Tracks, Lie's, and Exceptional Magic

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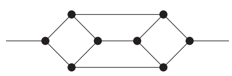
1 Introduction

Sometimes a solution to a mathematical problem is so beautiful that it can impede further progress for a whole century. So is the case with the Killing-Cartan classification of semi-simple Lie algebras [Killing 1888; Cartan 1952]. It is elegant, it is beautiful, and it says that the 3 classical families and 5 exceptional algebras are all there is, but what does that mean?

The construction of all Lie algebras outlined here (for a more detailed presentation, consult [Cvitanović 2004]) is an attempt to answer to this question. It is not a satisfactory answer – as a classification of semi-simple Lie groups it is incomplete – but it does offer a different perspective on the exceptional Lie algebras. The question that started the whole odyssey is: What is the group theoretic weight for Quantum Chromodynamic diagram



$$= ? \qquad (1.1)$$

A quantum-field theorist cares about such diagrams because they arise in calculations related to questions such as asymptotic freedom. The answer turns out to require quite a bit of group theory, and the result is better understood as the answer to a different question: Suppose someone came into your office and asked

> "On planet **Z**, mesons consist of quarks and antiquarks, but baryons contain 3 quarks in a *symmetric* color combination. What is the color group?"

If you find the particle physics jargon distracting, here is another way to posing the same question: "Classical Lie groups preserve bilinear vector norms. What Lie groups preserve trilinear, quadrilinear, and higher order invariants?"

The answer easily fills a book [Cvitanović 2004]. It relies on a new notation: invariant tensors ↔ "Feynman" diagrams, and a new computational method, diagrammatic from start to finish. It leads to surprising new relations: all exceptional Lie groups emerge together, in one family, and groups such as E_7 and $SO(4)$ are related to each other as "negative dimensional" partners.

Here we offer a telegraphic version of the "invariance groups" program. We start with a review of basic group-theoretic notions, in a somewhat unorthodox notation suited to the purpose at hand. A reader might want to skip directly to the interesting part, starting with sect. 3.

The big item on the "to do" list: *prove* that the resulting classification (primitive invariants → *all* semi-simple Lie algebras) is exhaustive, and *prove* the existence of F_4, E_6, E_7 and E_8 within this approach.

2 Lie groups, a review

Here we review some basic group theory: linear transformations, invariance groups, diagrammatic notation, primitive invariants, reduction of multiparticle states, Lie algebras.

2.1 Linear transformations

Consider an n-dimensional vector space $V \in \mathbb{C}$, and a group $\mathcal{G}$ acting linearly on V (consult any introduction to linear algebra [Gel′fand 1961; Lang 1971; Nomizu 1979]). A *basis* $\{\mathbf{e}^1, \cdots, \mathbf{e}^n\}$ is any linearly independent subset of V whose span is V. n, the number of basis elements is called the *dimension* of the vector space V. In calculations to be undertaken a vector $\mathbf{x} \in V$ is often specified by the n-tuple $(x_1, \cdots, x_n)^t$ in $\mathbb{C}^n$, its coordinates $\mathbf{x} = \sum \mathbf{e}^a x_a$ in a given basis. We rarely, if ever, actually fix an explicit basis, but thinking this way makes it easier to manipulate tensorial objects. Under a general linear transformation in $GL(n, \mathbb{C}) = \{G : \mathbb{C}^n \to \mathbb{C}^n \mid \det(G) \neq 0\}$ a basis set of V is mapped into another basis set by multiplication with a $[n \times n]$ matrix G with entries in $\mathbb{C}$, the *standard rep* of $GL(n, \mathbb{C})$,

$$\mathbf{e}'^{\,a} = \mathbf{e}^b (G^{-1})_b{}^a \,, \qquad x'_a = G_a{}^b x_b \,.$$

The space of all n-tuples $(x_1, x_2, \ldots, x_n)^t$, $x_i \in \mathbb{C}$ on which these matrices act is the *standard representation space* V.

Under left multiplication the column (row transposed) of basis vectors transforms as $\mathbf{e}'^t = G^\dagger \mathbf{e}^t$, where the *dual rep* $G^\dagger = (G^{-1})^t$ is the transpose of the inverse of G. This observation motivates introduction of a *dual* representation space $\bar{V}$, is the set of all linear forms on V over the field $\mathbb{C}$. This is also an n-dimensional vector space, a space on which $GL(n, \mathbb{C})$ acts via the dual rep $G^\dagger$.

If $\{\mathbf{e}^1, \cdots, \mathbf{e}^n\}$ is a basis of V, then $\bar{V}$ is spanned by the *dual basis* $\{\mathbf{f}_1, \cdots, \mathbf{f}_n\}$, the set of n linear forms $\mathbf{f}_a$ such that

$$\mathbf{f}_a(\mathbf{e}^b) = \delta_a^b \,,$$

where δ_a^b is the Kronecker symbol, $\delta_a^b = 1$ if $a = b$, and zero otherwise. The dual representation space coordinates, distinguished here by upper indices, $(y^1, y^2, \ldots, y^n)$, transform under $GL(n, \mathbb{C})$ as

$$y'^a = G^a{}_b y^b \,. \tag{2.1}$$

In the index notation $G^\dagger$ is represented by $G^a{}_b$, and G by $G_b{}^a$. For $GL(n, \mathbb{C})$ no complex conjugation is implied by the † notation; that interpretation applies only to unitary subgroups of $GL(n, \mathbb{C})$. In what follows we shall need the following notions:

The *defining rep* of group $\mathcal{G}$:

$$G : V \to V\,, \qquad [n \times n] \text{ matrices } G_a{}^b \in \mathcal{G}\,.$$

The *defining multiplet:* a "1-particle wave function" $q \in V$ transforms as

$$q'_a = G_a{}^b q_b\,, \qquad a, b = 1, 2, \dots, n\,.$$

The *dual multiplet:* "antiparticle" wave function $\bar{q} \in \bar{V}$ transforms as

$$q'^a = G^a{}_b q^b\,.$$

Tensors: multi-particle states transform as $V^p \otimes \bar{V}^q \to V^p \otimes \bar{V}^q$, for example

$$p'_a q'_b r'^c = G_a{}^f G_b{}^e G^c{}_d p_f q_e r^d\,.$$

Unless explicitly stated otherwise, repeated upper/lower index pairs are always summed over

$$G_a{}^b x_b \equiv \sum_{b=1}^{n} G_a{}^b x_b\,.$$

2.2 Invariants

A multinomial

$$H(\bar{q}, \bar{r}, \dots, s) = h_{ab\dots}{}^{\dots c}\, q^a r^b \dots s_c$$

is an *invariant* of the group $\mathcal{G}$ if for all $G \in \mathcal{G}$ and any set of vectors $q, r, s, \dots$ it satisfies

$$\textit{invariance condition:} \qquad H(G^\dagger \bar{q}, G^\dagger \bar{r}, \dots G s) = H(\bar{q}, \bar{r}, \dots, s)\,.$$

Take a finite list of *primitive invariants*:

$$\mathbf{P} = \{p_1, p_2, \dots, p_k\}\,.$$

(As it is difficult to state what a *primitive invariant* is before explaining what it is not, the definition is postponed to sect. 2.5.)

