Peeling of random planar maps*

Lecture notes for a mini-course given at the
Mini-School on Random Maps and the Gaussian Free Field
at École normale supérieure de Lyon

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14th May 2017

1 Planar maps and the peeling process

1.1 Planar maps

A multigraph is a graph in which we allow edges to form loops and pairs of vertices may be connected by multiple edges. We say a multigraph is properly embedded in the sphere (or plane) if it is drawn in such a way that the vertices are all disjoint and the edges intersect only at vertices.

Definition 1 (Planar map). *A finite planar map is a finite connected planar multigraph which is properly embedded in the sphere viewed up to orientation-preserving homeomorphisms.*

We'll always assume that our planar maps are *rooted*, meaning that an oriented edge is distinguished as the *root edge* of the map. The vertex at the origin of the root edge will be called the *root vertex*, and the face to the right of the root edge the *root face*. If $\mathfrak{m}$ is a planar map, we denote by $\mathsf{Vertices}(\mathfrak{m})$, $\mathsf{Edges}(\mathfrak{m})$ and $\mathsf{Faces}(\mathfrak{m})$ respectively the vertices, edges, and faces of the map. For a vertex v (respectively face f) we let $\deg(v)$ (respectively $\deg(f)$) be its degree, i.e. the number of incident edges with the convention that an edge that is incident at both ends (respectively sides) is counted twice. The *perimeter* of a planar map is the degree of its root face.

In these lectures we will exclusively deal with *bipartite* planar maps (although we comment on the non-bipartite case in Remark 4), which is equivalent to demanding that all faces have even degree.

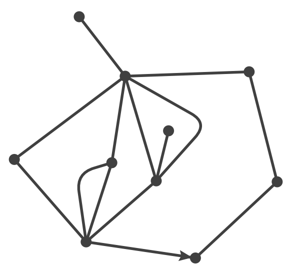


Figure 1: A bipartite rooted planar map with face degrees 2, 4, 4, 4, 6, 8 and perimeter 8.

*These lecture notes are mainly based on the papers [2] and [3]. For more details and proofs we advise the reader to consult the "cours Peccot" lecture notes by Curien [6].

1.2 Peeling process

A *planar map with holes* is a (rooted) planar map $\mathfrak{e}$ with a distinguished set of faces, that we call the *holes* of $\mathfrak{e}$, satisfying the following properties. The holes are required to be disjoint and simple, meaning that no two corners of any of the holes share a vertex. Moreover, the root face is not allowed to be a hole.

One should think of the holes as unexplored regions in a planar map. Exploration of a hole h of $\mathfrak{e}$ ("explored region") then corresponds to gluing inside h a planar map $\mathfrak{u}$ ("unexplored region") with perimeter equal to $\deg(h)$, see figure 2. This operation is well-defined for any such planar map $\mathfrak{u}$ because the hole is assumed to be simple. Moreover, if we specify by some (arbitrary but fixed) deterministic algorithm a distinguished edge on the boundary of the hole h to which the root edge of $\mathfrak{u}$ is to be glued, then the operation is *rigid* in the sense that different choices of $\mathfrak{u}$ lead to different results.

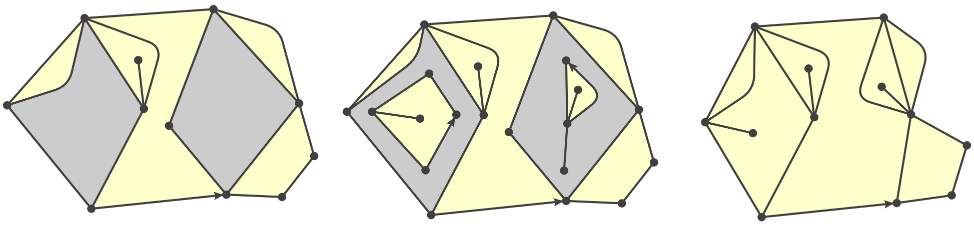


Figure 2: From left to right: a planar map $\mathfrak{e}$ with two holes (darker shaded faces); two planar maps with perimeters matching the degrees of the holes of $\mathfrak{e}$; the result of gluing. The map on the left is a submap of the one on the right.

