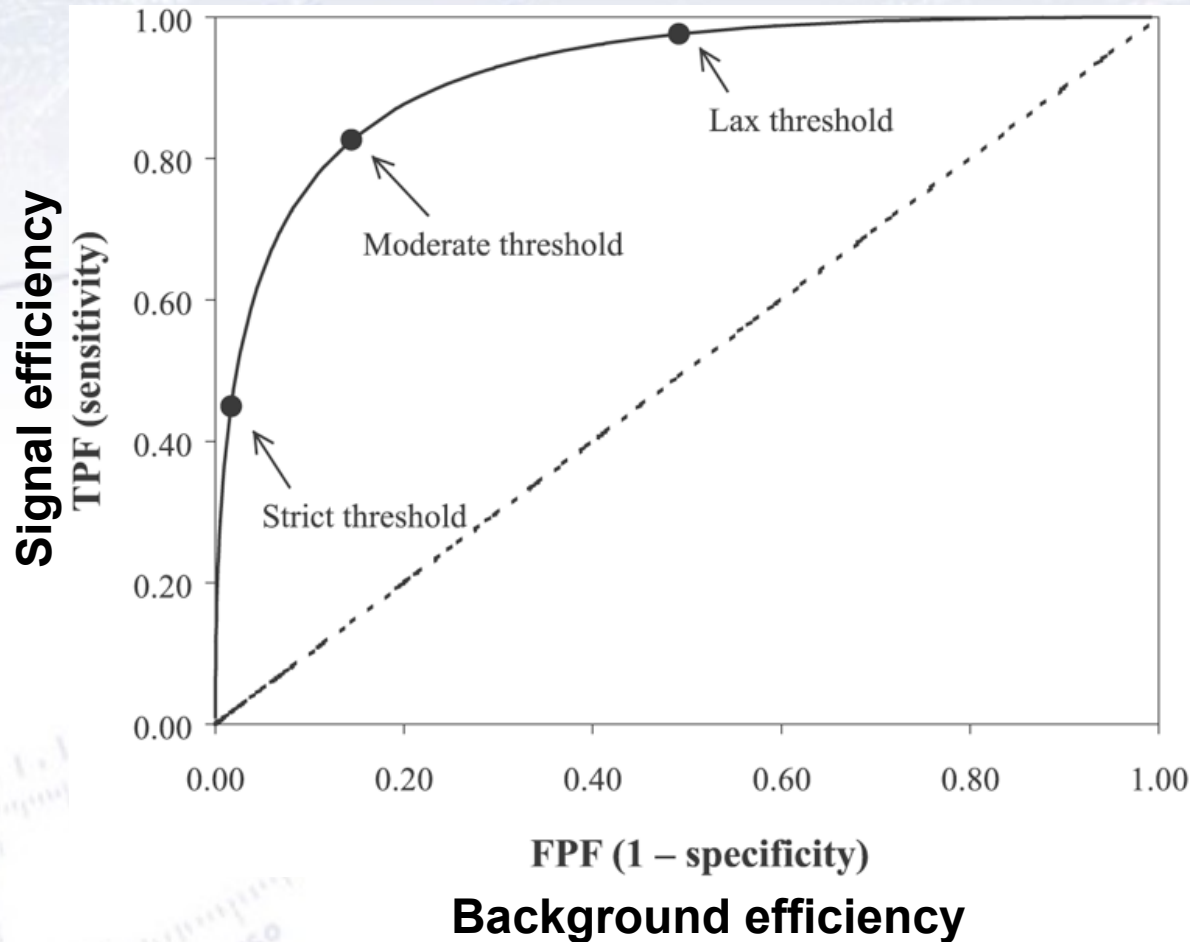
ROC-curves

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Useful ROC metrics

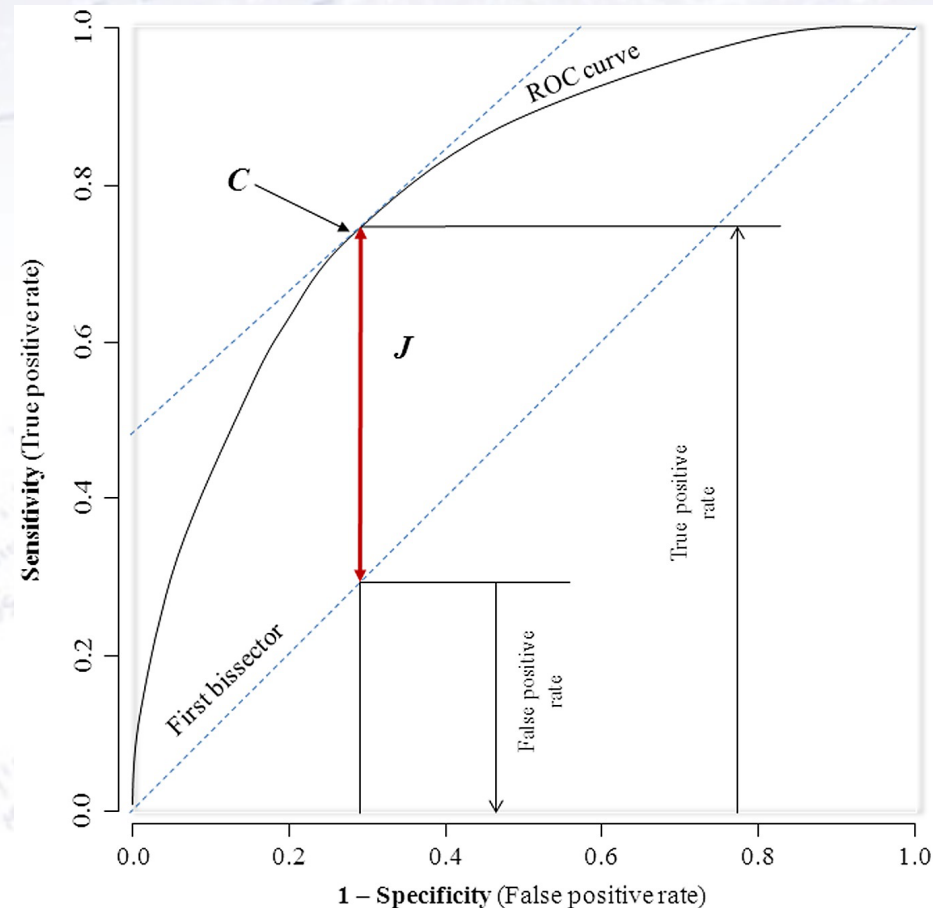
The performance of a test statistic is described fully by the ROC curve itself!

To summarise performance in one single number (i.e. easy to compare!), one used Area Under ROC curve.

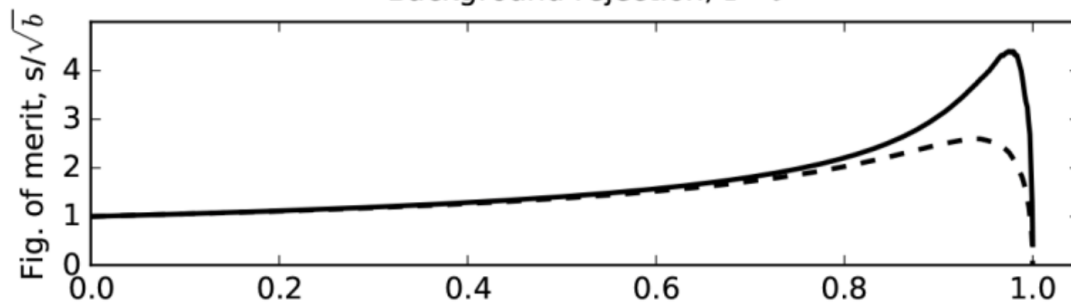
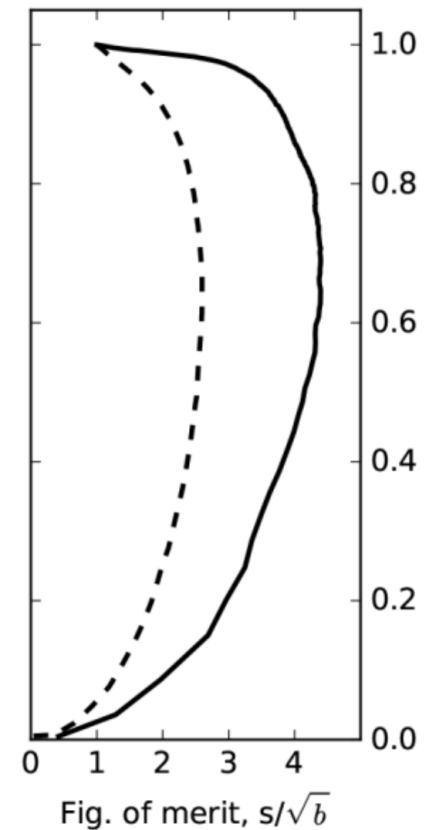
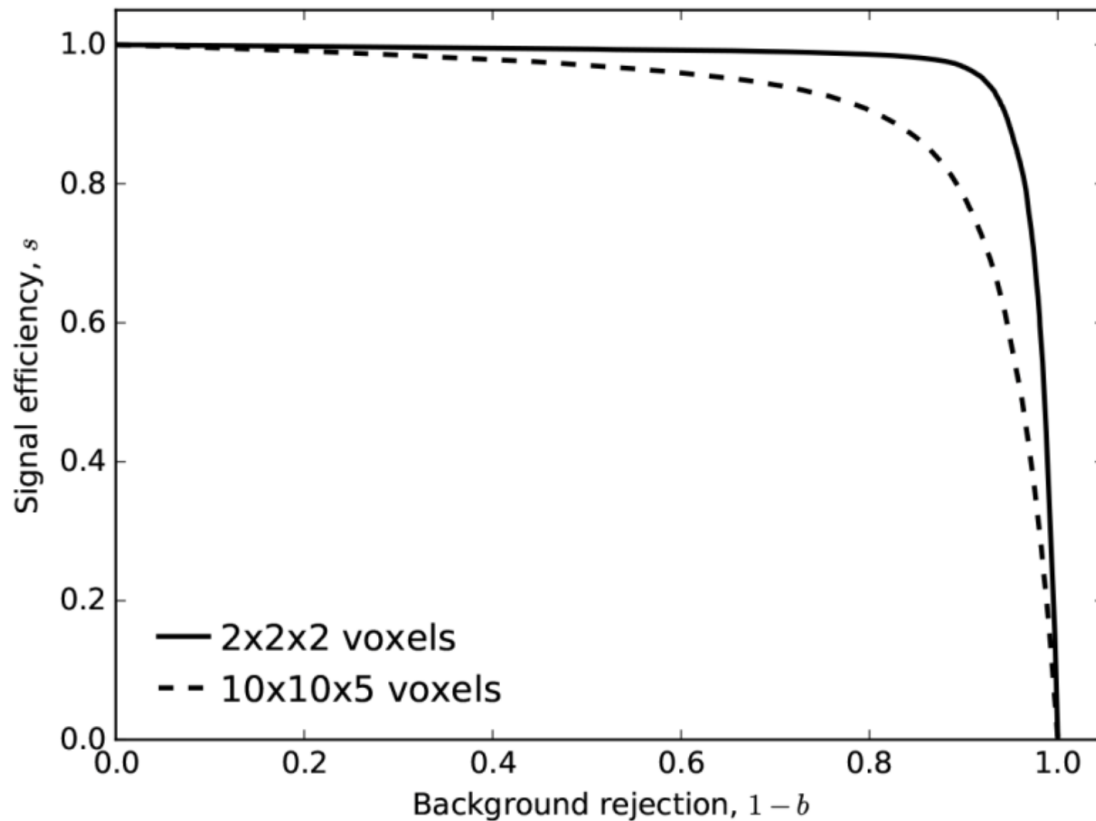
Alternatively, people use:

- Signal eff. for a given background eff.
- Background eff. for a given signal eff.
- Youden's index (J), defined as shown in the figure.

