Project in Applied Statistics

[redacted]
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This article uses two different experimental setups in the same location to determine the gravitational acceleration. The first experiment is a simple pendulum and the other is a ball on an incline. For each of the experiments all measurements are performed by each group member independently, and each person estimates their uncertainties on that measurement. For most parts these uncertaintites are underestimated and therefore replaced by RMS values in the data analysis. For the pendulum experiment $g =$ [redacted] $\frac{\mathrm{m}}{\mathrm{s}^2}$ is obtained and for the ball on incline experiment a value of $g =$ [redacted] $\frac{\mathrm{m}}{\mathrm{s}^2}$ is obtained. The pendulum experiment is therefore the most accurate and precise of the two experiments compared to the gravitational acceleration of [redacted] $\frac{\mathrm{m}}{\mathrm{s}^2}$ in Copenhagen, where the measurements where performed.

INTRODUCTION

In this project, two unique experiments are conducted in order to measure the gravitational acceleration g with the highest achievable precision. In the first experiment, the period of a pendulum of length L is measured:

$$T = 2\pi\sqrt{\frac{L}{g}} \tag{1.1}$$

From this equation, g can be determined:

$$g = L\left(\frac{2\pi}{T}\right)^2 \tag{1.2}$$

In the second experiment, the time it takes for a ball to roll down an incline is measured at five different points. Using the distances and times between the points of measurement, g can be derived as (assuming the acceleration to be constant):

$$g = \frac{a}{\sin(\theta + \Delta\theta)}\left[1 + \frac{2}{5}\frac{D^2}{D^2 - d^2}\right] \tag{1.3}$$

where a is the measured acceleration of the ball, θ is the angle of the incline with respect to the table, $\Delta\theta$ is the angle of the table with respect to level, D is the diameter of the ball, and d is the width of the track it is rolling on. For clarification, see figure 2.

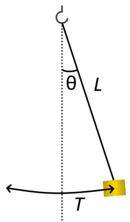

Figure 1: Schematic of the pendulum experiment.

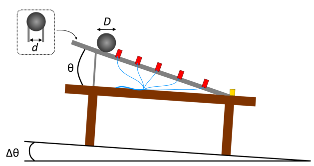

Figure 2: Schematic of the ball on incline experiment.

METHOD

All of the following measurements are conducted individually and independently by each of the four group members. To avoid biasing, results of other members' measurements are not disclosed until all members have completed their measurements. Along with each measurement, each group member must provide their best estimate on their uncertainty, taking into account the error on the equipment used.

Pendulum

The period of the pendulum is obtained by conducting timing measurements of the pendulum while it freely swings back and forth. As the small angle approximation has been used in equation 1.1, the pendulum motion is initiated by displacing it slightly from equilibrium, and letting go. The time is then measured using the provided stopwatch python-script over 25 periods of the pendulum's swings. The resulting data is then fitted with a straight line, and the period T is extracted as the slope of the fit, as seen in figure 3.

The height of the pendulum mass is measured using a caliper. The length of the pendulum, L, is measured in two ways: first using a tape measure, measured from the hook the pendulum wire is attached to, to the top of the pendulum. Secondly using a laser distance measurer. Here the distance from a spot as close as possible to the hook to the floor is measured, then subtracting the distance from the floor to the bottom of the pendulum. As the center-of-mass of the pendulum is assumed to be exactly in the center, half the height of the pendulum is then added/subtracted, respectively, to the two measuring methods. The influence of the hook on the pendulum weight is assumed negligible for the center-of-mass.

Ball on an incline

To calculate the gravitational acceleration g from the ball-on-an-incline experiment requires the measurement of five different quantities. First, the diameters D of the balls are measured using a caliper. Two balls of differing sizes are used in this experiment. The width of the track d is then measured using a caliper. The distance between the timing gates is measured using a long ruler and noting where the center of the lasers emitted from the gates hit the ruler. The ruler is kept in the same place, while each member reads off the positions. The angle of the incline θ is measured using a goniometer. To correct for the any errors related to the goniometer not being perfectly centered, the measurements are conducted twice, with the goniometer facing opposite directions. The table on which the incline stands will most likely have a minor angle $\Delta\theta$ to the floor on which it is placed. This minor angle will contribute to the measurement of the incline angle, and the actual measurement is $\theta \pm \Delta\theta$ ($\pm$ depending on the direction of the entire setup). Another way to measure the angle of the incline (without the contribution from the table's angle), is to measure a well-defined length and height in the incline, that should be exactly parallel and normal to the table, respectively. The angle of the incline can then be extracted using trigonometry.