Definition. An *invariance group* $\mathcal{G}$ is the set of all linear transformations which satisfy a finite number of *invariance conditions* (*ie*, preserve all primitive invariants $\in \mathbf{P}$)

$$p_1(x, \bar{y}) = p_1(Gx, G^\dagger \bar{y})\,, \qquad p_2(x, y, z, \dots) = p_2(Gx, Gy, Gz \dots)\,, \qquad \dots\,.$$

No other primitive invariants exist.

***Example**: orthogonal group $O(3)$*

Defining space: 3-dimensional Euclidean space of 3-component real vectors

$$x, y, \cdots \in V = \mathbb{R}^3\,, \qquad V = \bar{V}$$

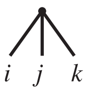
Primitive invariants:

$$\text{length} \qquad L(x, x) = \delta_{ij} x_i x_j$$

$$\text{volume} \qquad V(x, y, z) = \epsilon_{ijk} x_i y_j z_k$$

Invariant tensors:

$$\delta_{ij} = i \;\text{———}\; j\,, \qquad \epsilon_{ijk} = \qquad\qquad . \tag{2.2}$$



***Example**: unitary group $U(n)$*

Defining space: n-dimensional vector space of n-component complex vectors

$$x_a \in V = \mathbb{C}^n$$

Dual space: space of n-component complex vectors $x^a \in \bar{V} = \mathbb{C}^n$ transforming under $G \in \mathcal{G}$ as

$$x'^a = G^a{}_b x^b$$

Primitive invariants: a single primitive invariant, norm of a complex vector

$$N(\bar{x}, x) = |x|^2 = \delta^a_b x^b x_a = \sum_{a=1}^{n} x^*_a x_a\,.$$

The Kronecker $\delta^a_b = b \longleftarrow a$ is the *only* primitive invariant tensor. The invariance group $\mathcal{G}$ is the *unitary group* $U(n)$ whose elements satisfy $G^\dagger G = 1$:

$$x'^a y'_a = x^b (G^\dagger G)_b{}^c y_c = x^a y_a\,,$$

All invariance groups considered here will be subgroups of $U(n)$, *ie* have δ^a_b as one of their primitive invariant tensors.

2.3 Diagrammatic notation

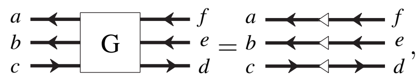
Depending on the context, we shall employ either the *tensorial index notation*

$$p'_a q'_b r'^c = G_{ab}{}^c{}_,{}_d{}^{ef} p_f q_e r^d\,, \qquad G_{ab}{}^c{}_,{}_d{}^{ef} = G_a{}^f G_b{}^e G^c{}_d\,,$$

or the *collective indices notation*

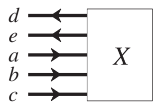
$$q'_\alpha = G_\alpha{}^\beta q_\beta \qquad \alpha = \left\{ {}^{\ \ c}_{ab} \right\} , \quad \beta = \left\{ {}^{\ ef}_{d} \right\} ,$$

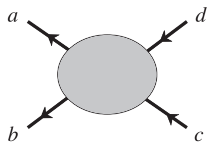
or the *diagrammatic notation*



whichever is most convenient for the purpose at hand.

We shall refer to diagrams representing agglomerations of invariant tensors as *birdtracks*, a group-theoretical version of Feynman diagrams, with invariant tensors corresponding to *vertices* (blobs with external legs)

$$X_\alpha = X^{abc}_{de} =$$ 

, $h^{cd}_{ab} =$ 

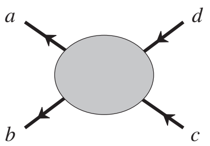
,

and index contractions corresponding to *propagators* (Kronecker deltas)

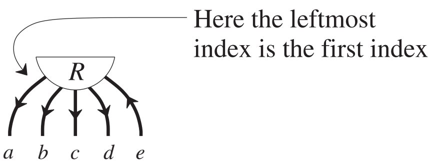
$$\delta^a_b = b \longleftarrow a \, .$$

Rules

(1) Direct arrows *from upper* indices "downward" *toward the lower* indices:

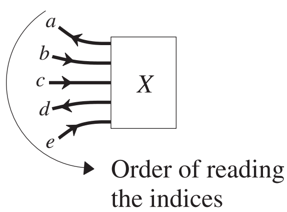
$$h^{cd}_{ab} =$$ 

(2) Indicate which in (out) arrow corresponds to the *first* upper (lower) index:

$$R^e_{abcd} =$$ 

.

(3) Read indices in the *counterclockwise* order around the vertex:

$$x^{bce}_{ad} =$$ 

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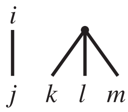
2.4 Composed invariants, tree invariants

Which rep is *"defining"?* The defining rep of group $\mathcal{G}$ is the $[n \times n]$ matrix rep acting on the defining vector space V. The defining space V *need not* carry the lowest dimensional rep of $\mathcal{G}$.

Definition. A *composed invariant tensor* is a product and/or contraction of invariant tensors.

Example: *$SO(3)$ composed invariant tensors*

$$\delta_{ij}\epsilon_{klm} =$$



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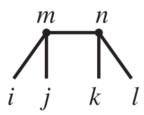
$$\epsilon_{ijm}\delta_{mn}\epsilon_{nkl} =$$



. (2.3)

Corresponding invariants:

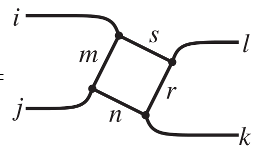
$$\textit{product } L(x,y)V(z,r,s)\,; \qquad \text{index } \textit{contraction } V(x,y,\frac{d}{dz})\,V(z,r,s)\,.$$

Definition. A *tree invariant* involves no loops of index contractions.

Example: *a tensor with an internal loop*

Tensors drawn in (2.3) are tree invariants. The tensor

$$h_{ijkl} = \epsilon_{ims}\epsilon_{jnm}\epsilon_{krn}\epsilon_{\ell sr} =$$



,

with internal loop indices m, n, r, s summed over, is *not* a tree invariant.

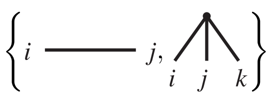
2.5 Primitive invariants

Definition. An invariant tensor is *primitive* if it cannot be expressed as a linear combination of tree invariants composed of other primitive invariant tensors.

Example: *$SO(3)$ invariant tensors*

The Kronecker delta and the Levi-Civita tensor (2.2) are the primitive invariant tensors of our 3-dimensional space:

$$\mathbf{P} =$$



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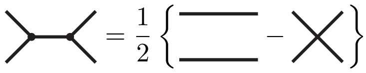
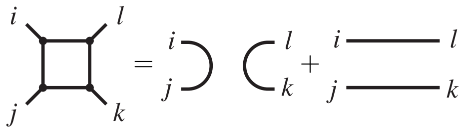
4-vertex loop is *not* a primitive, because the Levi-Civita relation



reduces it to a sum of tree contractions:



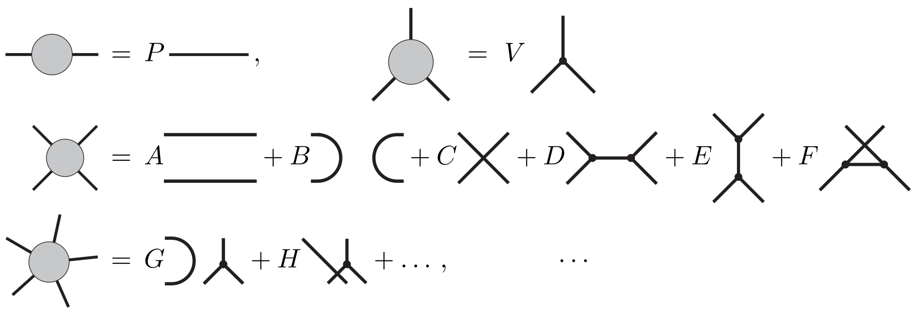
Let $T = \{\mathbf{t}_0, \mathbf{t}_1 \dots \mathbf{t}_r\}$ = a maximal set of r linearly independent tree invariants $\mathbf{t}_\alpha \in V^p \otimes \bar{V}^q$.