The optimal selection **depends entirely on your analysis at hand!**



Useful ROC metrics



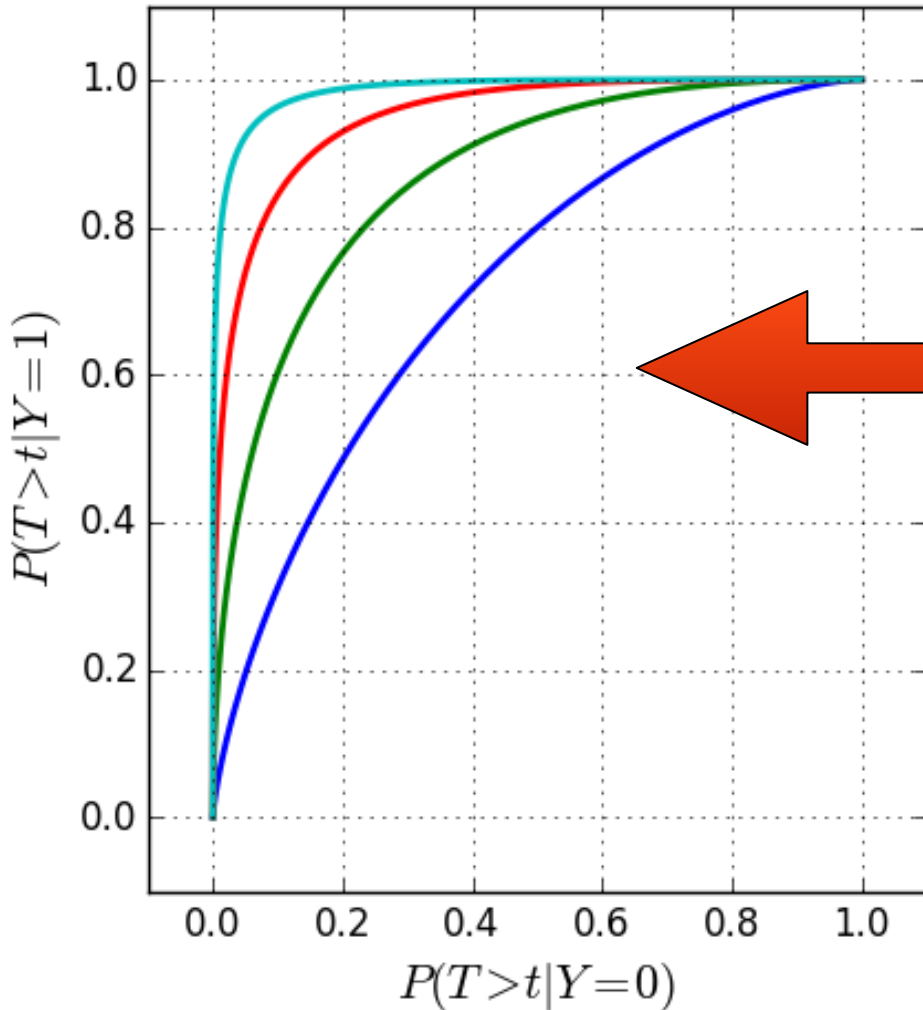
JINST, Vol12, Jan.2017

The background is a faded nautical chart. It features a compass rose at the top with a vertical line pointing to 0 degrees. Curved lines, representing magnetic isogonic lines, are drawn across the chart, labeled with values such as 20, 30, 40, 50, 60, 70, 80, 90, 100, 110, 120, 130, 140, 150, 160, 170, 180, 190, 200, 210, 220, 230, 240, 250, 260, 270, 280, 290, 300, and 310. The word "MAGNETIC" is printed in the upper left quadrant. In the upper right, the text "THE UNITED KINGDOM YACHTING CLUB" is visible. The chart also shows some latitude and longitude markings.