Definition 2 (Submap). *Given a planar map* $\mathfrak{m}$ *and a planar map* $\mathfrak{e}$ *with holes* $h_1, \ldots, h_k$ *we say that* $\mathfrak{e}$ *is a* submap *of* $\mathfrak{m}$*, denoted* $\mathfrak{e} \subset \mathfrak{m}$*, iff there exist planar maps* $\mathfrak{u}_1, \ldots, \mathfrak{u}_k$ *such that* $\mathfrak{m}$ *is obtained from* $\mathfrak{e}$ *by gluing* $\mathfrak{u}_i$ *into hole* h_i*.*

More generally, by allowing the maps $\mathfrak{u}_i$ to have holes we can make sense of $\mathfrak{e} \subset \mathfrak{e}'$ being a submap of another map $\mathfrak{e}'$ with holes. The rigidity of the gluing operation implies that if $\mathfrak{e} \subset \mathfrak{m}$ then the maps $\mathfrak{u}_1, \ldots, \mathfrak{u}_k$ are uniquely defined.

For a submap $\mathfrak{e} \subset \mathfrak{m}$ we denote by $\mathsf{Active}(\mathfrak{e}) \subset \mathsf{Edges}(\mathfrak{e})$ the set of ("active") edges incident to a hole. Given an active edge $e \in \mathsf{Active}(\mathfrak{e})$, we are going to define another map $\mathsf{Peel}(\mathfrak{e}, e, \mathfrak{m})$ with holes such that $\mathfrak{e} \subset \mathsf{Peel}(\mathfrak{e}, e, \mathfrak{m}) \subset \mathfrak{m}$, and which we say is obtained from $\mathfrak{e}$ by *peeling the edge* e. To this end let f be the unique face of $\mathfrak{m}$ that is incident to e (in $\mathfrak{m}$) but does not correspond to a face of $\mathfrak{e}$ that is already incident to e (in $\mathfrak{e}$). We distinguish to types of events:

- *Event* C_k: the face f is not a face of $\mathfrak{e}$ and its degree is $2k$. Then $\mathsf{Peel}(\mathfrak{e}, e, \mathfrak{m})$ is obtained from $\mathfrak{e}$ by gluing the face f to e inside the hole (without further identifications of the edges of f).

- *Event* G_{k_1,k_2}: the face f is already present in $\mathfrak{e}$. In this case e is glued to another edge e' incident to the same hole, namely the one incident (in $\mathfrak{e}$) to f that in $\mathfrak{m}$ was identified with e. The number of edges strictly in between e and e' when following the boundary while keeping the hole on the right (respectively left) is $2k_1$ (respectively $2k_2$).

In the event G_{k_1,k_2} with $k_1, k_2 > 0$ the hole of $\mathfrak{e}$ incident to e is divided into two holes of degrees $2k_1$ and $2k_2$.

Remark 1. *We call an edge or a vertex* inner *if it is not incident to a hole. It is easily verified that in each case the number of inner edges increases by one, while the number of inner vertices increases only with the event* G_{k_1,k_2} *when* $k_1 = 0$ *or* $k_2 = 0$ *(by one or two).*

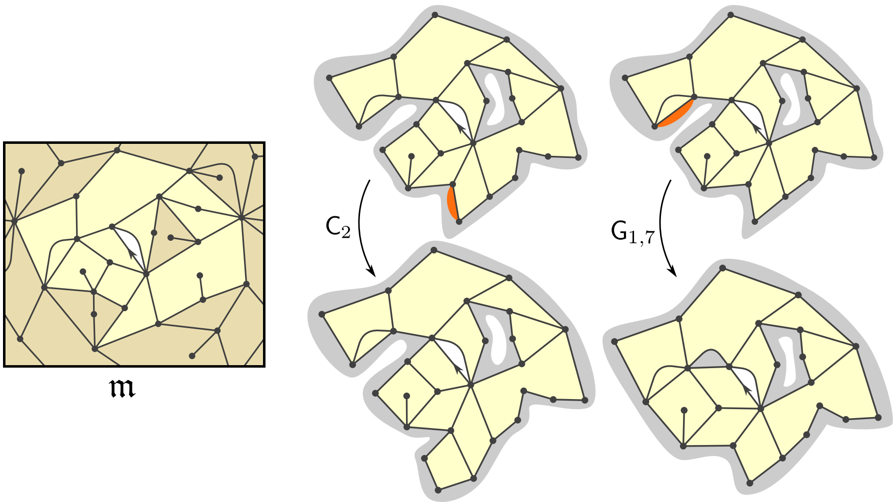


Figure 3: On the left a portion of a planar map $\mathfrak{m}$. On the right two events are shown resulting from peeling the active edge indicated in orange (notice that here the outer face is a hole too).