The measured acceleration of the ball a as it rolls down the incline is obtained by combining the measurements of the gate positions with timing measurements from the gate. Fitting the resulting data to the equation

$$x = \frac{1}{2}at^2 + vt + x_0, \tag{1.4}$$

the acceleration can thus be extracted. To measure the size of the angle of the table with respect to the floor, the entire setup is then turned 180 degrees. The table angle $\Delta\theta$ can then be extracted in two ways. Both methods require a new angle measurement using the goniometer, again done twice facing opposite directions. Using these measurements, the table angle is given by:

$$\theta_{m,norm} - \theta_{m,rev} = 2\Delta\theta \tag{1.5}$$

where θ_m are the measured angles. Taking time measurements of the ball going down the incline again, the angle can also be calculated as:

$$\Delta\theta = \frac{(a_{norm} - a_{rev})\sin(\theta)}{(a_{norm} + a_{rev})\cos(\theta)} \tag{1.6}$$

where a_{norm} and a_{rev} are the fitted accelerations for the setup in opposite directions.

RESULTS

A complete collection of all measurements and results are provided in the appendix.

Pendulum

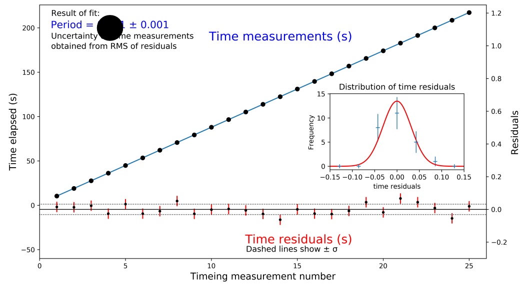

Figure 3: Plot of timing measurements with overlayed linear fit from which the period is extracted as the slope. Furthermore the residuals and their distribution is showed.

For the measuring tape a weighted average of the pendulum mass height and a weighted average of the pendulum length is added and gives L_{tape} = [redacted] m, where σ has been error propagated from the individual uncertainties. When doing a weighted average one should check whether the measurements are in agreement with each other, which was done with a χ^2-test, giving χ^2 = [redacted] and p = [redacted]. This χ^2 and p suggests that our measurements are in agreement with each other and that our uncertainties are neither over- or underestimated.
For the laser measuring device, a weighted average for both hook-to-floor distance, floor-to-pendulum distance and height of pendulum weight were found and then combined to give L_{laser} = [redacted] m with χ^2 = [redacted] and p = [redacted]. This high χ^2 and low p suggests that our uncertainties have been underestimated, which is maybe not that surprising as they were more or less guessed by us. Therefore L_{laser} is re-calculated as an arithmic mean with an uncertainty as the RMS, which gives L_{laser} = [redacted] m. The RMS is calculated with $\frac{1}{N-1}$ to take the low statistics into account. The values of L_{tape} and L_{laser} are within each others uncertainties, so a weighted average of the two values gives the length of the pendulum to be L = [redacted] m with χ^2 = [redacted] and p = [redacted]. This seems reasonable as the degrees of freedom is 1.
Fitting the timing measurements gave 4 different periods with different uncertainties, which were combined in a weighted average to give T = [redacted] s with χ^2 = [redacted] and p = [redacted]. One of the fits is seen in figure 3. The high χ^2 and low p suggests that there is a systematic error that is not being taken into account and that the uncertainties should therefore be inflated. The new systematic uncertainty is found by calculating the difference between all of the periods, finding the biggest difference and adding a systematic uncertainty such that all the periods are within each others uncertainties. Cal-

culating a new weighted average with the new uncertainties gives $T =$ [redacted] $\pm$ [redacted] $\pm$ [redacted] s with $\chi^2 =$ [redacted] and $p =$ [redacted]. This is a quite reasonable χ^2, as our degrees of freedom are 3 and therefore this period will be used for the calculation of g.
The value of g can now be computed according to eq. 1.2, which gives $g =$ [redacted] $\frac{\mathrm{m}}{\mathrm{s}^2}$, where the uncertainty has been calculated by the error propagation formula.

Ball on an incline

During the initial data analysis we found that using the estimated uncertainties on e.g. D, d and the angles, gave extremely large χ^2-values and associated probabilities p close to 0. We therefore opted to estimate uncertainties by calculating the RMS instead, which gave us much more sensible χ^2- and p-values. See tables in appendix.