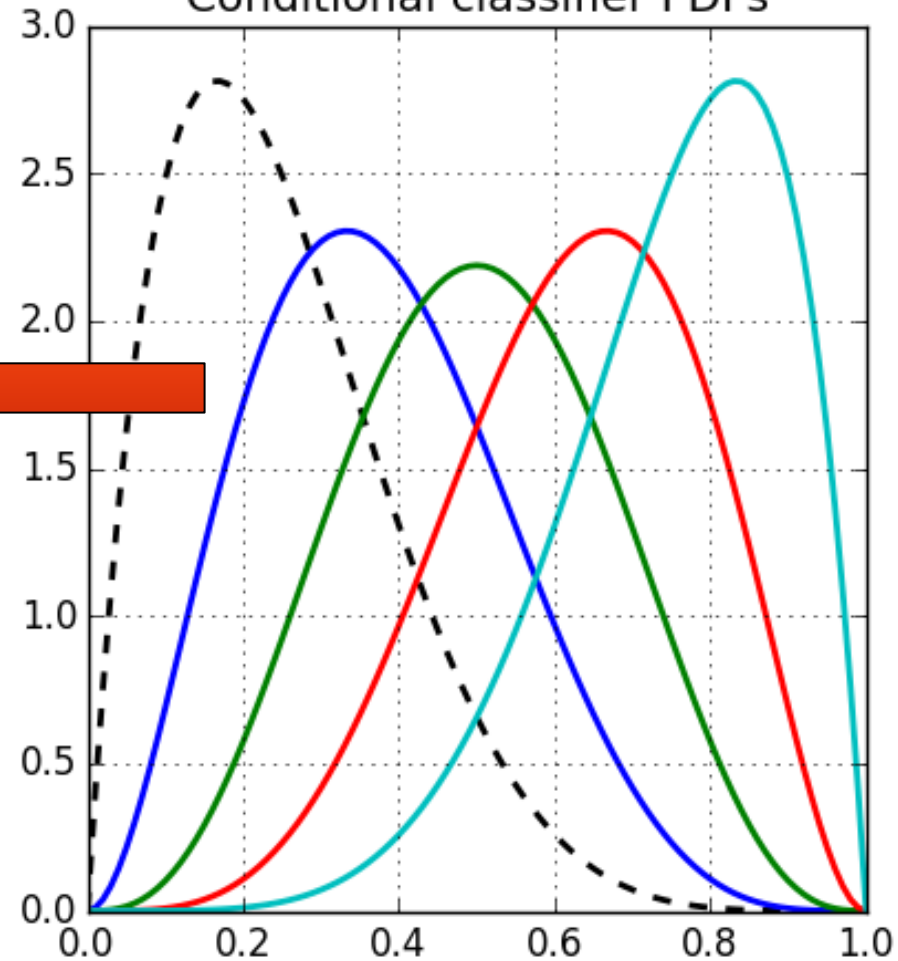
Example of ROC curves in use

Simple case

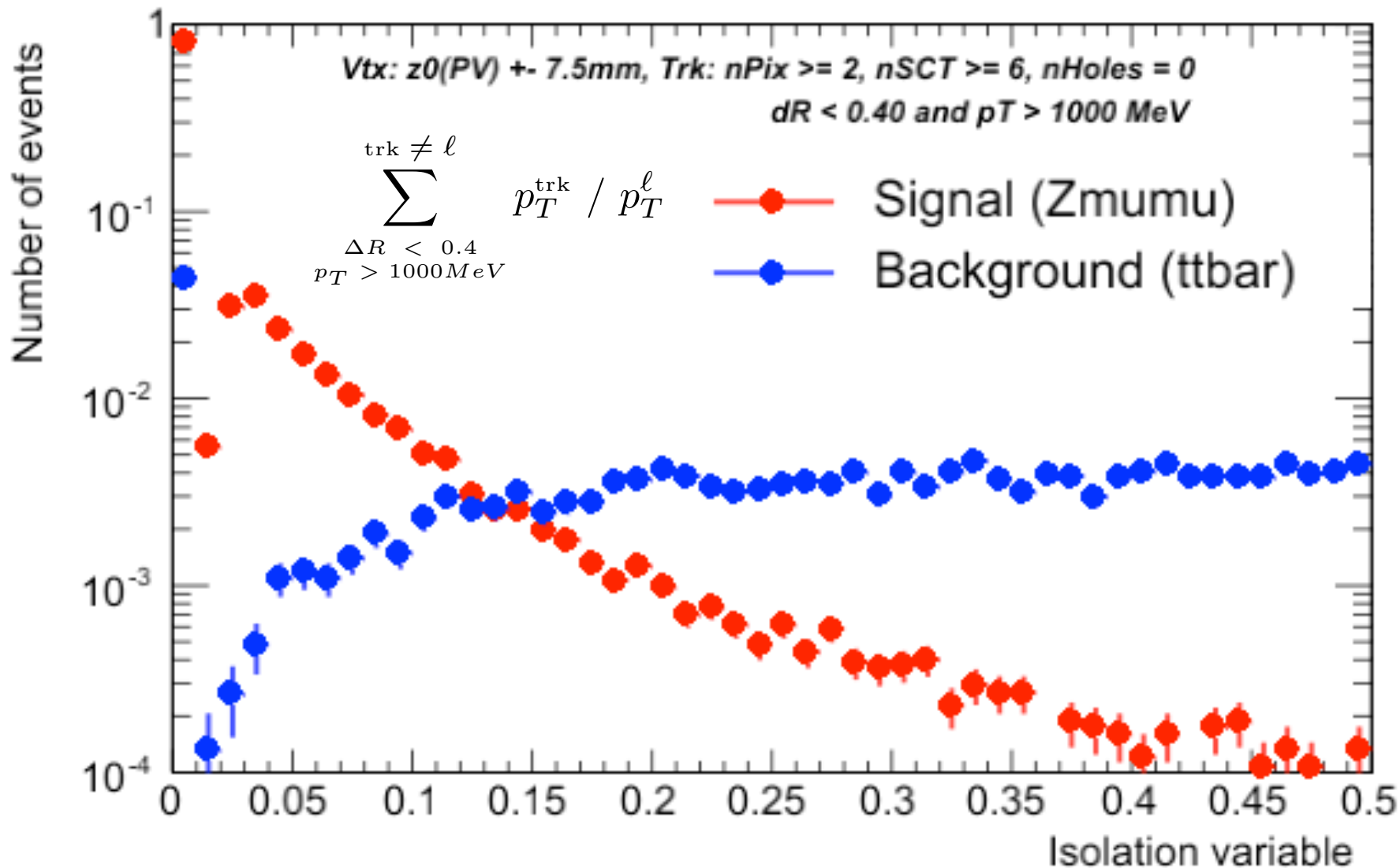
ROC curves



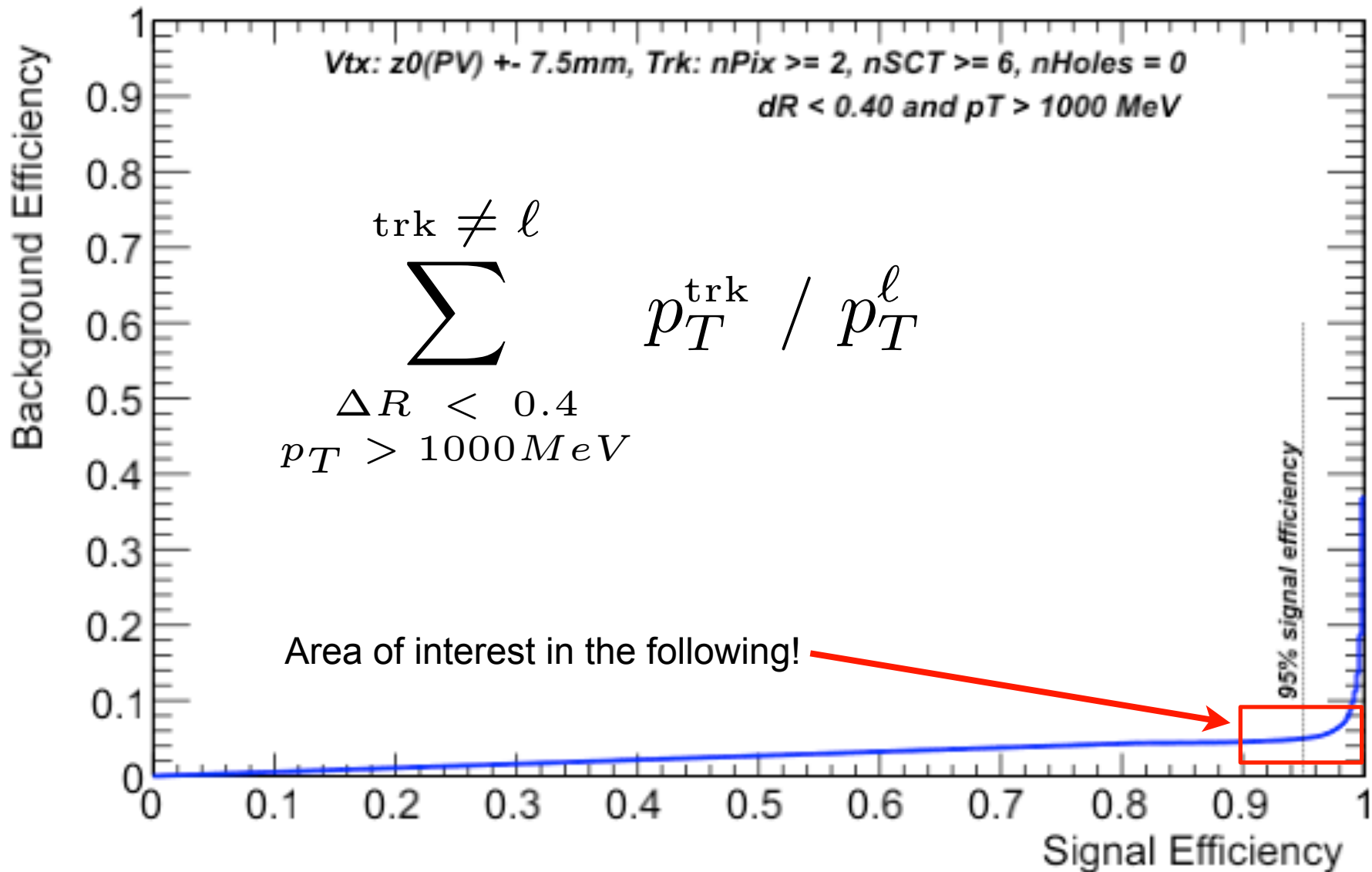
Conditional classifier PDFs



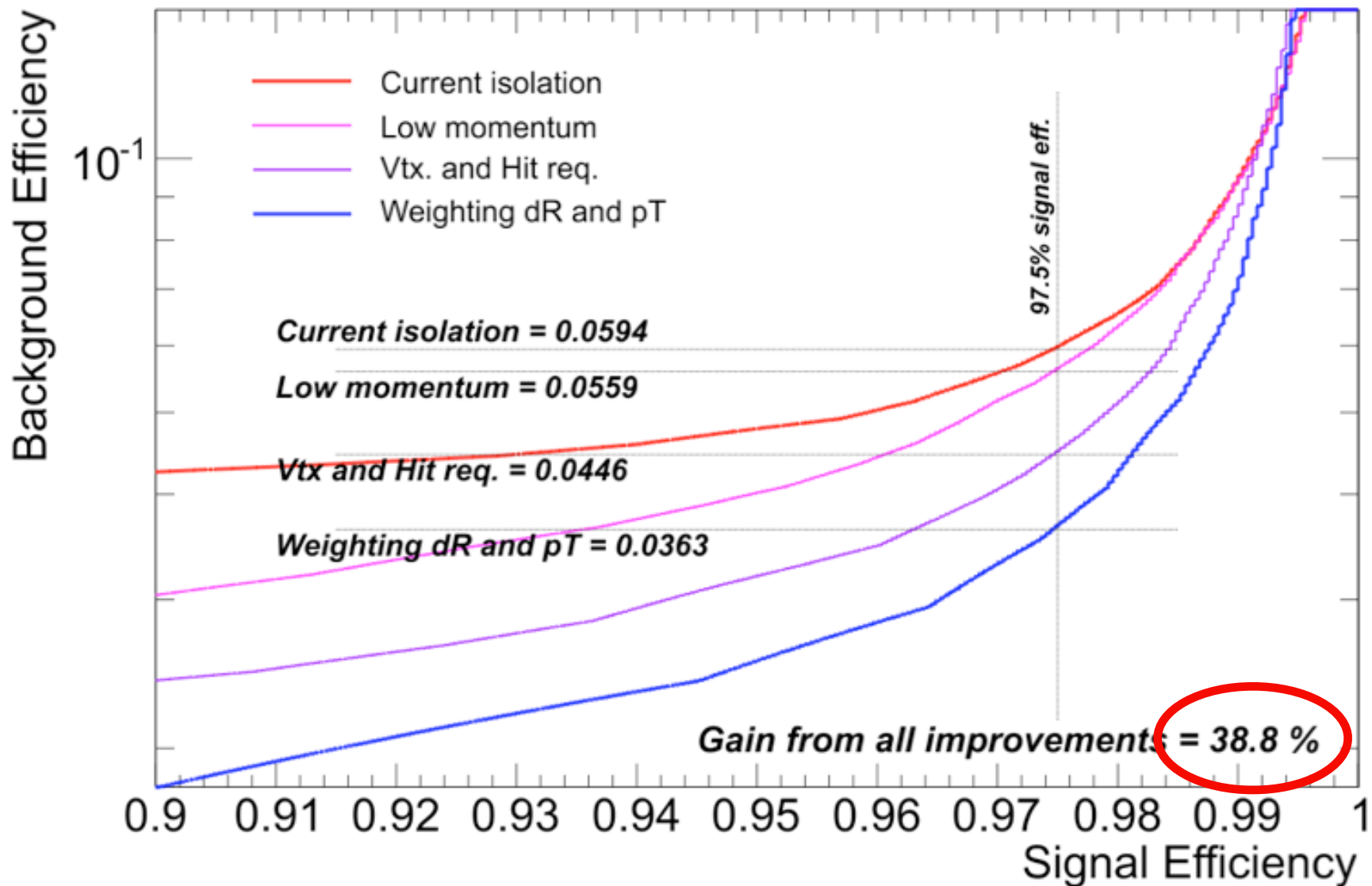
Basic steps - distributions



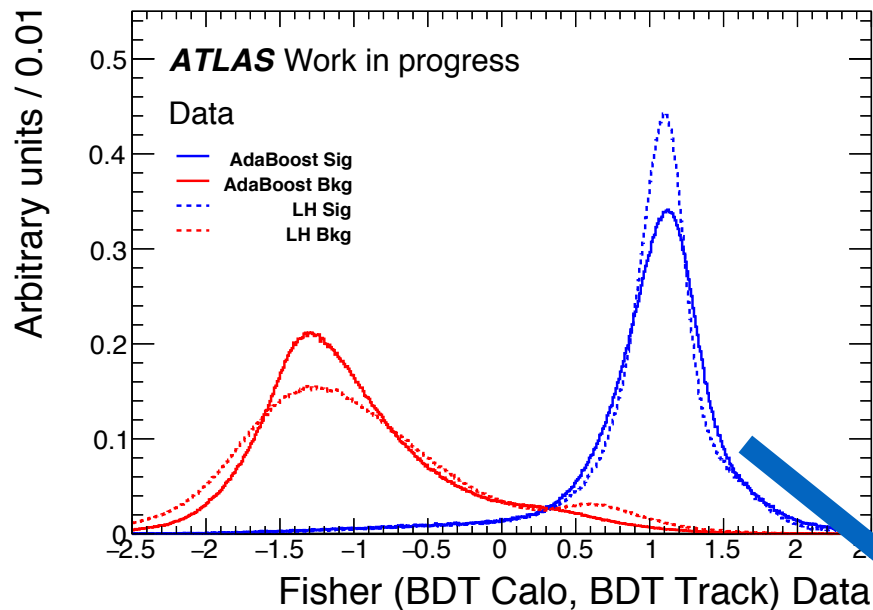
Basic steps - ROC curves



Overall improvement



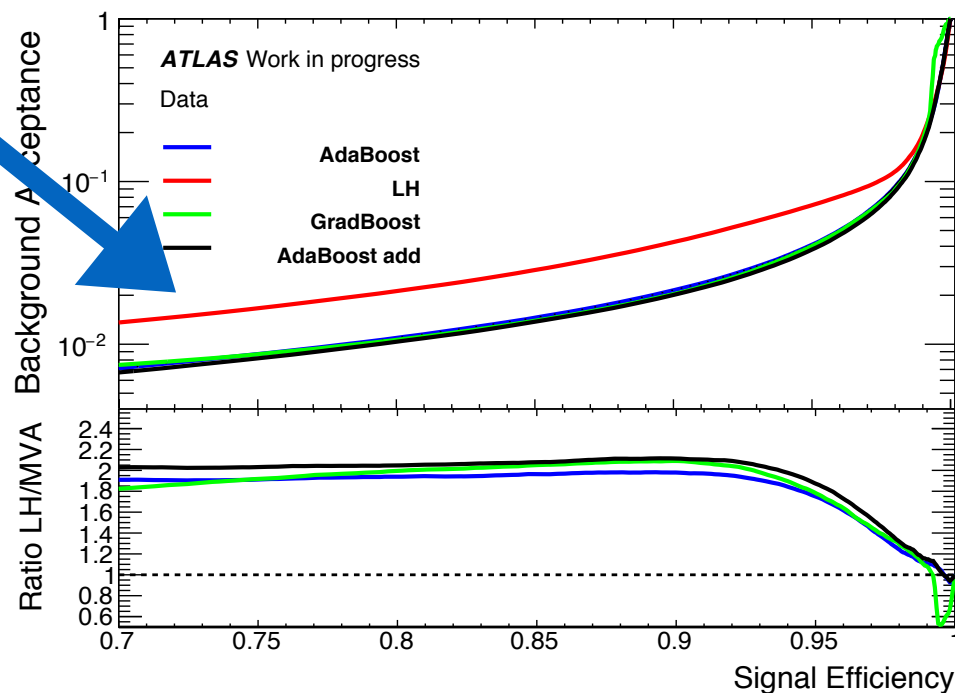
Recent example (electron PID)