By iterating this peeling operation we may explore the planar map $\mathfrak{m}$. To define such an exploration uniquely, suppose a *peeling algorithm* $\mathcal{A}$ is given that associates to any planar map $\mathfrak{e}$ with holes an element $\mathcal{A}(\mathfrak{e}) \in \mathrm{Active}(\mathfrak{e}) \cup \{\dagger\}$, where $\dagger$ is a cemetery point, which we interpret as the request to stop the exploration.

Setting $\mathrm{Peel}(\mathfrak{e}, \dagger, \mathfrak{m}) = \mathfrak{e}$ we are lead to defining

Definition 3 (Peeling exploration)**.** *For a planar map* $\mathfrak{m}$*, the* peeling exploration *of* $\mathfrak{m}$ *with algorithm* $\mathcal{A}$ *is the infinite sequence*

$$\mathfrak{e}_0 \subset \mathfrak{e}_1 \subset \cdots \subset \mathfrak{m},$$

where $\mathfrak{e}_{i+1} = \mathrm{Peel}(\mathfrak{e}_i, \mathcal{A}(\mathfrak{e}_i), \mathfrak{m})$ *for* $i \geq 0$*, and* $\mathfrak{e}_0$ *is the unique two-face map with a hole and root face of degree equal to the perimeter of* $\mathfrak{m}$*.*

Although we define the peeling exploration as an infinite sequence, it is easily seen that it stabilizes after a finite number of steps. Indeed, if the peeling algorithm never selects $\dagger$ when $\mathrm{Active}(\mathfrak{e})$ is non-empty then the number of steps n before the map is fully explored, i.e. $\mathfrak{e}_n = \mathfrak{m}$, is by Remark 1 exactly the number $n = |\mathrm{Edges}(\mathfrak{m})|$ of edges of $\mathfrak{m}$.

1.3 Targeted peeling process

For many applications it is useful to consider a peeling exploration of a planar map with some kind of target $\star$. In fact, in these lectures we will encounter three kinds of targets: a marked vertex ($\star = \bullet$), a marked face ($\star = \square$), or a "point at infinity" ($\star = \infty$). The latter case applies to infinite planar maps (of which we will see a formal definition later) which we require to have faces of finite degree and to be *one-ended*, meaning that the complement of any finite submap has exactly one infinite connected component.

Let $\mathfrak{m}_\star$ be such a planar map with a target and $\mathfrak{e} \subset \mathfrak{m}_\star$ a submap. Then we may naturally associate to $\mathfrak{e}$ the *filled-in* submap $\mathrm{Fill}(\mathfrak{e}, \mathfrak{m}_\star)$ in the following way. Recall that the maps $\mathfrak{u}_i$ to be glued into the holes h_i to obtain $\mathfrak{m}_\star$ from $\mathfrak{e}$ are well-defined by rigidity. For the types of targets we consider, at most one of the maps $\mathfrak{u}_i$ contains the target. We define $\mathrm{Fill}(\mathfrak{e}, \mathfrak{m}_\star)$ to be given by gluing $\mathfrak{u}_i$ into h_i for each of the maps $\mathfrak{u}_i$ not containing the target. The result is a planar map which either has a single hole or contains the target and equals $\mathfrak{m}_\star$.

Definition 4 (Filled-in peeling exploration)**.** *For a planar map* $\mathfrak{m}_\star$ *with a target* $\star \in \{\bullet, \square, \infty\}$*, the* peeling exploration *of* $\mathfrak{m}_\star$ *with algorithm* $\mathcal{A}$ *is the infinite sequence*

$$\mathfrak{e}_0 \subset \mathfrak{e}_1 \subset \cdots \subset \mathfrak{m}_\star,$$

where $\mathfrak{e}_{i+1} = \mathsf{Fill}(\mathsf{Peel}(\mathfrak{e}_i, \mathcal{A}(\mathfrak{e}_i), \mathfrak{m}_\star), \mathfrak{m}_\star)$ *for* $i \geq 0$*, and* $\mathfrak{e}_0$ *is the unique two-face map with a hole and root face of degree equal to the perimeter of* $\mathfrak{m}$*.*