Primitiveness assumption. Any invariant tensor $h \in V^p \otimes \bar{V}^q$ can be expressed as a linear sum over the basis set T.

$$h = \sum_{\alpha=0}^{r} h^\alpha \mathbf{t}_\alpha \,.$$

***Example**: invariant tensor basis sets*

Given primitives $P = \{\delta_{ij}, f_{ijk}\}$, any invariant tensor $h \in V^p$ (here denoted by a blob) is expressible as

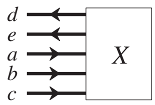
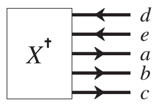


2.6 Reduction of tensor reps: Projection operators

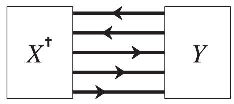
Dual of a tensor $T \to T^\dagger$ is obtained by

(a) exchanging the upper and the lower indices, *ie. reversing arrows*
(b) reversing the order of the indices, *ie. transposing* a diagram into its mirror image.

Example*: A tensor and its dual*

$X_\alpha = X^{abc}_{de} =$  , $\quad X^\alpha = X^{ed}_{cba} =$  .

Contraction of tensors $X^\dagger$ and Y

$X^\alpha Y_\alpha = X^{b_p \ldots b_1}_{a_q \ldots a_2 a_1} Y^{a_1 a_2 \ldots a_q}_{b_1 \ldots b_p} =$  .

Motivation for drawing a dual tensor as a flip of the initial diagram: contraction $X^\dagger X = |X|^2$ can be drawn in a plane.

For a defining space $V = \bar{V} = \mathbb{R}^n$ defined on reals there is no distinction between up and down indices, and lines carry no arrows

$\delta^j_i = \delta_{ij} = i$ —— j .

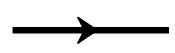
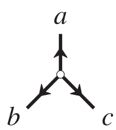
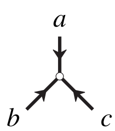
Invariant tensor $M \in V^{p+q} \otimes \bar{V}^{p+q}$ is a *self-dual*

$$M : V^p \otimes \bar{V}^q \to V^p \otimes \bar{V}^q$$

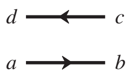
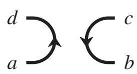
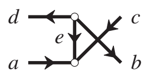
if it is invariant under transposition and arrow reversal.

Example*: symmetric cubic invariant*

Given the 3 *primitive invariant tensors*:

$\delta^b_a = a$  b , $\quad d_{abc} =$  , $\quad d^{abc} =$  .

(d_{abc} fully symmetric) one can construct only 3 *self-dual* tensors $M : V \otimes \bar{V} \to V \otimes \bar{V}$

$\delta^a_b \delta^c_d =$  , $\quad \delta^a_d \delta^c_b =$  , $\quad d^{ace} d_{ebd} =$  ,

all three self-dual under transposition and arrow reversal.

A Hermitian matrix M is diagonalizable by a unitary transformation C

$$CMC^\dagger = \begin{pmatrix} \lambda_1 & 0 & 0 & & \ldots \\ 0 & \lambda_1 & 0 & & \\ 0 & 0 & \lambda_1 & & \\ & & & \lambda_2 & \\ \vdots & & & & \ddots \end{pmatrix} .$$

Removing a factor $(M-\lambda_j \mathbf{1})$ from the characteristic equation $\prod(M-\lambda_i \mathbf{1}) = 0$ yields a *projection operator:*

$$P_i = \prod_{j\neq i} \frac{M-\lambda_j \mathbf{1}}{\lambda_i - \lambda_j} = C^\dagger \begin{pmatrix} 0 & & & & & & \\ & \ddots & & & & 0 & \\ & & 0 & & & & \\ & & & \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & & \\ \vdots & & \ddots & \vdots \\ 0 & & \dots & 1 \end{pmatrix} & & & \\ & & & & 0 & & \\ & 0 & & & & \ddots & \\ & & & & & & 0 \end{pmatrix} C$$

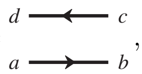
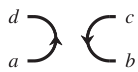
for each distinct eigenvalue of M.

Example: *$U(n)$ invariant matrices*

$U(n)$ is the invariance group of the norm of a complex vector $|x|^2 = \delta^a_b x^b x_a$,

only *primitive invariant tensor*: $\delta^a_b = a \longrightarrow b$

Can construct 2 *invariant hermitian matrices* $M \in V^2 \otimes \bar{V}^2$:

identity : $\mathbf{1}^{a\ c}_{d,b} = \delta^a_b \delta^c_d =$  , trace : $T^{a\ c}_{d,b} = \delta^a_d \delta^c_b =$  .

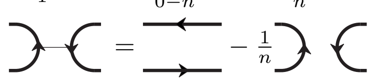
The *characteristic equation* for T in tensor, birdtrack, matrix notation:

$$T^{a\ f}_{d,e} T^{e\ c}_{f,b} = \delta^a_d \delta^f_e \delta^e_f \delta^c_b = n\, T^{a\ c}_{d,b}\,,$$

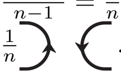
$$T^2 = nT\,.$$

where $\delta^e_e = n =$ the dimension of the defining vector space V. The roots of the characteristic equation $T^2 = nT$ are $\lambda_1 = 0$, $\lambda_2 = n$. The corresponding *projection operators* decompose $U(n) \to SU(n) \oplus U(1)$:

$SU(n)$ adjoint rep: $P_1 = \frac{T-n\mathbf{1}}{0-n} = \mathbf{1} - \frac{1}{n}T$



$U(n)$ singlet: $P_2 = \frac{T-0\cdot\mathbf{1}}{n-1} = \frac{1}{n}T$

$= \frac{1}{n}$  .

2.7 Infinitesimal transformations

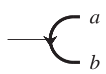
Infinitesimal unitary transformation, its action on the dual space:

$$G_a{}^b = \delta_a^b + i\epsilon_j (T_j)_a^b\,, \qquad G^a{}_b = \delta_b^a - i\epsilon_j (T_j)_b^a\,, \qquad |\epsilon_j| \ll 1\,.$$

is parametrized by

$$N = \textit{dimension} \text{ of the group (Lie algebra, adjoint rep)} \leq n^2$$

real parameters ϵ_j. The adjoint representation matrices $\{T_1, T_2, \cdots, T_N\}$ are *generators* of infinitesimal transformations, drawn as

$$\frac{1}{\sqrt{a}}(T_i)_b^a = i$$

$$a, b = 1, 2, \ldots, n\,, \quad i = 1, 2, \ldots, N\,,$$

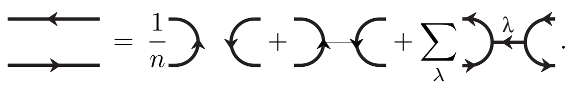
where a is an (arbitrary) overall normalization. The adjoint representation Kronecker delta will be drawn as a thin straight line

$$\delta_{ij} = i \;\text{———}\; j\,, \qquad i, j = 1, 2, \ldots, N\,.$$

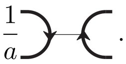
The decomposition of $V \otimes \bar{V}$ into (ir)reducible subspaces always contains the adjoint subspace:

$$\mathbf{1} = \frac{1}{n}T + P_A + \sum_{\lambda \neq A} P_\lambda$$

$$\delta_d^a \delta_b^c = \frac{1}{n}\delta_b^a \delta_d^c + (P_A)_b^a{}_d^c + \sum_{\lambda \neq A} (P_\lambda)_b^a{}_d^c$$