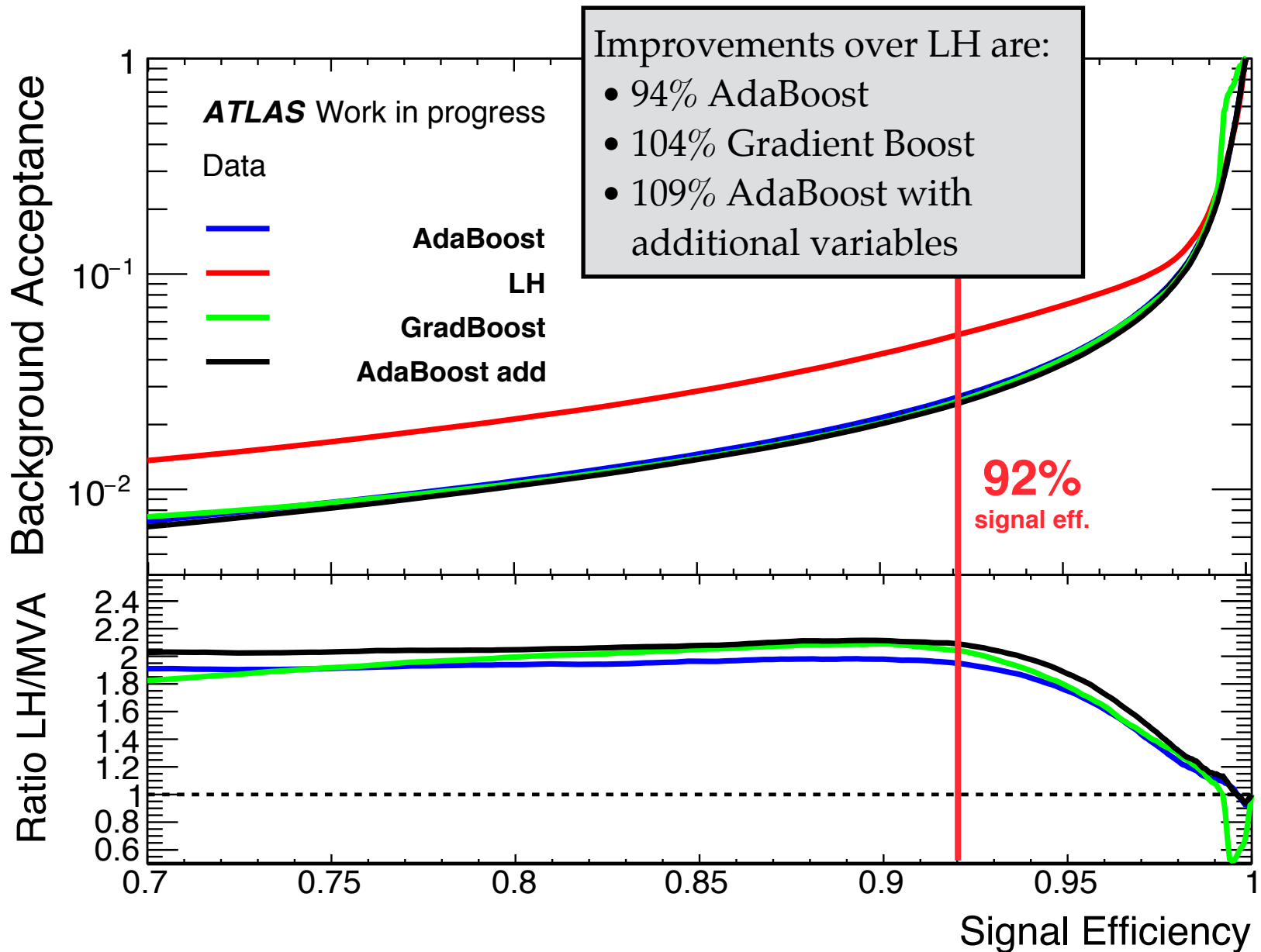
In addition to the ROC curves, the ratio of these are also shown, to illustrate the improvement as a function of operating point.

The three new methods clearly improve on the existing result.

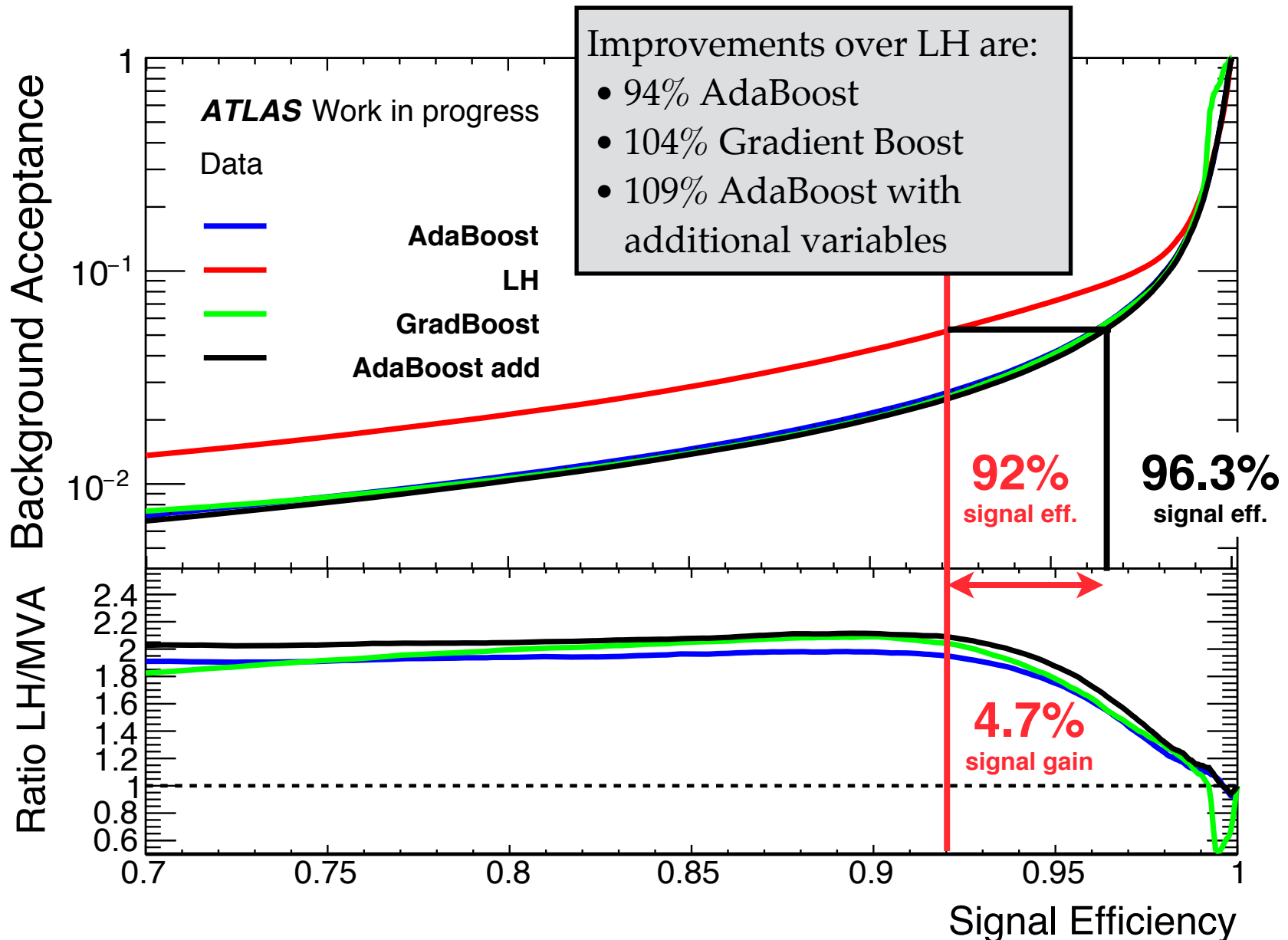
This example is from identifying electrons using Machine Learning in the ATLAS experiment. It is the result of applying ML on data, which solves the problem of differences between data and simulation.