By measuring the angle of the track with the goniometer and turning the experiment 180° we determined the angle of $d\theta =$ [redacted]°. To cross-check this value, we use (1.6), and get $d\theta =$ [redacted]°. Since these two obviously do not agree, we inflated our errors with a systematic uncertainty of $\pm$ [redacted]° to get $\chi^2 =$ [redacted] and $p =$ [redacted]. The weighted average of these two values for $d\theta$ was calculated to be $d\theta =$ [redacted]°.

To find the time at which a ball passes through a gate, we wrote a script that extracted a subset of the timing measurements, corresponding to the totality period, when the laser light is completely obscured by a passing ball, see figure 5. We believe the subset is approximately uniform, so we take the passing times and their errors, to be the mean $\mu = \frac{b-a}{2}$, where b and a is the end and start time respectively, and error on the mean $\frac{\sigma}{\sqrt{N}}$, where $\sigma = \frac{b-a}{\sqrt{12}}$, of a uniform distribution.

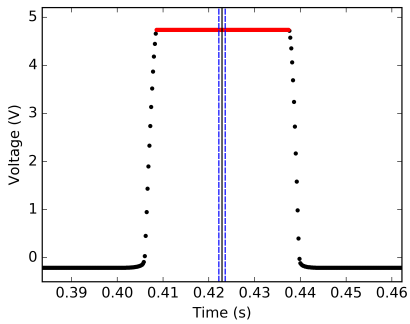

Figure 4: Example voltage peak as the ball passes a timing gate. A subset of the timing data, corresponding to the totality period, is shown in red. The mean and error on the mean of this subset are shown by the solid black line and the blue dashed lines, respectively.

We collected 16 data sets in total, half of which were collected after rotating the entire experiment 180°, which are further split between using a big- and medium-sized balls. A distance-time graph can now be constructed. We fit the data with a quadratic polynomial using `iminuit`[1], and extract the acceleration, see figure 5. We get four accelerations in total: 2 for each ball size, and 2 for each direction of the experiment. Using these accelerations, along with weighted averages of the remaining measurements, gives us $g =$ [redacted] $\frac{\mathrm{m}}{\mathrm{s}^2}$.

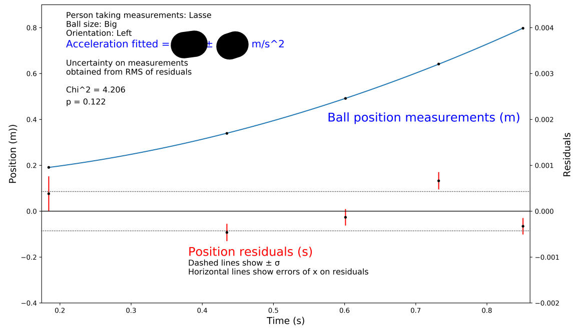

Figure 5: One distance-time graph with the quadratic fit overlayed on top for Lasse with a big ball before the experiment was reversed. $\chi^2 = 4.206$ and $p = 0.122$ between the data and the fit. Bottom part of the graph shows the time residuls.

DISCUSSION

Obtaining $g =$ [redacted] $\frac{\mathrm{m}}{\mathrm{s}^2}$ from the pendulum experiment and $g =$ [redacted] $\frac{\mathrm{m}}{\mathrm{s}^2}$ from the ball on incline experiment is respectively 0.5σ and 2.8σ away from the official gravitational accelerlation in Copenhagen of [redacted] $\frac{\mathrm{m}}{\mathrm{s}^2}$, calculated by using the International Gravity Formula and the values of the GRS80 reference system [2].
For the pendulum experiment the result is therfore both quite accurate and precise, whereas the ball on incline experiment is less accurate and precise. One of the reasons for this is due to the fact that there are a lot more variables to measure in the ball on incline experiment, which leads to more terms in the error propagation formula. The error propagation formula for the pendulum experiment is

$$\sigma_{g,pend}^2 = \frac{64L^2\pi^4}{T^6}\sigma_T^2 + \frac{16\pi^4}{T^4}\sigma_L^2 \tag{1.7}$$