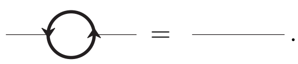


where the adjoint rep projection operators is drawn in terms of the generators:

$$(P_A)_b^a{}_d^c = \frac{1}{a}(T_i)_b^a (T_i)_d^c = \frac{1}{a}$$


The arbitrary normalization a cancels out in the projection operator orthogonality condition

$$\mathrm{tr}\,(T_i T_j) = a\,\delta_{ij}$$



2.8 Invariance under infinitesimal transformations

By definition, an invariant tensor h is unchanged under an infinitesimal transformation

$$G_\alpha{}^\beta h_\beta = (\delta_\alpha{}^\beta + \epsilon_j (T_j)_\alpha{}^\beta) h_\beta + O(\epsilon^2) = h_\alpha \,,$$

so the generators of infinitesimal transformations *annihilate* invariant tensors

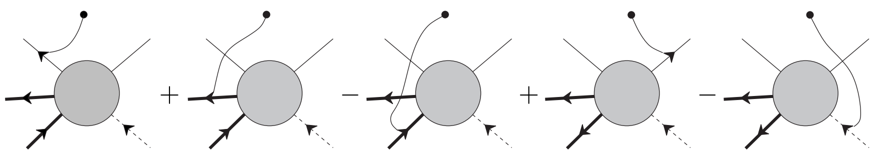
$$T_i h = 0 \,.$$

The *tensorial index notation* is cumbersome:

$$p'_a q'_b r'^c = G_a{}^f G_b{}^e G^c{}_d \, p_f q_e r^d$$

$$G_a{}^f G_b{}^e G^c{}_d = \delta_a^f \delta_b^e \delta_d^c + \epsilon_j ((T_j)_a^f \delta_b^e \delta_d^c + \delta_a^f (T_j)_b^e \delta_d^c - \delta_a^f \delta_b^e (T_j)_d^c) + O(\epsilon^2) \,,$$

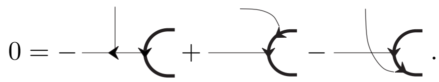
but diagrammatically the *invariance condition* is easy to grasp. The sum



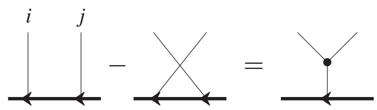
vanishes, *i.e.* the group acts as a derivative.

2.9 Lie algebra

The generators T_i are themselves invariant tensors, so they also must satisfy the invariance condition,

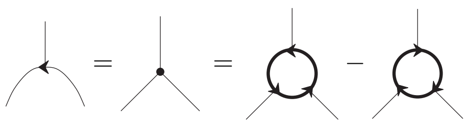


Redraw, replace the adjoint rep generators by the structure constants: and you have the *Lie algebra* commutation relation



$$T_i T_j \;-\; T_j T_i \;=\; i C_{ijk} T_k \,. \qquad (2.4)$$

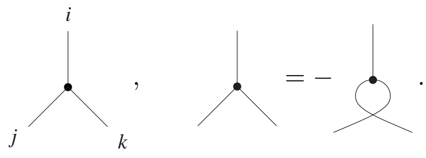
For a generator of an infinitesimal transformation acting on the adjoint rep, $A \to A$, it is convenient to replace the arrow by a full dot



$$(T_i)_{jk} \equiv -i C_{ijk} \;=\; -\mathrm{tr}\,[T_i, T_j] T_k \,,$$

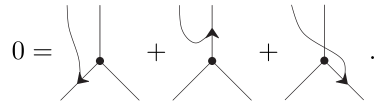
where the dot stands for a fully antisymmetric structure constant iC_{ijk}. Keep track of the overall signs by always reading indices *counterclockwise* around a vertex

$-iC_{ijk} =$

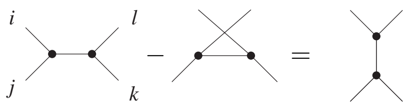


(2.5)

The invariance condition for structure constants C_{ijk} is



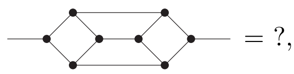
Redraw with the dot-vertex to obtain the *Jacobi relation*



$$C_{ijm}C_{mkl} - C_{ljm}C_{mki} = C_{iml}C_{jkm}\,. \tag{2.6}$$

Example: *Evaluation of any $SU(n)$ graph*

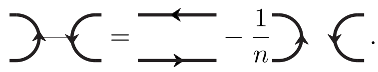
Remember (1.1),



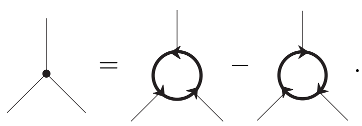
the one graph that launched this whole odyssey?

We saw that the adjoint rep projection operator for the invariance group of the norm of a complex vector $|x|^2 = \delta^a_b x^b x_a$ is

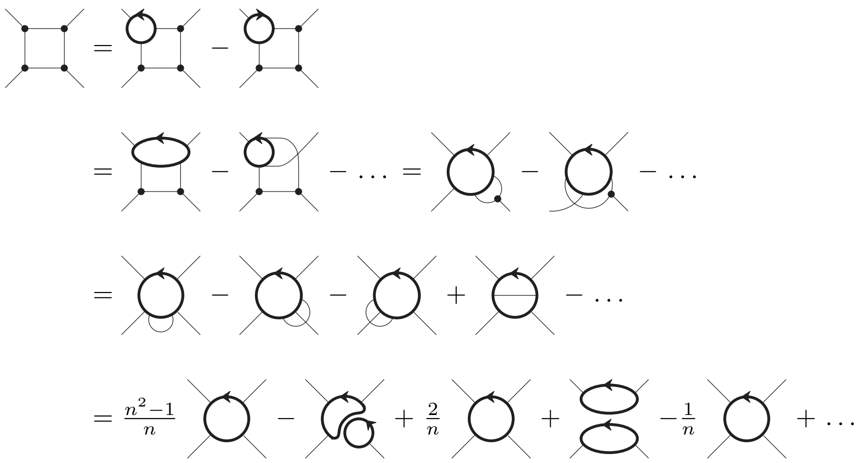
$SU(n)$:



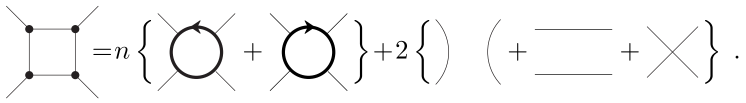
Eliminate C_{ijk} 3-vertices using



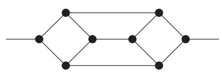
Evaluation is performed by a recursive substitution, the algorithm easily automated



arriving at



Collecting everything together, we finally obtain

$SU(n):$  $= 2n^2(n^2 + 12)$ ——— .