Let us consider the i'th peeling step $\mathfrak{e}_{i+1} = \mathsf{Fill}(\mathsf{Peel}(\mathfrak{e}_i, \mathcal{A}(\mathfrak{e}_i), \mathfrak{m}_\star), \mathfrak{m}_\star)$ assuming that $\mathcal{A}(\mathfrak{e}_i) \neq \dagger$. Then we can identify (with a slight abuse of notation) the following events:

- *Event* C_k: corresponding to the discovery of a (unmarked) face of degree $2k$; no filling-in occurs.
- *Event* M_k (only when $\star = \square$): when the marked face is explored, then the remaining hole of degree $2k$ is filled in.
- *Event* $\mathsf{G}_{k_1,*}$: peeling leads to event G_{k_1,k_2} and the hole of degree $2k_1$ is filled in.
- *Event* G_{*,k_2}: peeling leads to event G_{k_1,k_2} and the hole of degree $2k_2$ is filled in.

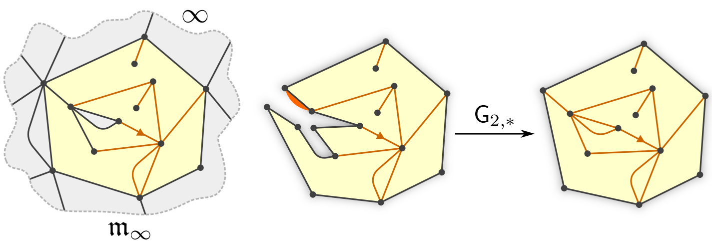


Figure 4: On the left a portion of an infinite planar map $\mathfrak{m}_\infty$. On the right an event is shown resulting from peeling the active edge indicated in orange.

A natural process associated to the filled-in peeling exploration is the *perimeter process* (P_i) obtained by setting P_i equal to the half-degree of the unique hole of $\mathfrak{e}_i$ as long as $\mathfrak{e}_i \neq \mathfrak{m}_\star$. If $\mathfrak{e}_i = \mathfrak{m}_\star$, i.e. the exploration has finished, we will set by convention $P_i = 0$ in the case of a marked vertex and $P_i = -k$ in the case of a marked face of degree $2k$. We then find

$$P_{i+1} - P_i = \begin{cases} k - 1 & \text{on the event } \mathsf{C}_k \ (k \geq 1) \\ -k - 1 & \text{on the events } \mathsf{G}_{k,*}, \mathsf{G}_{*,k}, \mathsf{M}_k \ (k \geq 0). \end{cases}$$

In particular, the exploration stops precisely when (P_i) takes its first non-positive value.

2 The perimeter process of a Boltzmann map

2.1 Boltzmann maps

We wish to apply the peeling exploration to a broad class of (bipartite) planar maps known as the Boltzmann planar maps. Let $\mathbf{q} = (q_i)_{i\geq 1}$ be a sequence of non-negative weights. We use it to put a measure $w_{\mathbf{q}}$ on the set $\mathcal{M}$ of (finite, bipartite) planar maps by setting for $\mathfrak{m} \in \mathcal{M}$

$$w_{\mathbf{q}}(\mathfrak{m}) = \prod_{f \in \mathsf{Faces}(\mathfrak{m}) \setminus \{f_r\}} q_{\deg(f)/2},$$

where f_r is the root face of $\mathfrak{m}$. For $\ell \geq 1$ we let $\mathcal{M}^{(\ell)} \subset \mathcal{M}$ be the set of all maps with perimeter 2ℓ, and $\mathcal{M}_\bullet^{(\ell)}$ the set of such maps with a marked vertex (also called *pointed* maps). Then we may introduce the *disk functions*

$$W^{(\ell)}(\mathbf{q}) := w_\mathbf{q}(\mathcal{M}^{(\ell)}) \qquad \text{and} \qquad W_\bullet^{(\ell)}(\mathbf{q}) := w_\mathbf{q}(\mathcal{M}_\bullet^{(\ell)}).$$

For convenience we take $\mathcal{M}^{(0)}$ and $\mathcal{M}_\bullet^{(0)}$ to contain a single degenerate planar map consisting of a single vertex and no edges or faces, such that $W^{(0)}(\mathbf{q}) = W_\bullet^{(0)}(\mathbf{q}) = 1$. We will simply write $W^{(\ell)}$ and $W_\bullet^{(\ell)}$ when the sequence $\mathbf{q}$ is clear from the context.