Looking in table I it can be seen that the first term is the largest and if one wanted a more precise measurement one should aim for better timing measurements. Initially there was a smaller uncertainty on T, but it had to be inflated due to inconsistencies in the measurements. One should therefore work on taking better timing measurements, e.g. filming the experiment, or having more people take the timing measurements and then use an

arithmic mean of their periods and the RMS as the uncertainty. This was also attempted for our calculations, but lead to a less accurate and less precise result and therefore the inflation of uncertainties was chosen instead.
The error propagation formula for the ball on incline is given by

$$\sigma^2_{g,ball} = \left(1 + \frac{2D^2}{5(D^2 - d^2)}\right) \frac{1}{sin(\theta \pm \Delta\theta)^2} \sigma^2_a + a^2 \left(1 + \frac{2D^2}{5(D^2 - d^2)}\right) \frac{cos(\theta \pm \Delta\theta)^2}{sin(\theta \pm \Delta\theta)^4} \sigma^2_\theta + a^2 \left(1 + \frac{2D^2}{5(D^2 - d^2)}\right) \frac{cos(\theta \pm \Delta\theta)^2}{sin(\theta \pm \Delta\theta)^4} \sigma^2_{\Delta\theta} + \frac{a^2}{sin(\theta + \Delta\theta)^2} \left(\frac{4D}{5(D^2 - d^2} - \frac{4D^3}{5(D^2 - d^2)}\right)^2 \sigma^2_D + \frac{a^2 16 D^4 d^2}{25 sin(\theta + \Delta\theta)^2 (D^2 - d^2)^4} \sigma^2_d \tag{1.8}$$

As seen in table I the largest contribution comes from the third term, which is the contribution for the uncertainty on the table angle. A more precise result could therefore be obtained by minimising this uncertainty, e.g. with more measurements, which would lead to a smaller RMS. This angle was also not very well determined, as we got two very different answers from the cross-checks.
Ultimately one could calculate a final g from the values from the two experiments. Taking a weighted mean then gives $g =$ [redacted] $\frac{\mathrm{m}}{\mathrm{s}^2}$ with a $\chi^2 =$ [redacted] and $p =$ [redacted]. It is a small p value, but still above 1% and we choose therefore not to inflate the uncertainties. This final value is then equal to the official value of g to this number of decimal points.

Variable	Value	Error	Impact
Pendulum			
Period T	[redacted] s	[redacted] s	[redacted] $\mathrm{m}^2/\mathrm{s}^4$
Length L	[redacted] m	[redacted] m	[redacted] $\mathrm{m}^2/\mathrm{s}^4$
Resulting g	[redacted] $\mathbf{m/s^2}$	[redacted] $\mathbf{m/s^2}$	
Ball on incline (big ball, normal)			
Acceleration a	[redacted] m	[redacted] m	[redacted] $\mathrm{m}^2/\mathrm{s}^4$
Incline Angle θ	[redacted]°	[redacted]°	[redacted] $\mathrm{m}^2/\mathrm{s}^4$
Table Angle $\Delta\theta$	[redacted]°	[redacted]°	[redacted] $\mathrm{m}^2/\mathrm{s}^4$
Diameter Ball D	[redacted] mm	[redacted] mm	[redacted] $\mathrm{m}^2/\mathrm{s}^4$
Diameter Rail d	[redacted] mm	[redacted] mm	[redacted] $\mathrm{m}^2/\mathrm{s}^4$
Resulting g	[redacted] $\mathbf{m/s^2}$	[redacted] $\mathbf{m/s^2}$	

Table I: Variables used in the two experiments, their values, uncertainties, and the impact they have on the uncertainty on g. The impact is the size of the respective term in the error propagation formulas (1.7 and 1.8) associated with that variable. It should be noted that the resulting g is the weighted average of four different g's from different ball sizes and directions. The variables given in this table is for just one of the g's as an example.

CONCLUSION

In conclusion the gravitational acceleration was determined to be $g =$ [redacted] $\frac{\mathrm{m}}{\mathrm{s}^2}$ for the pendulum experiment and [redacted] $\frac{\mathrm{m}}{\mathrm{s}^2}$ for the ball on incline experiment. Comparing to the official value of $g =$ [redacted] $\frac{\mathrm{m}}{\mathrm{s}^2}$, we can conclude that the pendulum experiment gave more accurate and precise measurements. Taking a weighted average of the two accelerations from the two experiments, we obtain a value a final value for g of [redacted] $\frac{\mathrm{m}}{\mathrm{s}^2}$, which is in agreement with the official value.