Any $SU(n)$ graph, no matter how complicated, is eventually reduced to a polynomial in traces of $\delta_a^a = n$, the dimension of the defining rep.

2.10 A brief history of birdtracks

Semi-simple Lie groups are here presented in an unconventional way, as "birdtracks". This notation has two lineages; a group-theoretical lineage, and a quantum field theory lineage:

Group-theoretical lineage

1930: Wigner [Wigner 1959]: all group theory weights in atomic, nuclear, and particle physics can be reduced to $3n$-j coefficients.
1956: I. B. Levinson [Levinson 1956]: presents the Wigner $3n$-j coefficients in graphical form, appears to be the first paper to introduce diagrammatic notation for any group-theoretical problem. See Yutsis, Levinson and Vanagas [Yutsis et al. 1964] for a full exposition. For the most recent survey, see G. E. Stedman [Stedman 1990].

Quantum field-theoretic lineage

1949: R. P. Feynman [Feynman 1949]: beautiful sketches of the very first "Feynman diagrams" .
1971: R. Penrose's [Penrose 1971a; Penrose 1971b] drawings of symmetrizers and antisymmetrizers.
1974: G. 't Hooft ['t Hooft 1974] double-line notation for $U(n)$ gluons.
1976: P. Cvitanović [Cvitanović 1976; Cvitanović 1977b] birdtracks for classical and exceptional Lie groups.

In the quantum groups literature graphs composed of 3-vertices are called trivalent. The Jacobi relation (2.6) in diagrammatic form [Cvitanović 1976] appears in literature for the first time in 1976. This set of diagrams has since been given moniker IHX [Bar-Natan 1995]. who refers to the full antisymmetry of structure constants (2.5) as the "AS relation", and to the Lie algebra commutator (2.4) as the "STU relation", by analogy to the Mandelstam's scattering cross-channel variables (s, t, u).

A reader might ask: "These are Feynman diagrams. Why rename them birdtracks?" In quantum field theory Feynman diagrams are a *mnemonic device*, an aid in writing down an integral, which then has to be evaluated by other means. *"Birdtracks"* are a *calculational method*: all calculations are carried out in terms of diagrams, from start to finish. Left behind are blackboards and pages of squiggles of kind that made my colleague Bernice Durand exclaim: "What are these birdtracks!?" and thus give them the name.

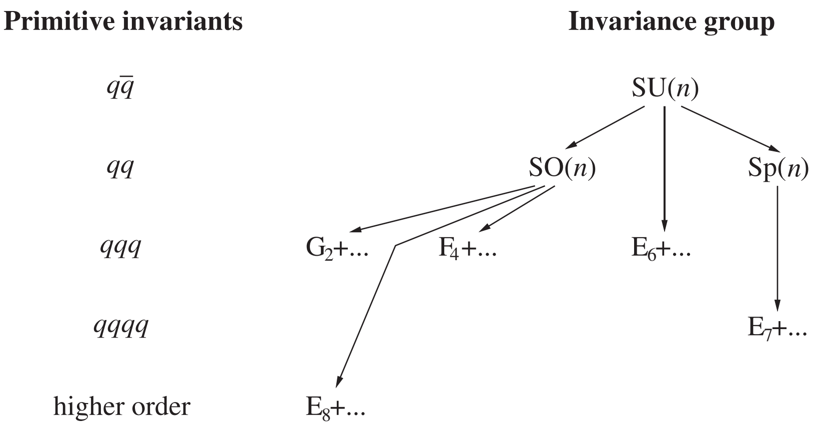
3 Lie groups as invariance groups

We proceed to classify groups that leave trilinear or higher invariants. The strategy:

i) define an invariance group by specifying a list of *primitive invariants*
ii) *primitiveness* and *invariance* conditions → algebraic relations between primitive invariants
iii) construct *invariant matrices* acting on tensor product spaces,
iv) construct *projection operators* for reduced rep from characteristic equations for invariant matrices.
v) determine allowed realizations from *Diophantine conditions* on representation dimensions.

When the next invariant is added, the group of invariance transformations of the previous invariants splits into two subsets; those transformations which preserve the new invariant, and those which do not. Such successive decompositions yield Diophantine conditions on rep dimensions, so constraining that they limit the possibilities to a few which can be easily identified.

The logic of this construction schematically indicated by the chains of subgroups



The arrows indicate the primitive invariants which characterize a particular group.

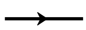
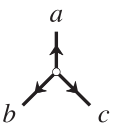
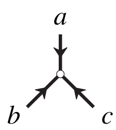
As a warm-up, we derive the "E_6 family" as a family of groups that preserve a symmetric cubic invariant.

3.1 E_6 primitives

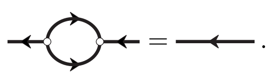
What invariance group preserves norms of complex vectors, as well as a symmetric cubic invariant

$$D(p,q,r) = D(q,p,r) = D(p,r,q) = d^{abc} p_a q_b r_c \ ?$$

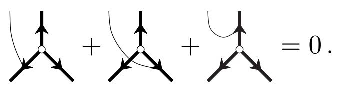
i) *primitive invariant tensors*:

$\delta_a^b = a$  $b\,,\quad d_{abc} =$  $,\quad d^{abc} = (d_{abc})^* =$  .

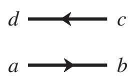
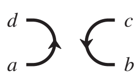
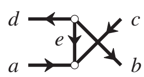
ii) *primitiveness*: $d_{aef}d^{efb}$ is proportional to δ_b^a, the only primitive 2-index tensor. This can be used to fix the d_{abc}'s normalization:



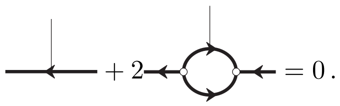
invariance condition:



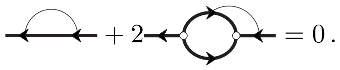
iii) *all invariant self-dual matrices* in $V \otimes \bar{V} \to V \otimes \bar{V}$:

$\delta_b^a \delta_d^c =$  $,\quad \delta_d^a \delta_b^c =$  $,\quad d^{ace}d_{ebd} =$  .

Contract the invariance condition with d^{abc}:

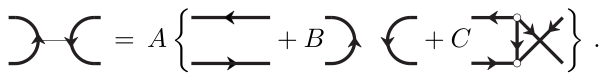


Contract with $(T_i)_a^b$ to get an invariance condition on the adjoint projection operator P_A:

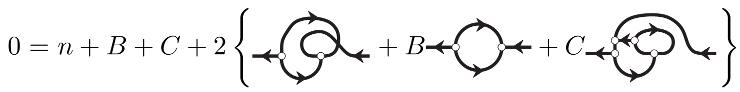


Adjoint projection operator in the invariant tensor basis (A, B, C to be fixed):

$$(T_i)_b^a (T_i)_c^d \;=\; A(\delta_c^a \delta_b^d + B\delta_b^a \delta_c^d + C d^{ade} d_{bce})$$



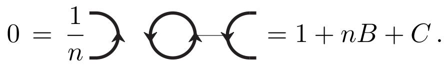
Substituting P_A



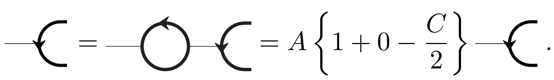
$$0 = B + C + \frac{n+2}{3}\,.$$

iv) *projection operators* are orthonormal: P_A is orthogonal to the singlet projection operator P_1, $0 = P_A P_1$.