For the purposes of these lectures we will use the following definition of admissibility:

Definition 5 (Admissibility). *The weight sequence* $\mathbf{q}$ *is said to be* admissible *if* $W_\bullet^{(\ell)}(\mathbf{q}) < \infty$ *for all* $\ell \geq 1$ *and* $W_\bullet^{(\ell+1)}(\mathbf{q})/W_\bullet^{(\ell)}(\mathbf{q})$ *converges as* $\ell \to \infty$.

Since $W^{(\ell)}(\mathbf{q}) < W_\bullet^{(\ell)}(\mathbf{q})$ admissibility implies also that $W^{(\ell)}(\mathbf{q}) < \infty$, and therefore we may normalize $w_\mathbf{q}$ into probability measures on $\mathcal{M}^{(\ell)}$ and $\mathcal{M}_\bullet^{(\ell)}$, called the (unmarked and marked) *Boltzmann planar maps* of perimeter 2ℓ. We shall denote the corresponding distributions by $\mathbb{P}_\mathbf{q}^{(\ell)}$ and $\mathbb{P}_{\bullet,\mathbf{q}}^{(\ell)}$, i.e.

$$\mathbb{P}_\mathbf{q}^{(\ell)}(\mathfrak{m}) = \frac{w_\mathbf{q}(\mathfrak{m})}{W^{(\ell)}(\mathbf{q})} \qquad \text{and} \qquad \mathbb{P}_{\bullet,\mathbf{q}}^{(\ell)}(\mathfrak{m}_\bullet) = \frac{w_\mathbf{q}(\mathfrak{m}_\bullet)}{W_\bullet^{(\ell)}(\mathbf{q})}. \tag{1}$$

Remark 2. *It can be shown that the admissibility condition of Definition 5 is equivalent to* $W_\bullet^{(\ell)} < \infty$ *for some* $\ell \geq 1$*, and one may even replace* $W_\bullet^{(\ell)} < \infty$ *by* $W^{(\ell)} < \infty$*. We will however not prove this here. See e.g. [8] or [6].*

2.2 The law of the perimeter process

Let us fix an admissible weight sequence $\mathbf{q}$ and a peeling algorithm $\mathcal{A}$ that never selects † unless the full map is explored. We are going to consider the filled-in peeling exploration of the Boltzmann map $\mathfrak{m}_\bullet$ with a marked vertex and perimeter $2\ell \geq 2$.

Proposition 1. *Under* $\mathbb{P}_\bullet^{(\ell)}$ *the filled-in peeling exploration* $(\mathfrak{e}_i)_{i\geq 0}$ *of* $\mathfrak{m}_\bullet$ *is a Markov chain. Conditionally on the half-degree* p *of the hole of* $\mathfrak{e}_i$ *the probability of event* C_k *with* $k \geq 1$ *occurring at the* i*'th peeling step is*

$$\frac{W_\bullet^{(p+k-1)}}{W_\bullet^{(p)}} q_k$$

while the events $\mathrm{G}_{k,*}$ *and* $\mathrm{G}_{*,k}$ *with* $k \geq 0$ *occur each with probability*

$$\frac{W_\bullet^{(p-k-1)}}{W_\bullet^{(p)}} W^{(k)}.$$

Conditionally on one of the latter events with $k \geq 1$ *the map used to fill in the hole is a (unmarked) Boltzmann map of perimeter* $2k$ *independent of* $\mathfrak{e}_i$*.*

Proof. Let us look at the first step $\mathfrak{e}_0 \to \mathfrak{e}_1$ of the peeling exploration. Using rigidity the probability of event C_k for $k \geq 1$ is easily seen to be given by

$$\mathbb{P}_\bullet^{(\ell)}(\mathrm{C}_k) = \frac{q_k \sum_{\mathfrak{m}'_\bullet \in \mathcal{M}_\bullet^{(\ell+k-1)}} w_\mathbf{q}(\mathfrak{m}'_\bullet)}{W_\bullet^{(\ell)}} = \frac{q_k W_\bullet^{(\ell+k-1)}}{W_\bullet^{(\ell)}}. \tag{2}$$