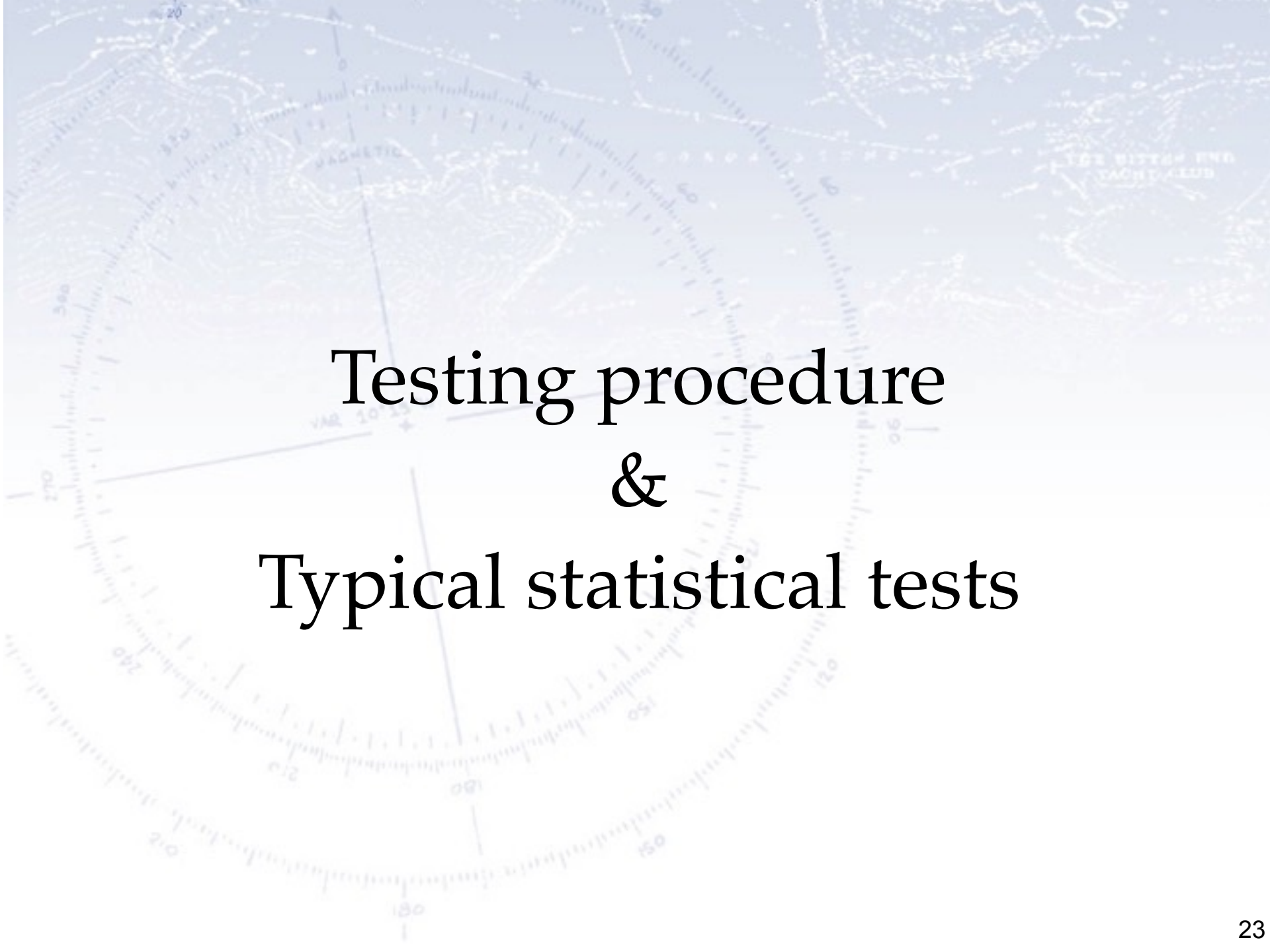


Combined result



Combined result

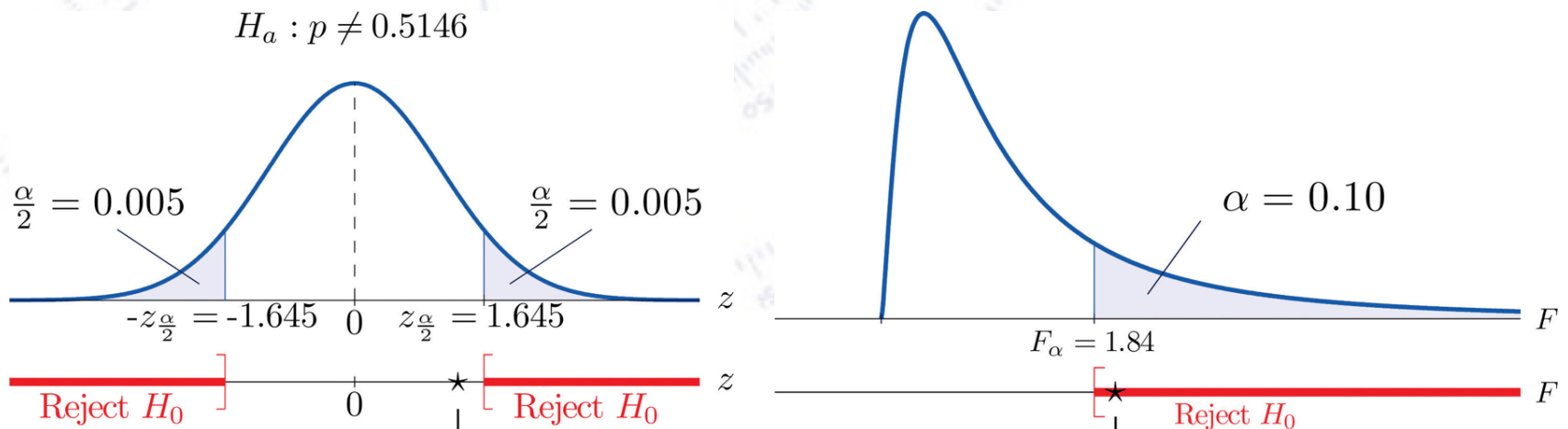




Testing procedure & Typical statistical tests

Testing procedure

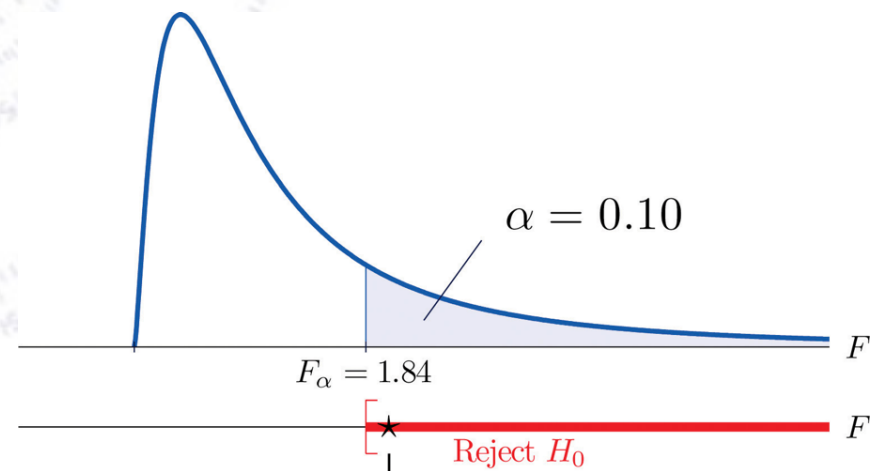
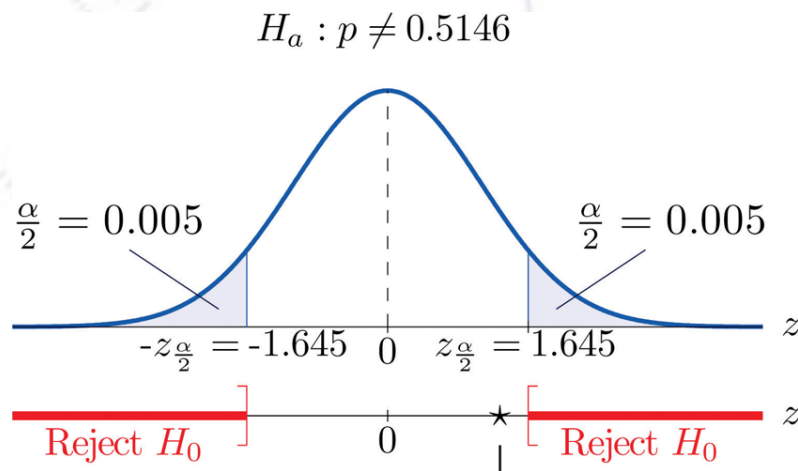
1. Consider an **initial (null) hypothesis**, of which the truth is unknown.
2. State null and **alternative hypothesis**.
3. Consider statistical **assumptions** (independence, distributions, etc.)
4. Decide for appropriate test and state relevant **test statistic**.
5. **Derive the test statistic** distribution under null and alternative hypothesis.
In standard cases, these are well known (Poisson, Gaussian, Student's t, etc.)
6. **Select a significance level (α)**, that is a probability threshold below which null hypothesis will be rejected (typically from 5% (biology) and down (physics)).
7. Compute from (otherwise blinded) observations / data **value of test statistic t** .
8. From t calculate **probability of observation under null hypothesis (p-value)**.
9. **Reject null hypothesis** for alternative if **p-value is below significance level**.



Testing procedure

1. Consider an **initial (null) hypothesis**, of which the truth is unknown.
2. State null and **alternative hypothesis**.
3. Consider statistical **assumptions** (independence, distributions, etc.)
4. Decide for appropriate test and state relevant **test statistic**.
5. **Derive test statistic** (e.g., t -test, F -test, etc.)
6. **Select a significance level α** (e.g., 0.05, 0.10, etc.)
7. **Compute the test statistic** (e.g., t -statistic, F -statistic, etc.)
8. From t calculate **probability of observation under null hypothesis (p-value)**.
9. **Reject null hypothesis** for alternative if **p-value is below significance level**.

1. State hypothesis.
2. Set the criteria for a decision.
3. Compute the test statistic.
4. Make a decision.



Hypothesis testing philosophy

In hypothesis testing, you can never **prove** a hypothesis.

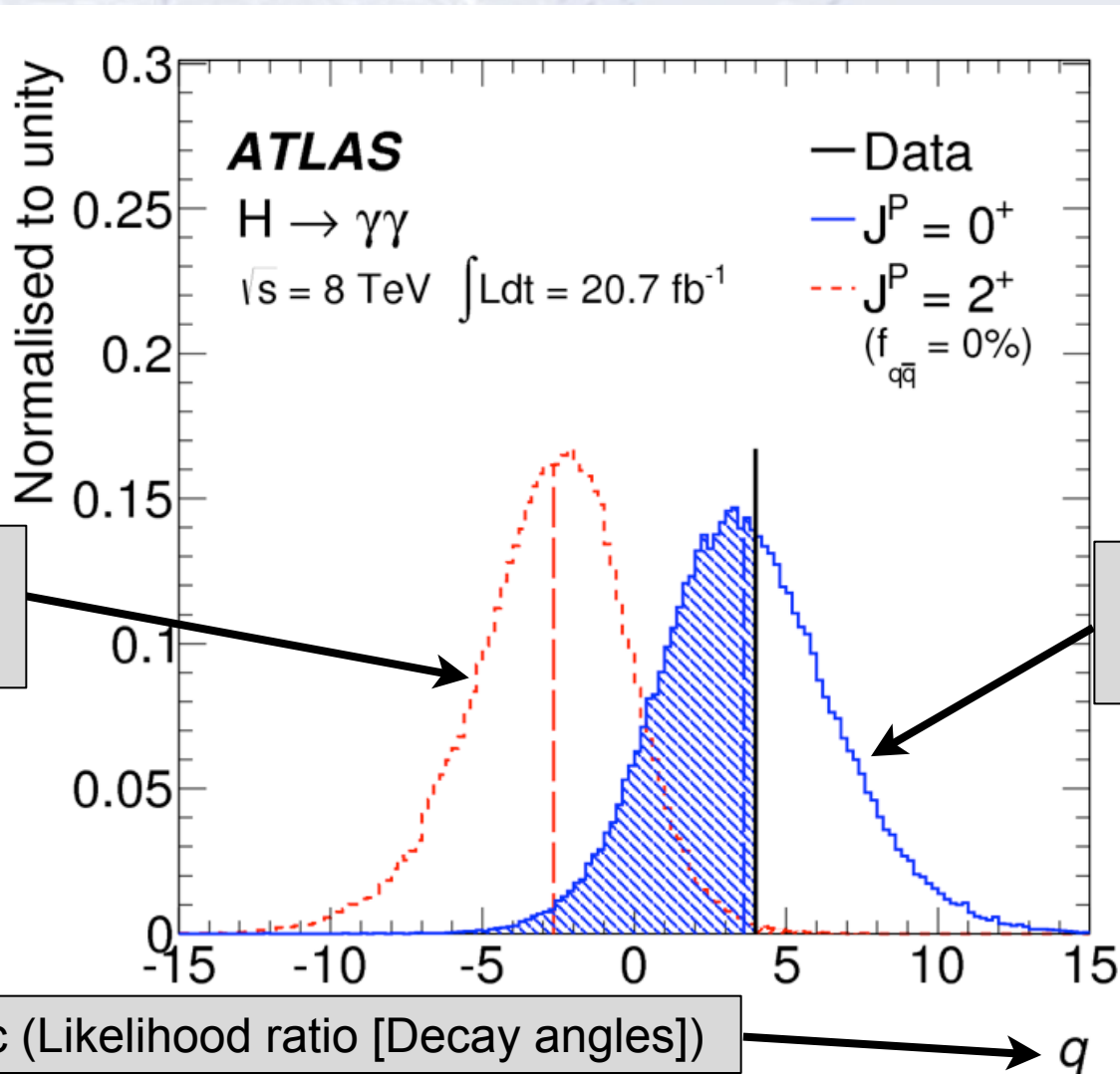
You can **accept** a hypothesis, but this does not exclude accepting other hypothesis.

However, you can **reject** a hypothesis on the basis that it's probability of being correct (p-value) is too small.

Thus, in hypothesis testing, the line of reasoning is to state a hypothesis *opposite* of what you want to show, and then try to **reject** this hypothesis.

Example of hypothesis test

The spin of the newly discovered “Higgs-like” particle (spin 0 or 2?):



Neyman-Pearson Lemma

Consider a **likelihood ratio** between the null and the alternative model:

$$D = -2 \ln \frac{\text{likelihood for null model}}{\text{likelihood for alternative model}}$$

The Neyman-Pearson lemma (loosely) states, that this is the most powerful test there is.

In reality, the problem is that it is not always easy to write up a likelihood for complex situations!

However, there are many tests derived from the likelihood...

Likelihood ratio problem

While the **likelihood ratio** is in principle both simple to write up and powerful:

$$D = -2 \ln \frac{\text{likelihood for null model}}{\text{likelihood for alternative model}}$$

...it turns out that determining the exact distribution of the likelihood ratio is often very hard.

To know the two likelihoods one might use a Monte Carlo simulation, representing the distribution by an n-dimensional histogram (since our observable, x , can have n dimensions). But if we have M bins in each dimension, then we have to determine M^n numbers, which might be too much.

However, a convenient result (Wilk's Theorem) states that as the sample size approaches infinity, **the test statistic D will be χ^2 -distributed with N_{dof} equal to the difference in dimensionality of the Null and the Alternative (nested) hypothesis.**

Alternatively, one can choose a simpler (and usually fully acceptable test)...

Common statistical tests

- **One-sample test** compares sample (e.g. mean) to known value:
Example: Comparing sample to known constant ($\mu_{\text{exp}} = 2.91 \pm 0.01$ vs. $c = 2.99$).
$$z = \frac{\bar{x} - \mu_0}{\sigma(\bar{x})}$$
- **Two-sample test** compares two samples (e.g. means).
Example: Comparing sample to control ($\mu_{\text{exp}} = 4.1 \pm 0.6$ vs. $\mu_{\text{control}} = 0.7 \pm 0.4$).
$$z = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\sigma(\bar{x}_1)^2 + \sigma(\bar{x}_2)^2}}$$
- **Paired test** compares paired member difference (to control important variables).
Example: Testing environment influence on twins to control genetic bias ($\mu_{\text{diff}} = 0.81 \pm 0.29$ vs. 0).
- **Chi-squared test** evaluates adequacy of model compared to data.
Example: Model fitted to (possibly binned) data, yielding p-value = $\text{Prob}(\chi^2 = 45.9, N_{\text{dof}} = 36) = 0.125$
- **Kolmogorov-Smirnov test** compares if two distributions are compatible.
Example: Compatibility between function and sample or between two samples, yielding p-value = 0.87
- **Wald-Wolfowitz runs test** is a binary check for independence.
- **Fisher's exact test** calculates p-value for contingency tables.
- **F-test** compares two sample variances to see, if grouping is useful.

Which test to use?

In principle all statistical tests can be used on every problem, but they are not all equally powerful, and some might also be biased (low stat.) or otherwise unfit. Finally, they may not all be equally easy to implement!

The figure of merit is typically the **Power of a Test***, defined as $(1 - \beta)$, complement of the false negative rate, β .

This is thus the test's probability of correctly rejecting the null hypothesis.

Example:

This is a powerful test: Thus, since the result is negative, we can confidently say that the null hypothesis is not rejected (e.g. the patient does not have the condition).

In medical science, it is typically important to have a powerful test (i.e. low β), while in criminal science it is a low type I error rate (i.e. low α), convicting innocents.