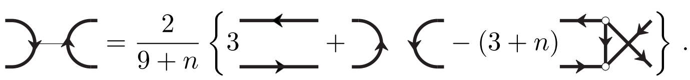
This yields the second relation on the coefficients:



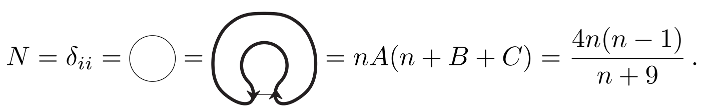
Normalization fixed by $P_A P_A = P_A$:



The three relations yield the adjoint projection operator for the E_6 *family*:



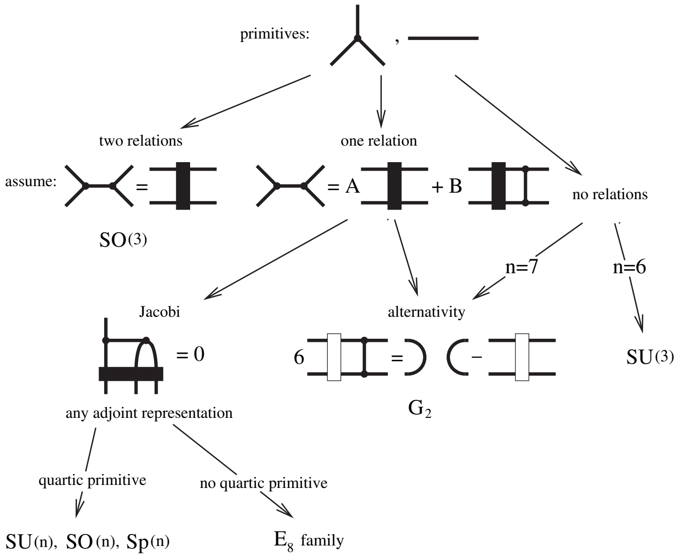
The dimension of the adjoint rep is given by:



As the defining and adjoint rep dimensions n and N are integers, this formula is a *Diophantine condition*, satisfied by a small family of invariance groups, the E_6 row in the Magic Triangle of fig. 1, with E_6 corresponding to $n = 27$ and $N = 78$.

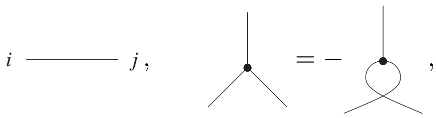
4 G_2 and E_8 families of invariance groups

We classify next all groups that leave invariant a symmetric quadratic invariant and an antisymmetric cubic invariant



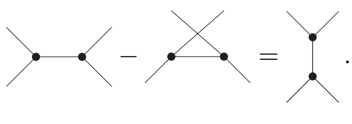
Assumption of no relation between the three 4-index invariant tree tensors constructed by the 3 distinct ways of contracting two f_{abc} tensors leads to the G_2 *family of invariance groups* [Cvitanović 2004], interesting its own right, but omitted here for brevity. If there is a relation between the three such tensors, symmetries this relation is necessarily the Jacobi relation.

The E_8 *family of invariance groups* follows if the primitive invariants are *symmetric* quadratic, *antisymmetric cubic*



(4.1)

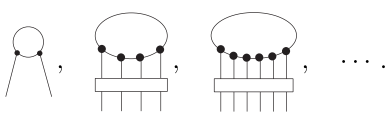
and the Jacobi relation is satisfied:



(4.2)

The task the we face is:

(i) enumerate *all Lie groups that leave these primitives invariant.*
(ii) demonstrate that we can reduce all loops



(4.3)

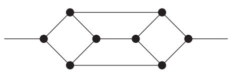
Accomplished so far: The Diophantine conditions yield all of the E_8 family Lie algebras, and no stragglers.

"To do":

(i) so far *no proof* that there exist no further Diophantine conditions.
(ii) The projection operators for E_8 family enable us to evaluate diagrams with internal loops of *length 5 or smaller*, but we have *no proof* that *any* vacuum bubble can be so evaluated.

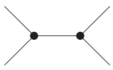
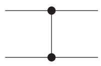
4.1 Two-index tensors

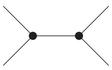
Remember

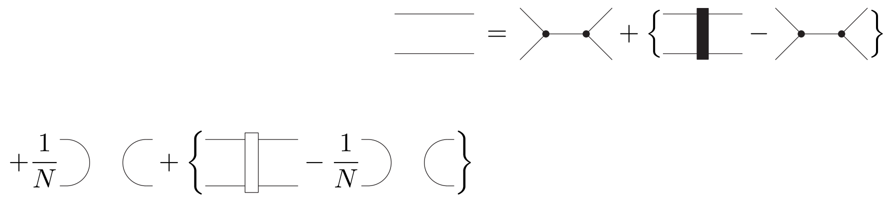
 $= ?,$

the graph that launched this whole odyssey?

A loop with four structure constants is reduced by reducing the $A \otimes A \to A \otimes A$ space. By the Jacobi relation there are only two linearly independent tree invariants in A^4 constructed from the cubic invariant:

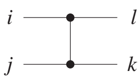
 and 

 induces a decomposition of $\wedge^2 A$ antisymmetric tensors:



$$\mathbf{1} = \mathbf{P}_{\square} + \mathbf{P}_{\square\atop\square} + \mathbf{P}_{\bullet} + \mathbf{P}_{s} \, .\nu \qquad (4.4)$$

The $A \otimes A \to A \otimes A$ matrix

$\mathbf{Q}_{ij,kl} =$ 

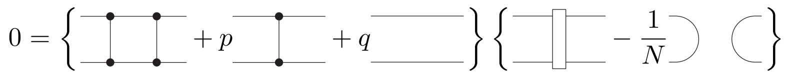
.

can decompose only the symmetric subspace $\mathrm{Sym}^2 A$.

What next? The key is the *primitiveness assumption*: any invariant tensor is a linear sum over the tree invariants constructed from the quadratic and the cubic invariants, *i.e. no quartic* primitive invariant exists in the adjoint rep.

4.2 Primitiveness assumption

By the primitiveness assumption, the 4-index loop invariant $\mathbf{Q}^2$ is expressible in terms of $\mathbf{Q}_{ij,k\ell}$, $C_{ijm}C_{mk\ell}$ and δ_{ij}, hence on the traceless symmetric subspace



$$0 = (\mathbf{Q}^2 + p\mathbf{Q} + q\mathbf{1})\mathbf{P}_s\,.$$

The assumption that there exists no primitive quartic invariant is the *defining relation* for the E_8 family.