BIBLIOGRAPHY

[1] Https://iminuit.readthedocs.io/en/latest/about.html.
[2] Https://en.wikipedia.org/wiki/Theoretical_gravity

APPENDIX

Pendulum measurements

Period [s]		
Value	Statistical	Systematical
[redacted]	± [redacted]	± [redacted]
[redacted]	± [redacted]	± [redacted]
[redacted]	± [redacted]	± [redacted]
[redacted]	± [redacted]	± [redacted]
Mean: [redacted]	± [redacted]	± [redacted]

Table II: Measurements of the pendulum period extracted from fits of the data

Mass height [cm]	
Value	Error
[redacted]	± [redacted]
[redacted]	± [redacted]
[redacted]	± [redacted]
[redacted]	± [redacted]
Mean: [redacted]	± [redacted]

Table III: Measurements of the mass height using a caliper

Laser, top → ground [m]	
Value	Error
[redacted]	± [redacted]
[redacted]	± [redacted]
[redacted]	± [redacted]
[redacted]	± [redacted]
Mean: [redacted]	± [redacted]

Table IV: Measurements of the distance from the top of the pendulum wire to the ground using the laser

Laser, ground → bottom [cm]	
Value	Error
[redacted]	± [redacted]
[redacted]	± [redacted]
[redacted]	± [redacted]
[redacted]	± [redacted]
Mean: [redacted]	± [redacted]

Table V: Measurements of the distance from the ground to the bottom of the mass using the laser

Pendulum length (tape) [m]	
Value	Error
[redacted]	± [redacted]
[redacted]	± [redacted]
[redacted]	± [redacted]
[redacted]	± [redacted]
Mean: [redacted]	± [redacted]

Table VI: Measurements of the pendulum length using a tape measure. Note that these go to the top of the swinging mass, not the center-of-mass, which must be included in the final pendulum length

Ball on incline measurements

Ball diameters [mm]	
Value	Error
Big ball	
[redacted]	± [redacted]
[redacted]	± [redacted]
[redacted]	± [redacted]
[redacted]	± [redacted]
Mean: [redacted]	± [redacted]
Medium ball	
[redacted]	± [redacted]
[redacted]	± [redacted]
[redacted]	± [redacted]
[redacted]	± [redacted]
Mean: [redacted]	± [redacted]

Table VII: Diameter measurements of the three different balls

Rail width [mm]	
Value	Error
[redacted]	± [redacted]
[redacted]	± [redacted]
[redacted]	± [redacted]
[redacted]	± [redacted]
Mean: [redacted]	± [redacted]

Table VIII: Measurements of the rail width using a caliper

Incline angle [°]			
Value	Error	Value	Error
Track normal			
Gonio norm		Gonio rev	
[redacted]	± [redacted]	[redacted]	± [redacted]
[redacted]	± [redacted]	[redacted]	± [redacted]
[redacted]	± [redacted]	[redacted]	± [redacted]
[redacted]	± [redacted]	[redacted]	± [redacted]
Overall mean: [redacted] ± [redacted]			
Track reversed			
Gonio norm		Gonio rev	
[redacted]	± [redacted]	[redacted]	± [redacted]
[redacted]	± [redacted]	[redacted]	± [redacted]
[redacted]	± [redacted]	[redacted]	± [redacted]
[redacted]	± [redacted]	[redacted]	± [redacted]
Overall mean: [redacted] ± [redacted]			

Table IX: Measurements of the angle using the goniometer

Lengths used for θ_{trig} [cm]	
Value	Error
Length	
[redacted]	± [redacted]
[redacted]	± [redacted]
[redacted]	± [redacted]
[redacted]	± [redacted]
Mean: [redacted] ± [redacted]	
Height	
[redacted]	± [redacted]
[redacted]	± [redacted]
[redacted]	± [redacted]
[redacted]	± [redacted]
Mean: [redacted] ± [redacted]	
Height offset	
[redacted]	± [redacted]
[redacted]	± [redacted]
[redacted]	± [redacted]
[redacted]	± [redacted]
Mean: [redacted]	

Table X: Length measurements of the triangle formed by the incline

Gate positions [cm]	
Value	Error
Gate 1	
[redacted]	± [redacted]
[redacted]	± [redacted]
[redacted]	± [redacted]
[redacted]	± [redacted]
Mean: [redacted] ± [redacted]	
Gate 2	
[redacted]	± [redacted]
[redacted]	± [redacted]
[redacted]	± [redacted]
[redacted]	± [redacted]
Mean: [redacted] ± [redacted]	
Gate 3	
[redacted]	± [redacted]
[redacted]	± [redacted]
[redacted]	± [redacted]
[redacted]	± [redacted]
Mean: [redacted] ± [redacted]	
Gate 4	
[redacted]	± [redacted]
[redacted]	± [redacted]
[redacted]	± [redacted]
[redacted]	± [redacted]
Mean: [redacted] ± [redacted]	
Gate 5	
[redacted]	± [redacted]
[redacted]	± [redacted]
[redacted]	± [redacted]
[redacted]	± [redacted]
Mean: [redacted] ± [redacted]	

Table XI: Measurements of the gate positions.