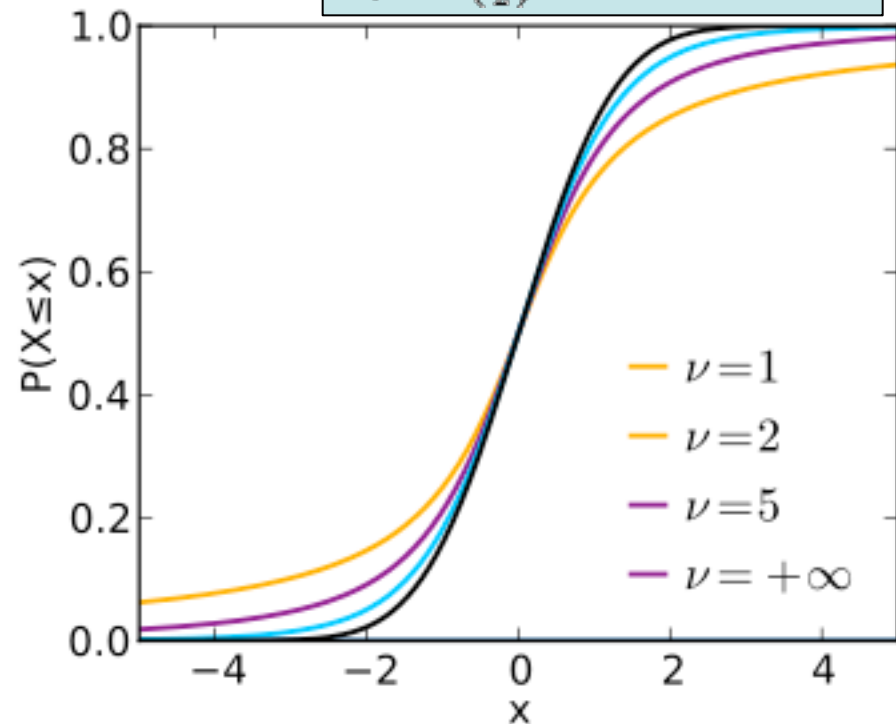
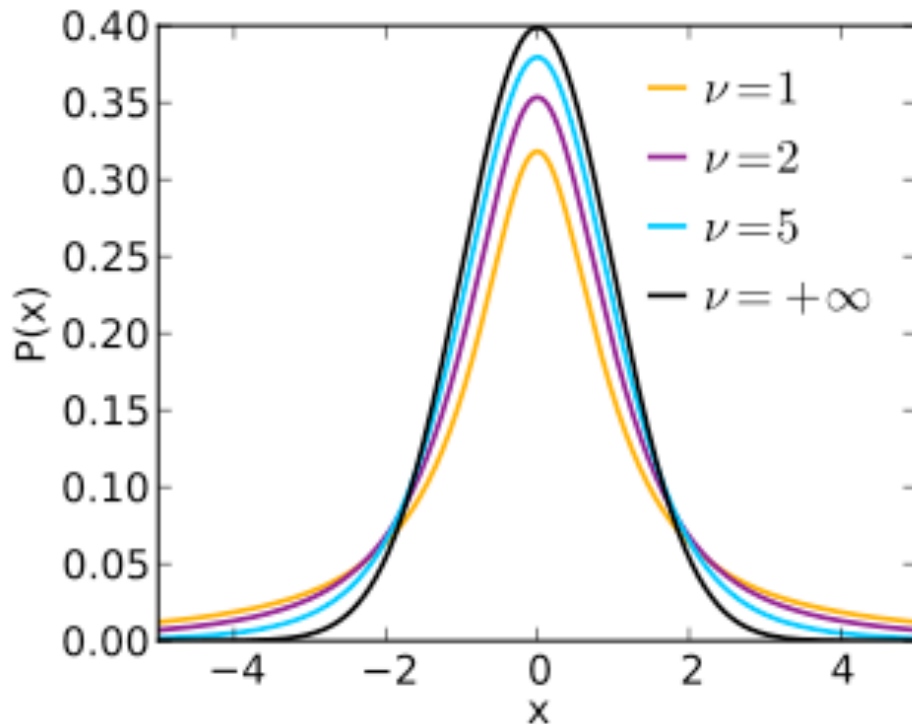
In the end, choosing a test comes down to **experience, importance of power, ease of use**, and even standards in the field of research in question.

* Power of a test is often termed sensitivity in biostatistics.

Student's t-distribution

Discovered by William Gosset (who signed "student"), student's t-distribution takes into account lacking knowledge of the variance.

$$\frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\nu\pi}\Gamma\left(\frac{\nu}{2}\right)} \left(1 + \frac{x^2}{\nu}\right)^{-\frac{\nu+1}{2}}$$



When variance is unknown, estimating it from sample gives additional error:

Gaussian:

$$z = \frac{x - \mu}{\sigma}$$

Student's:

$$t = \frac{x - \mu}{\hat{\sigma}}$$

Simple tests (Z- or T-tests)

- **One-sample test** compares sample (e.g. mean) to known value:
Example: Comparing sample to known constant ($\mu_{\text{exp}} = 2.91 \pm 0.01$ vs. $c = 3.00$).
$$z = \frac{\bar{x} - \mu_0}{\sigma(\bar{x})}$$
- **Two-sample test** compares two samples (e.g. means).
Example: Comparing sample to control ($\mu_{\text{exp}} = 4.1 \pm 0.6$ vs. $\mu_{\text{control}} = 0.7 \pm 0.4$).
$$z = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\sigma(\bar{x}_1)^2 + \sigma(\bar{x}_2)^2}}$$
- **Paired test** compares paired member difference (to control important variables).
Example: Testing environment influence on twins to control genetic bias ($\mu_{\text{diff}} = 0.81 \pm 0.29$ vs. 0).

Things to consider:

- Variance known (Z-test) vs. Variance unknown (T-test).
Rule-of-thumb: If $N > 10-20$ or σ known then Z-test, else T-test.
- One-sided vs. two-sided test.
Rule-of-thumb: If you want to test for difference, then use two-sided. If you care about specific direction of difference, use one-sided.

Two-Tailed Versus One-Tailed Hypothesis Tests

Figure A:
Two-Tailed Test

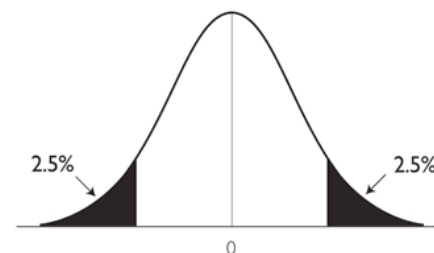
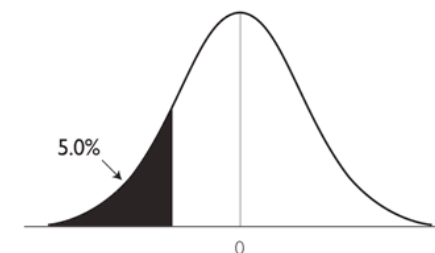


Figure B:
One-Tailed Test
(Left-Tailed Test)



Chi-squared test

Without any further introduction...

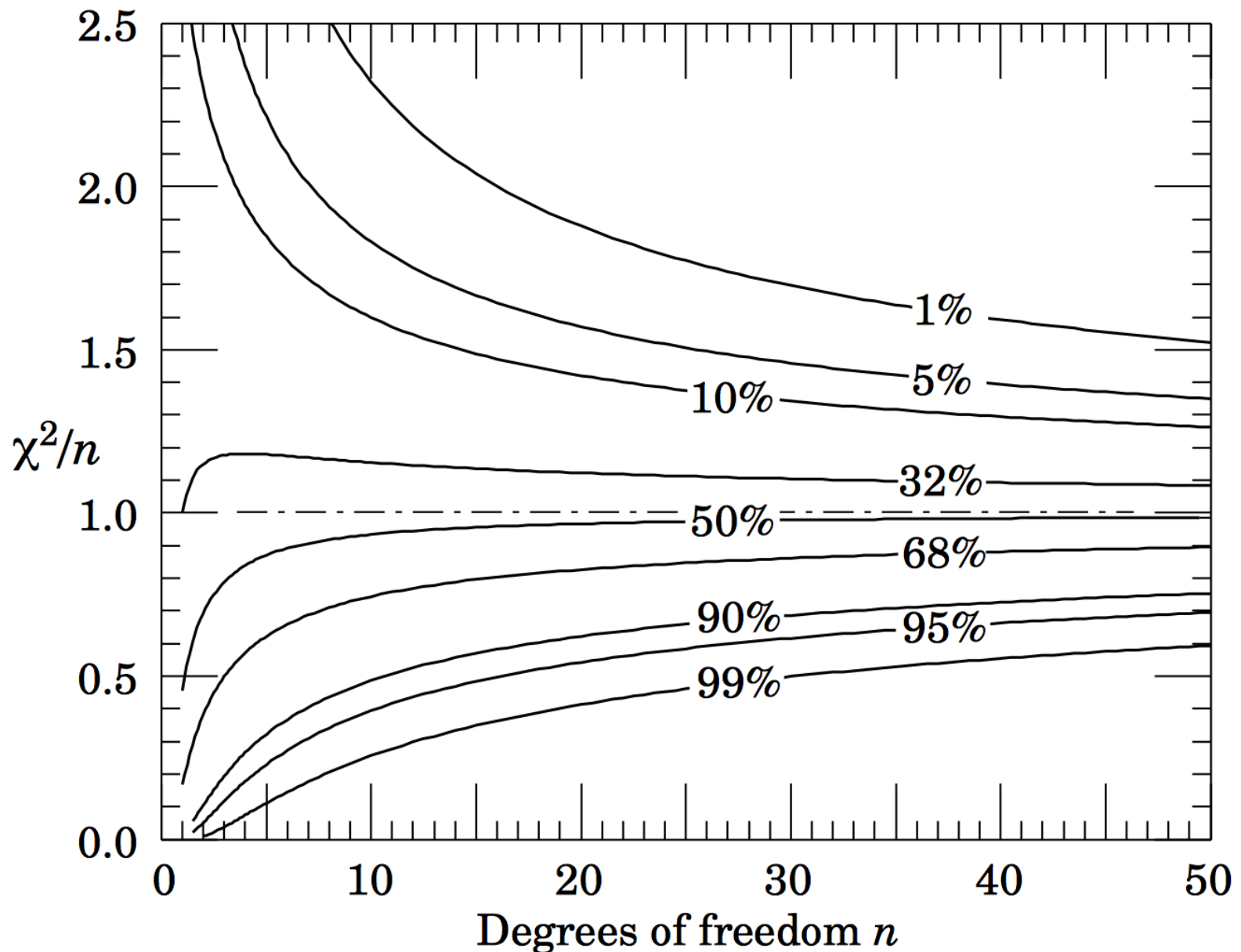
$$\chi^2(\bar{\theta}) = \sum_{i=1}^N \frac{(y_i - \lambda(x_i; \bar{\theta}))^2}{\sigma_i^2}$$

- **Chi-squared test** evaluates adequacy of model compared to data.

Example: Model fitted to (possibly binned) data, yielding p-value = $\text{Prob}(\chi^2 = 45.9, N_{\text{dof}} = 36) = 0.125$

If the p-value is small, the hypothesis is unlikely...