Coefficients p, q follow from symmetry and the Jacobi relation, yielding the characteristic equation for $\mathbf{Q}$

$$\left(\mathbf{Q}^2 - \frac{1}{6}\mathbf{Q} - \frac{5}{3(N+2)}\mathbf{1}\right)\mathbf{P}_s = (\mathbf{Q} - \lambda\mathbf{1})(\mathbf{Q} - \lambda^*\mathbf{1})\mathbf{P}_s = 0\,.$$

Rewrite the condition on an eigenvalue of $\mathbf{Q}$,

$$\lambda^2 - \frac{1}{6}\lambda - \frac{5}{3(N+2)} = 0\,,$$

as formula for N:

$$N + 2 = \frac{5}{3\lambda(\lambda - 1/6)} = 60\left(\frac{6 - \lambda^{-1}}{6} - 2 + \frac{6}{6 - \lambda^{-1}}\right)\,.$$

As we shall seek for values of λ such that the adjoint rep dimension N is an integer, it is convenient to re-parametrize the two eigenvalues as

$$\lambda = \frac{1}{6}\frac{1}{1 - m/6} = -\frac{1}{m-6}\,, \qquad \lambda^* = \frac{1}{6}\frac{1}{1 - 6/m} = \frac{1}{6}\frac{m}{m-6}\,.$$

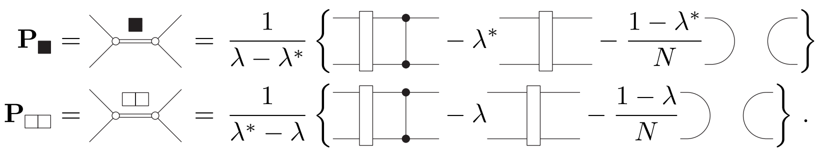
In terms of the parameter m, the dimension of the adjoint representation is given by

$$N = -122 + 10m + 360/m\,. \tag{4.5}$$

As N is an integer, allowed m are rationals $m = P/Q$, P and Q relative primes. It turns out that we need to check only a handful of rationals $m > 6$.

4.3 Further Diophantine conditions

The associated projection operators:



reduce the $A\otimes A$ space into irreps of dimensions:

$$\begin{aligned} d_{\square\square} = \operatorname{tr}\mathbf{P}_{\square\square} &= \frac{(N+2)(1/\lambda+N-1)}{2(1-\lambda^*/\lambda)} \\ &= \frac{5(m-6)^2(5m-36)(2m-9)}{m(m+6)}\,, \end{aligned} \tag{4.6}$$

$$d_{\blacksquare} = \frac{270(m-6)^2(m-5)(m-8)}{m^2(m+6)}\,. \tag{4.7}$$

To summarize: $A\otimes A$ decomposes into 5 irreducible reps

$$\mathbf{1} = \mathbf{P}_{\square} + \mathbf{P}_{\substack{\square\\ \square}} + \mathbf{P}_{\bullet} + \mathbf{P}_{\square\square} + \mathbf{P}_{\blacksquare}\,.$$

The decomposition is parametrized by a rational m and is possible only if dimensions N and $d_{\square\square}$ are integers. From the decomposition of the $\mathrm{Sym}^3 A$ if follows, by the same line of reasoning, that there is a rep of dimension

$$d_{\textcolor{gray}{\blacksquare}} = \frac{5(m-5)(m-8)(m-6)^2(2\,m-15)(5\,m-36)}{m^3(3+m)(6+m)}\,(36-m)\,. \tag{4.8}$$

Table 1. All solutions of the Diophantine conditions (4.5), (4.6), (4.7) and (4.8): the $m=30$ solution still survives this set of conditions.

m	5	8	9	10	12	15	18	24	30	36
N	0	3	8	14	28	52	78	133	190	248
d_5	0	0	1	7	56	273	650	1,463	1,520	0
$d_{\textcolor{gray}{\blacksquare}}$	0	-3	0	64	700	4,096	11,648	40,755	87,040	147,250
$d_{\square\blacksquare}$	0	0	27	189	1,701	10,829	34,749	152,152	392,445	779,247

Our homework problem is done: the reduction of the adjoint rep 4-vertex loops for *all* exceptional Lie groups. The main result of all this heavy birdtracking is, however, much more interesting than the problem we set out to solve:

The solutions of $A\otimes A \to A\otimes A$ Diophantine conditions yield *all* exceptional Lie algebras, see table 1. $N > 248$ is excluded by the positivity of $d_{\textcolor{gray}{\blacksquare}}$, $N = 248$ is special, as $\mathbf{P}_{\textcolor{gray}{\blacksquare}} = 0$ implies existence of a tensorial identity on the $\mathrm{Sym}^3 A$ subspace. I eliminate *(somewhat indirectly)* the $m = 30$ case by the semi-simplicity condition; Landsberg and Manivel [Landsberg and Manivel 2002c] identify the $m = 30$ solution as a non-reductive Lie algebra.

5 Exceptional magic

After "some algebra" F_4 and E_7 families emerge in a similar fashion. A closer scrutiny of the solutions to all $V \otimes \bar{V} \to V \otimes \bar{V}$ Diophantine conditions appropriately re-parametrized

m	8	9	10	12	15	18	20	24	30	36	40	$\cdots$	360
F_4			0	0	3	8	.	21	.	52	.	$\cdots$	.
E_6		0	0	2	8	16	.	35	36	78	.	$\cdots$	.
E_7	0	1	3	9	21	35	.	66	99	133	.	$\cdots$	.
E_8	3	8	14	28	52	78	.	133	190	248	.	$\cdots$	.

leads to a surprise: all of them are the one and the same condition

$$N = \frac{(\ell - 6)(m - 6)}{3} - 72 + \frac{360}{\ell} + \frac{360}{m}$$

which magically arranges all exceptional families into the *Magic Triangle.* The triangle is called "magic", because it contains the Magic Square [Vinberg 1994; Freudenthal 1964a].

						0 / 0	3 $\mathbf{A}_1$ 2
					0 / 0	1 $\mathbf{U}(1)$ 1	8 $\mathbf{A}_2$ 3
				0 / 0	0 / 1	3 $\mathbf{A}_1$ 3	14 $\mathbf{G}_2$ 7
			0 / 0	0 / 1	2 $2\mathbf{U}(1)$ 2	9 $3\mathbf{A}_1$ 4	28 $\mathbf{D}_4$ 8
		0 / 0	0 / 2	3 $\mathbf{A}_1$ 5	8 $\mathbf{A}_2$ 8	21 $\mathbf{C}_3$ 14	52 $\mathbf{F}_4$ 26
	0 / 0	0 / 1	2 $2\mathbf{U}(1)$ 3	8 $\mathbf{A}_2$ 6	16 $2\mathbf{A}_2$ 9	35 $\mathbf{A}_5$ 15	78 $\mathbf{E}_6$ 27
0 / 0	1 $\mathbf{U}(1)$ 2	3 $\mathbf{A}_1$ 4	9 $3\mathbf{A}_1$ 8	21 $\mathbf{C}_3$ 14	35 $\mathbf{A}_5$ 20	66 $\mathbf{D}_6$ 32	133 $\mathbf{E}_7$ 56
3 $\mathbf{A}_1$ 3	8 $\mathbf{A}_2$ 8	14 $\mathbf{G}_2$ 14	28 $\mathbf{D}_4$ 28	52 $\mathbf{F}_4$ 52	78 $\mathbf{E}_6$ 78	133 $\mathbf{E}_7$ 133	248 $\mathbf{E}_8$ 248

Fig. 1. All solutions of the Diophantine conditions place the defining and adjoint reps exceptional Lie groups into a triangular array. Within each entry: the number in the upper left corner is N, the dimension of the corresponding Lie algebra, and the number in the lower left corner is n, the dimension of the defining rep. The expressions for n for the top four rows are guesses.

5.1 A brief history of exceptional magic

There are many different strands woven into “exceptional magic” described only in small part in this monograph. I will try to summarize few of the steps along the way, the ones that seem important to me – with apologies to anyone whose work I have overseen.

1894: in his thesis Cartan [Cartan 1914] identifies G_2 as the group of octonion isomorphisms, and notes that E_7 has a skew-symmetric quadratic and a symmetric quartic invariant.
1907: Dickinson characterizes E_6 as a 27-dimensional group with a cubic invariant.
1934: Jordan, von Neumann and Wigner [Jordan et al. 1934] introduce octonions and Jordan algebras into physics, in a failed attempt at formulating a new quantum mechanics which would explain the neutron, discovered in 1932.
1954–66: First noted by Rosenfeld [Rosenfeld 1956], the *Magic Square* was rediscovered by Freudenthal, and made rigorous by Freudenthal and Tits [Freudenthal 1954 ; Tits 1966]. A mathematician’s history of the octonion underpinning of exceptional Lie groups is given in a delightful review by Freudenthal [Freudenthal 1964b].
1976: Gürsey and collaborators [Gürsey and Sikivie 1976] take up octonionic formulations in a failed attempt of formulating a quantum mechanics of quark confinement.
1975–77: *Primitive invariants* construction of all semi-simple Lie algebras [Cvitanović 1976] and the Magic Triangle [Cvitanović 1977b], except for the E_8 family.
1979: E_8 *family* primitiveness assumption (*no quartic primitive* invariant), inspired by *Okubo*’s observation [Okubo 1979] that the quartic Dynkin index vanishes for the exceptional Lie algebras.
1979: E_7 *symmetry in extended supergravities* discovered by Cremmer and Julia [Cremmer and Julia 1979].
1981: *Magic Triangle*, the E_7 family and its $SO(4)$-family of “negative dimensional” relatives published [Cvitanović 1981a]. The total number of citations in the next 20 years: *3 (three).*
1981: *Magic Triangle in extended supergravities* constructed by Julia [Julia 1981]. Appears unrelated to the Magic Triangle described here.
1987–2001: *Angelopoulos* [Angelopoulos 2001] classifies Lie algebras by the spectrum of the Casimir operator acting on $A\otimes A$, and, *inter alia*, obtains the same E_8 family.
1995 : *Vogel* [Vogel 1995] notes that for the exceptional groups the dimensions and casimirs of the $A\otimes A$ adjoint rep tensor product decomposition $\mathbf{P}_{\square}+\mathbf{P}_{\substack{\square\\\square}}+\mathbf{P}_{\bullet}+\mathbf{P}_{\square\square}+\mathbf{P}_{\blacksquare}$ are rational functions of the quadratic Casimir a (related to my parameter m by $a = 1/m - 6$).
1996: *Deligne* [Deligne 1996] conjectures that for A_1, A_2, G_2, F_4, E_6, E_7 and E_8 the dimensions of higher tensor reps $\otimes A^k$ could likewise be expressed as rational functions of parameter a.

1996: *Cohen and de Man* [Cohen and de Man 1996] verify by computer algebra the Deligne conjecture for all reps up to $\otimes A^4$. They note that "*miraculously* for all these rational functions both numerator and denominator factor in $Q[a]$ as a product of linear factors". This is immediate in the derivation outlined above.
1999: *Cohen* and *de Man* [Cohen and de Man 1999] derive the projection operators and dimension formulas of sect. 4 for the E_8 family by the same birdtrack computations (they cite [Cvitanović 2004], not noticing that the calculation was already in the current draft of the webbook).
2001–2003: *J. M. Landsberg and L. Manivel* [Landsberg and Manivel 2002c; Landsberg and Manivel 2001; Landsberg and Manivel 2002b; Landsberg and Manivel 2002a] utilize projective geometry and triality to interpret the Magic Triangle, recover the known dimension and decomposition formulas, and derive an infinity of higher-dimensional rep formulas.
2002: *Deligne* and *Gross* [Deligne and Gross 2002] derive the Magic Triangle by a method different from the derivation outlined here.

6 Epilogue

"Why did you do this?" you might well ask.

Here is an answer.

It has to do with a conjecture of finiteness of gauge theories, which, by its own twisted logic, led to this sidetrack, birdtracks and exceptional magic:

> *If gauge invariance of QED guarantees that all UV and IR divergences cancel, why not also the finite parts?*

And indeed; when electron magnetic moment diagrams are grouped into gauge invariant subsets, a rather surprising thing happens [Cvitanović 1977a]; while the finite part of each Feynman diagram is of order of 10 to 100, every subset computed so far adds up to approximately

$$\pm\frac{1}{2}\left(\frac{\alpha}{\pi}\right)^n .$$

If you take this numerical observation seriously, the "zeroth" order approximation to the electron magnetic moment is given by

$$\frac{1}{2}(g-2) = \frac{1}{2}\frac{\alpha}{\pi}\frac{1}{\left(1-\left(\frac{\alpha}{\pi}\right)^2\right)^2} + \text{“corrections”}.$$

Now, this is a great heresy - my colleagues will tell you that Dyson [Dyson 1952] has shown that the perturbation expansion is an asymptotic series, in the sense that the nth order contribution should be exploding combinatorially

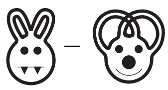
$$\frac{1}{2}(g-2) \approx \cdots + n^n \left(\frac{\alpha}{\pi}\right)^n + \cdots ,$$

and not growing slowly like my estimate

$$\frac{1}{2}(g-2) \approx \cdots + n \left(\frac{\alpha}{\pi}\right)^n + \cdots .$$

I kept looking for a simpler gauge theory in which I could compute many orders in perturbation theory and check the conjecture. We learned how to count Feynman diagrams. I formulated a planar field theory whose perturbation expansion is convergent [Cvitanović 1981b]. I learned how to compute the group weights of Feynman diagrams in non-Abelian gauge theories [Cvitanović 1976]. By marrying Poincaré to Feynman we found a new perturbative expansion more compact than the standard Feynman diagram expansions [Cvitanović et al. 1999]. No dice. To this day I still do not know how to prove or disprove the conjecture.

QCD quarks are supposed to come in three colors. This requires evaluation of SU(3) group theoretic factors, something anyone can do. In the spirit of Teutonic completeness, I wanted to check all possible cases; what would happen if the nucleon consisted of 4 quarks, doodling

 $= n(n^2-1)$,

and so on, and so forth. In no time, and totally unexpectedly, all exceptional Lie groups arose, not from conditions on Cartan lattices, but on the same geometrical footing as the classical invariance groups of quadratic norms, $SO(n)$, $SU(n)$ and $Sp(n)$.

6.1 Magic ahead

For many years nobody, truly nobody, showed a glimmer of interest in the exceptional Lie algebra parts of my construction, so there was no pressure to publish it before completing it:

By completing it I mean finding the algorithms that would reduce any bubble diagram to a number, for any semi-simple Lie algebra. The task is accomplished for G_2, but for F_4, E_6, E_7 and E_8 it is still an open problem. This, perhaps, is only matter of algebra (all of my computations were done by hand, mostly on trains and in airports), but the truly frustrating unanswered question is:

Where does the Magic Triangle come from? Why is it symmetric across the diagonal? Some of the other approaches explain the symmetry, but my derivation misses it. Most likely the starting idea - to classify all simple Lie groups from the primitiveness assumption - is flawed. Is there a mother of all Lie algebras, some analytic function (just as the *Gamma* function extends combinatorics on n objects into complex plane) which yields the Magic Triangle for a set of integer parameter values?

And then there is a practical issue of unorthodox notation: transferring birdtracks from hand drawings to LaTeX took another 21 years. In this I was rescued by Anders Johansen who undertook drawing some 4,000 birdtracks needed to complete [Cvitanović 2004], of elegance far outstripping that of the old masters.

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