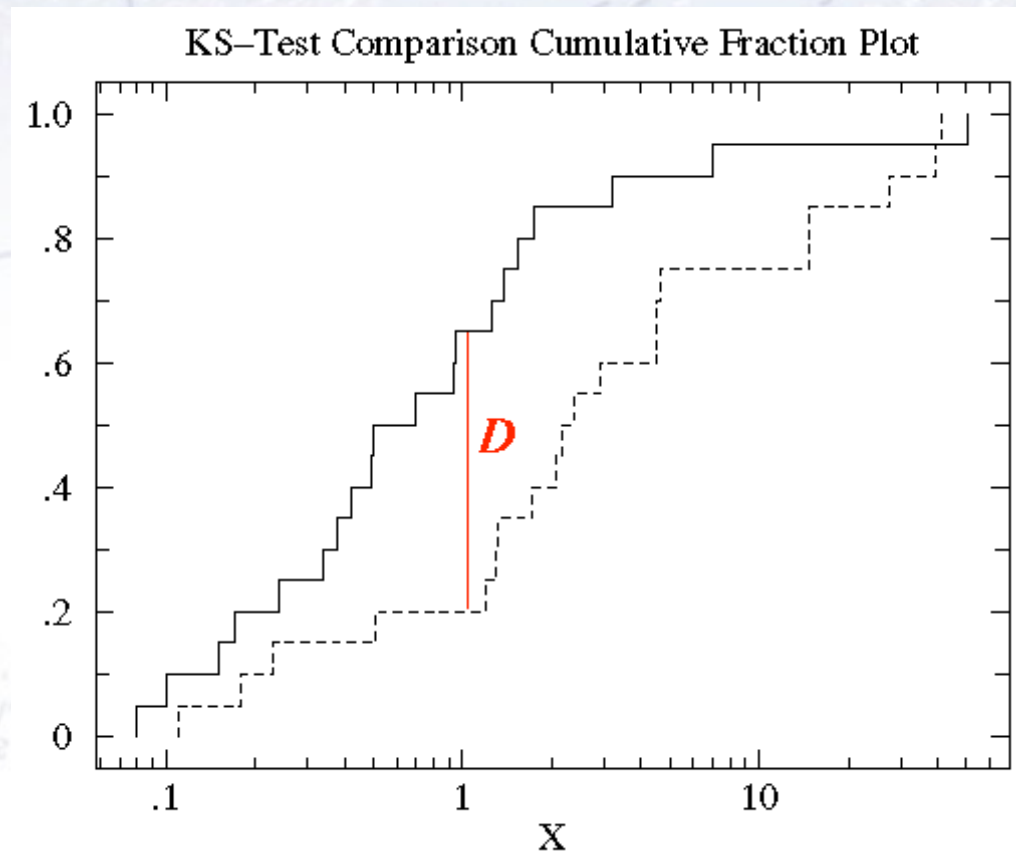
Chi-squared test



Kolmogorov-Smirnov test

- **Kolmogorov-Smirnov test** compares if two distributions are compatible.

Example: Compatibility between function and sample or between two samples, yielding p-value = 0.87

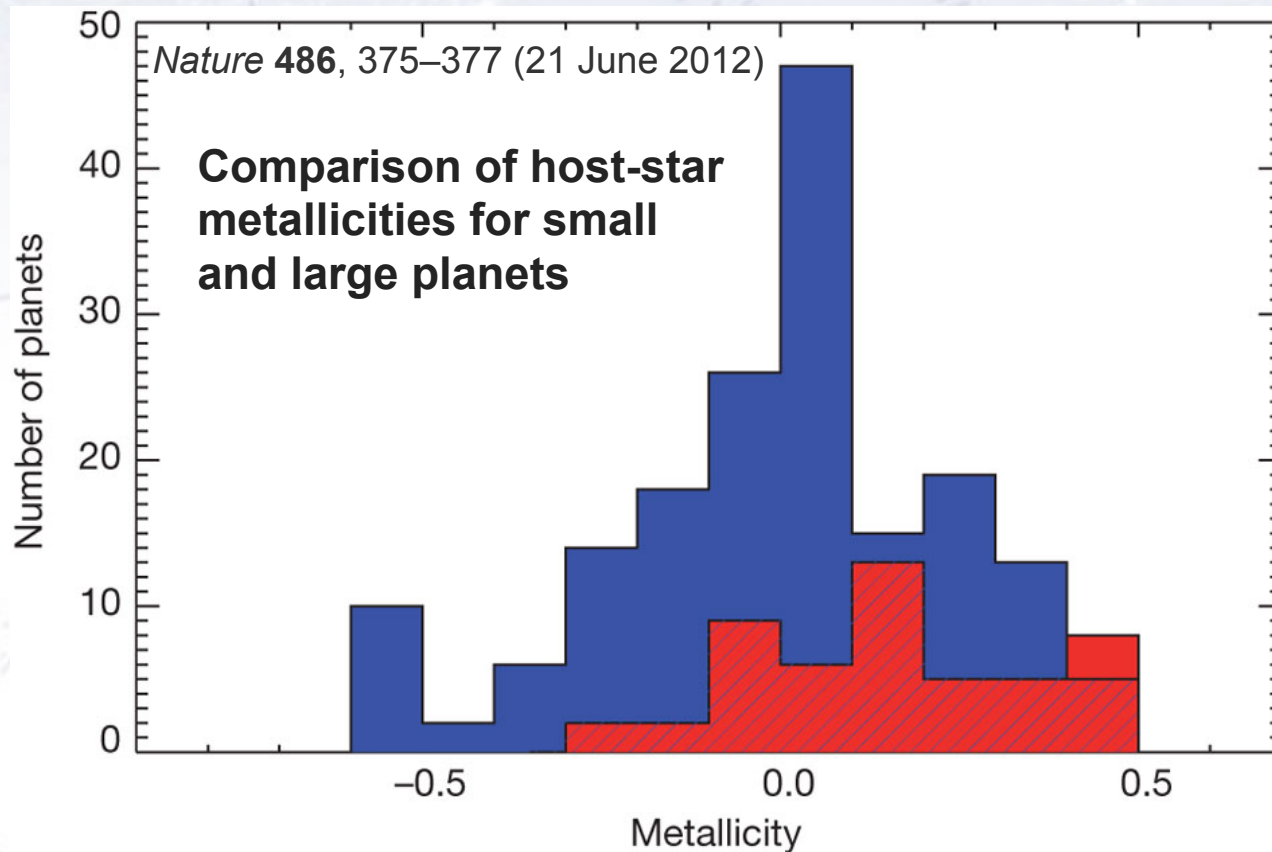


The Kolmogorov test measures the maximal distance between the integrals of two distributions and gives a probability of being from the same distribution.

Kolmogorov-Smirnov test

- **Kolmogorov-Smirnov test** compares if two distributions are compatible.

Example: Compatibility between function and sample or between two samples, yielding p-value = 0.87

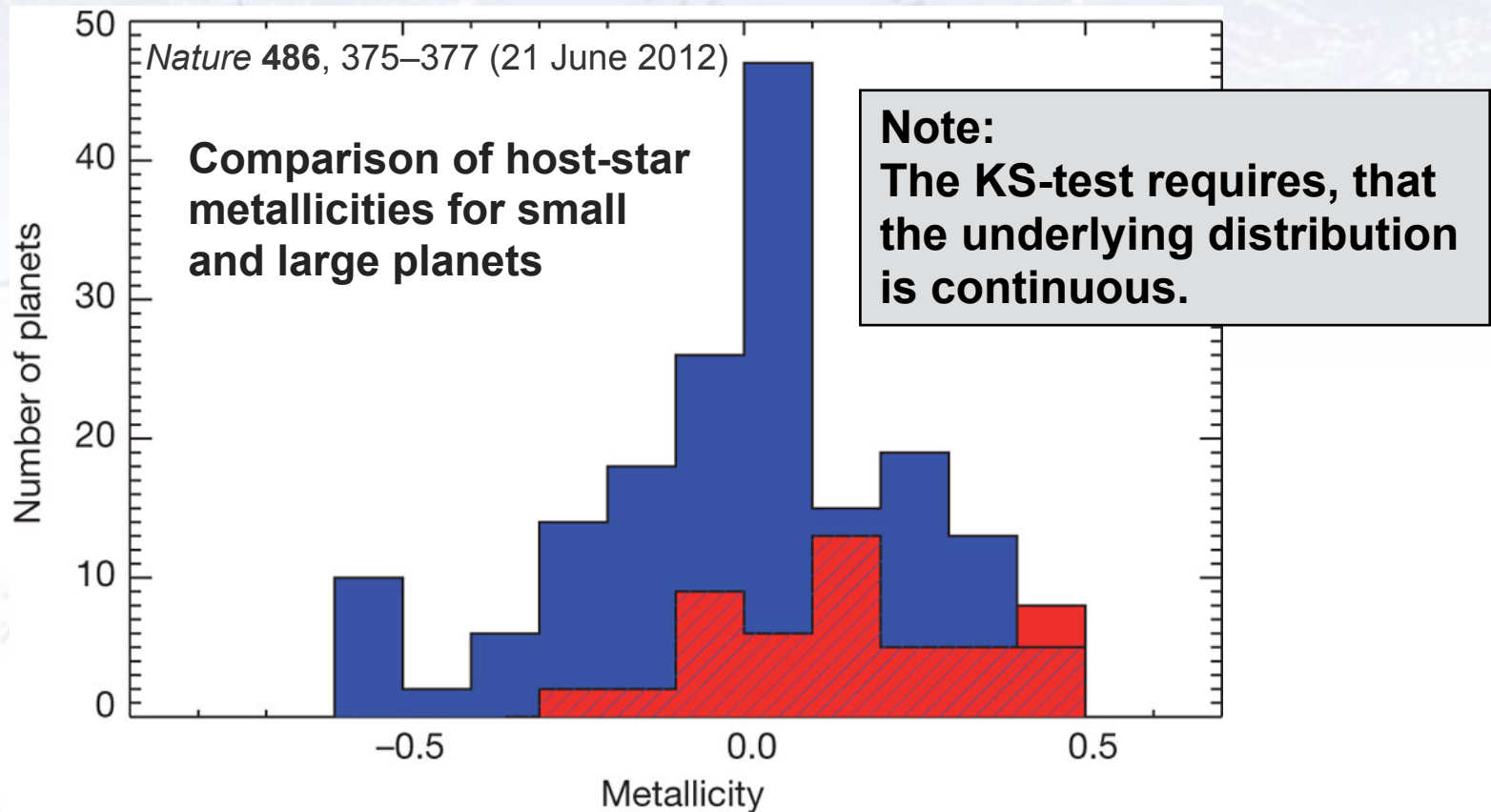


“A Kolmogorov–Smirnov test shows that the probability that the two distributions are not drawn randomly from the same parent population is greater than 99.96%; that is, the two distributions differ by more than 3.5σ ”. [Quote from figure caption]

Kolmogorov-Smirnov test

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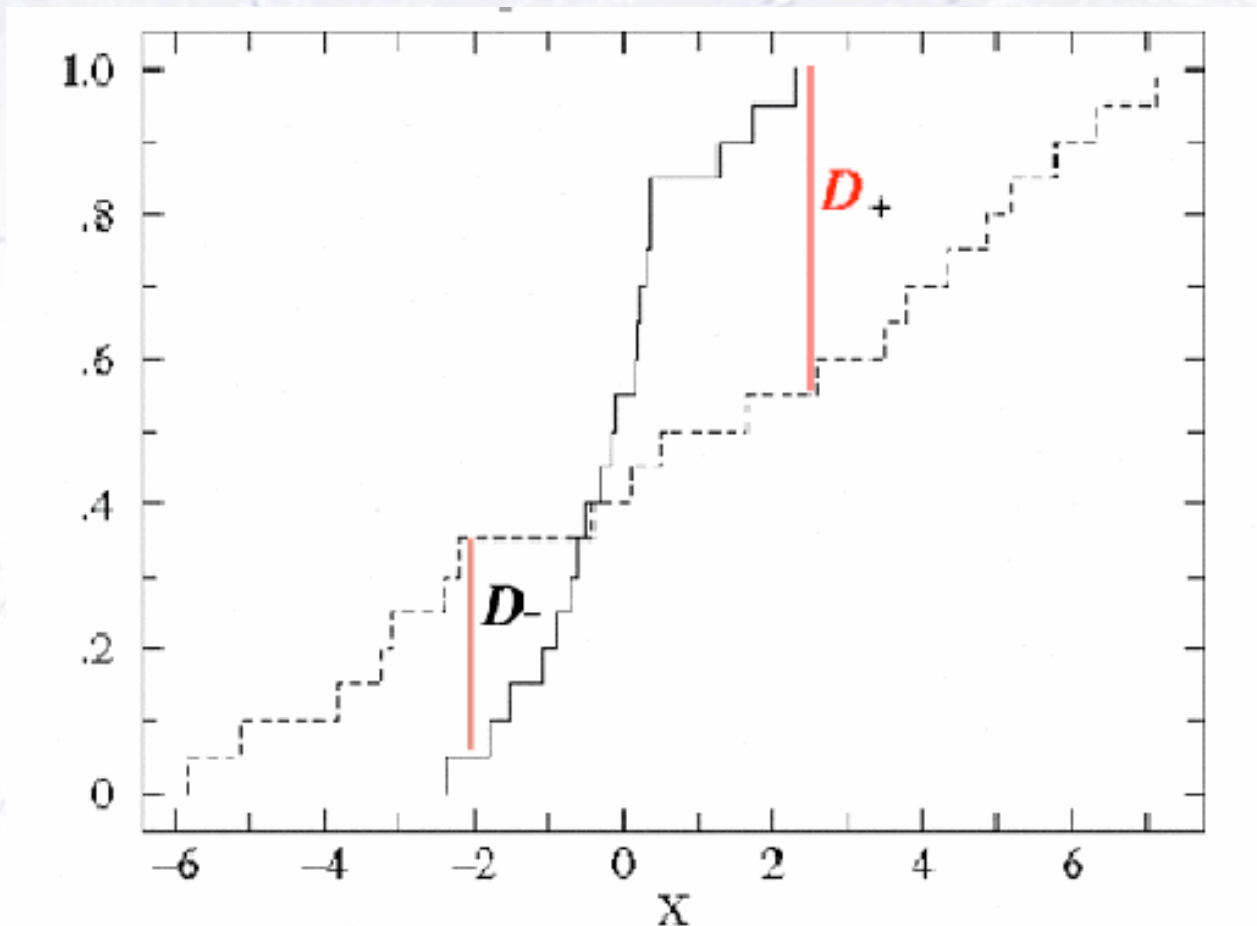
Example: Compatibility between function and sample or between two samples, yielding p-value = 0.87



“A Kolmogorov–Smirnov test shows that the probability that the two distributions are not drawn randomly from the same parent population is greater than 99.96%; that is, the two distributions differ by more than 3.5σ ”. [Quote from figure caption]

Kuiper test

Is a similar test, but it is more specialised in that it is good to detect SHIFTS in distributions (as it uses the maximal signed distance in integrals).



Common statistical tests

- **One-sample test** compares sample (e.g. mean) to known value.
Example: Comparing sample to known constant ($\mu_{\text{exp}} = 2.91 \pm 0.01$ vs. $c = 3.00$).
$$z = \frac{\bar{x} - \mu_0}{\frac{\sigma}{\sqrt{n}}}$$
- **Two-sample test** compares two samples (e.g. means).
Example: Comparing sample to control ($\mu_{\text{exp}} = 4.1 \pm 0.6$ vs. $\mu_{\text{ctrl}} = 3.7 \pm 0.4$).
$$z = \frac{(\bar{x}_1 - \bar{x}_2) - d_0}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}}$$
- **Paired test** compares paired member difference (to control important variables).
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Example: Compatibility between function and sample or between two samples, yielding p-value = 0.87

**These tests you should know by heart!
Those below are for general education,
and you should just know about them
(and the last one is not curriculum).**

- **Wald-Wolfowitz runs test** is a binary check for independence.
- **Fisher's exact test** calculates p-value for contingency tables.
- **F-test** compares two sample variances to see, if grouping is useful.

Wald-Wolfowitz runs test

Barlow, 8.3.2, page 153

A different test to the Chi2 (and in fact a bit orthogonal!) is the Wald-Wolfowitz runs test.

It measures the number of “runs”, defined as sequences of same outcome (only two types).

Example:

++++- - - -++++- - - -+++++

If random, the mean and variance is known:

$$\mu = \frac{2 N_+ N_-}{N} + 1$$

$$\sigma^2 = \frac{2 N_+ N_- (2 N_+ N_- - N)}{N^2 (N - 1)} = \frac{(\mu - 1)(\mu - 2)}{N - 1}$$

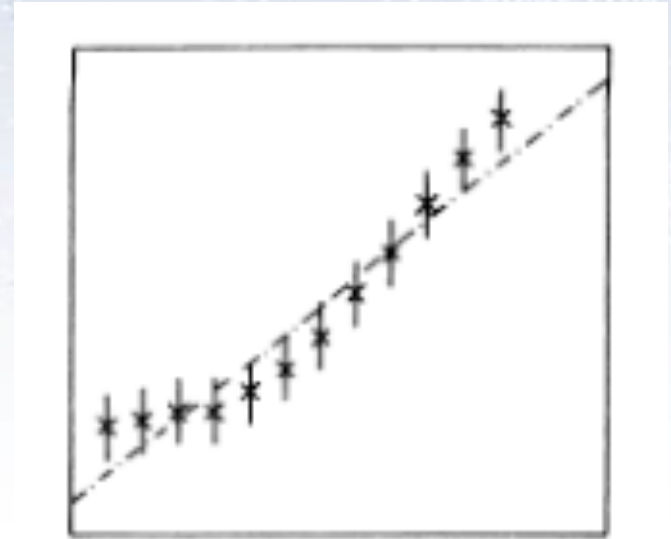


Fig. 8.3. A straight line through twelve data points.

$N = 12, N_+ = 6, N_- = 6$
 $\mu = 7, \sigma = 1.76$
 $(7-3)/1.65 = 2.4 \sigma (\sim 1\%)$

Note: The WW runs test requires $N > 10-15$ for the output to be approx. Gaussian! 41

Fisher's exact test

When considering a **contingency table** (like below), one can calculate the probability for the entries to be uncorrelated. This is **Fisher's exact test**.

	Row 1	Row 2	Row Sum
Column 1	A	B	A+B
Column 2	C	D	C+D
Column Sum	A+C	B+D	N

$$p = \frac{\binom{A+C}{A} \binom{B+D}{B}}{\binom{N}{A+B}} = \frac{(A+B)! (C+D)! (A+C)! (B+D)!}{A! B! C! D! N!}$$

Simple way to test categorical data (Note: Barnard's test is "possibly" stronger).

Fisher's exact test - example

Consider data on men and women dieting or not. The data can be found in the below table:

	Men	Women	<i>Row total</i>
Dieting	1	9	10
Non-dieting	11	3	14
<i>Column total</i>	12	12	24

Is there a correlation between dieting and gender?

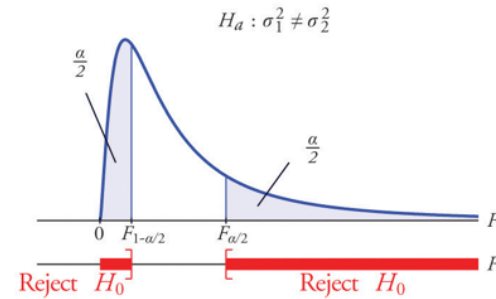
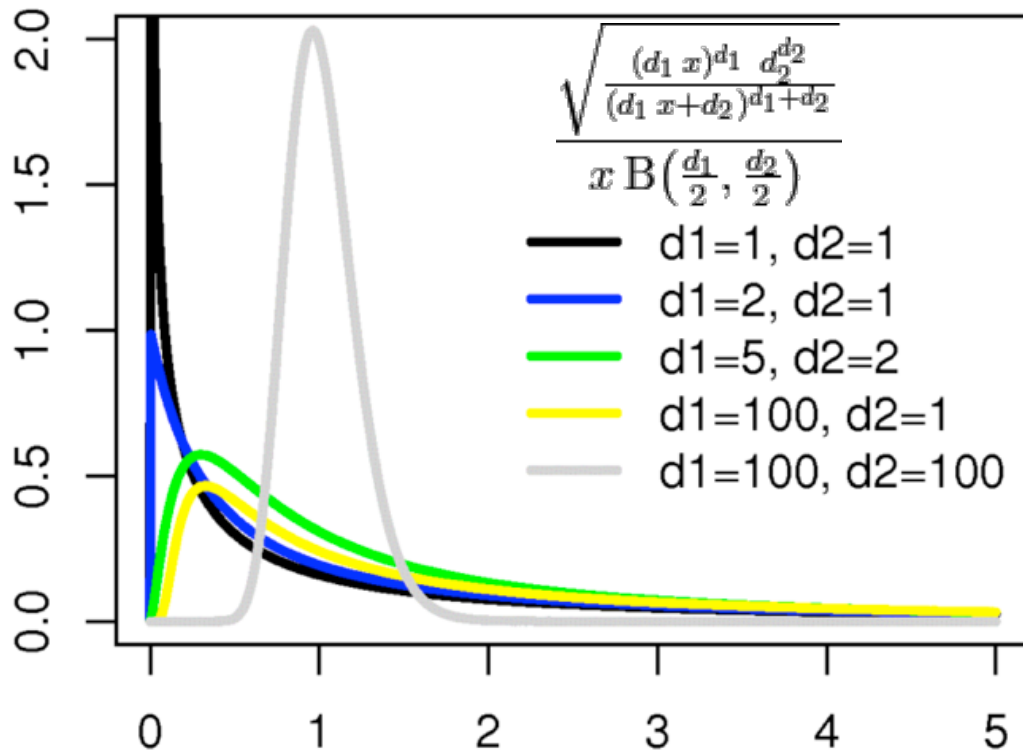
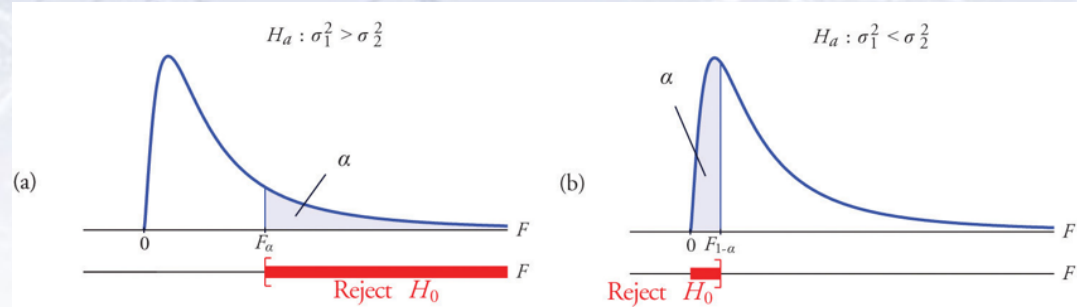
The Chi-square test is not optimal, as there are (several) entries, that are very low (< 5), but Fisher's exact test gives the answer:

$$p = \binom{10}{1} \binom{14}{11} / \binom{24}{12} = \frac{10! 14! 12! 12!}{1! 9! 11! 3! 24!} \simeq 0.00135$$

F-test

To test for differences between variances in two samples, one uses the F-test:

$$F = \frac{S_X^2}{S_Y^2}$$



Note that this is a two-sided test. One is generally testing, if the two variances are the same.

How many sigmas?

The number of sigmas (or p-value) required to make a claim should perhaps vary, according to the target of the data analysis.

Louis Lyons has below given his take on it (aimed at particle physics searches).

Search	Degree of surprise	Impact	LEE	Systematics	Number of σ
Higgs search	Medium	Very high	Mass	Medium	5
Single top	No	Low	No	No	3
SUSY	Yes	Very high	Very large	Yes	7
B_s oscillations	Medium/low	Medium	Δm	No	4
Neutrino oscillations	Medium	High	$\sin^2(2\theta), \Delta m^2$	No	4
$B_s \rightarrow \mu\mu$	No	Low/Medium	No	Medium	3
Pentaquark	Yes	High/very high	M, decay mode	Medium	7
$(g - 2)_\mu$ anomaly	Yes	High	No	Yes	4
H spin $\neq 0$	Yes	High	No	Medium	5
4 th generation q, l, ν	Yes	High	M, mode	No	6
$v_\nu > c$	Enormous	Enormous	No	Yes	>8
Dark matter (direct)	Medium	High	Medium	Yes	5
Dark energy	Yes	Very high	Strength	Yes	5
Grav waves	No	High	Enormous	Yes	7

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The more extraordinary the claim, the more extraordinary the evidence needed!