Similarly the probability of event $\mathrm{G}_{*,k}$ (respectively $\mathrm{G}_{k,*}$) for $0 \leq k < p-1$ is

$$\mathbb{P}_\bullet^{(\ell)}(\mathrm{G}_{*,k}) = \frac{\sum_{\mathfrak{m}' \in \mathcal{M}^{(k)}} w_\mathbf{q}(\mathfrak{m}') \sum_{\mathfrak{m}'_\bullet \in \mathcal{M}_\bullet^{(\ell-k-1)}} w_\mathbf{q}(\mathfrak{m}'_\bullet)}{W_\bullet^{(\ell)}} = \frac{W^{(k)} W_\bullet^{(\ell-k-1)}}{W_\bullet^{(\ell)}}, \tag{3}$$

which shows that conditionally on $G_{*,k}$ (respectively $G_{k,*}$) the map $\mathfrak{m}'$ filling in the hole of degree $2k$ is an independent Boltzmann planar map of perimeter $2k$. This shows the claimed properties for the first step. Moreover, it follows from (2) (respectively (3)) that conditionally on the event C_k (respectively $\mathsf{G}_{*,k}$ or $\mathsf{G}_{k,*}$) the unexplored part after the first peeling step is distributed as $\mathfrak{m}_\bullet$ under $\mathbb{P}_\bullet^{(\ell+k-1)}$ (respectively $\mathbb{P}_\bullet^{(\ell-k-1)}$). Hence the general result follows from iterating the calculation. □

In particular we find that the perimeter process (P_i) is a Markov process with transition probabilities

$$\mathbb{P}_\bullet^{(\ell)}(P_{i+1} = p + k | P_i = p > 0) = \frac{W_\bullet^{(p+k)}}{W_\bullet^p} \cdot \begin{cases} q_{k+1} & \text{for } 0 \le k \\ 2W^{(-k-1)} & \text{for } -l \le k < 0, \end{cases} \tag{4}$$

and $P_{i+1} = 0$ whenever $P_i = 0$.

2.3 Intermezzo: Wiener-Hopf factorization

Before we continue studying the perimeter process, we need to discuss some generalities on random walks on $\mathbb{Z}$. Let ν be a probability measure on $\mathbb{Z} \cup \{\dagger\}$ with $\nu(0) < 1$, and (X_i) a sequence of i.i.d. random variables with law ν. For $\ell \in \mathbb{Z}$ we introduce a random walk (S_n) started at ℓ by setting

$$S_n := \ell + X_1 + \cdots + X_n,$$

with the convention that $S_n = \dagger$ if $X_i = \dagger$ for some $i = 1, \ldots, n$, and we denote the corresponding probability by $\mathbf{P}_\ell$. If $\nu(\mathbb{Z}) < 1$ then the random walk is *defective*, which means that at each step the walk has a probability of $\nu(\dagger)$ to be killed (i.e. sent to a cemetery state $\dagger$). If the walk is not defective, we say it is *proper*.

The *weak ascending ladder epochs* $(T_i^\geq)$ of the walk (S_n) correspond to the successive times at which the walk attains its running maximum, i.e. iteratively we set $T_0^\geq = 0$ and

$$T_{i+1}^\geq = \inf\{n > T_i^\geq : S_n \geq S_{T_i^\geq}\},$$

which is taken to be ∞ when $S_n < S_{T_i^\geq}$ or $S_n = \dagger$ for all $n > T_i^\geq$. The *weak ascending ladder process* $(H_i^\geq)$ is then given by $H_i^\geq = S_{T_i^\geq}$ provided $T_i^\geq < \infty$ and $H_i^\geq = \dagger$ otherwise. Similarly we introduce the *strict descending ladder epochs* $(T_i^<)$ by setting $T_0^< = 0$ and

$$T_{i+1}^< = \inf\{n > T_i^< : S_n < S_{T_i^<}\}.$$

The *strict descending ladder process* $(H_i^<)$ is then given by $H_i^< = -S_{T_i^<}$ provided $T_i^< < \infty$ and $H_i^< = \dagger$ otherwise. Under $\mathbf{P}_0$ both $(H_i^\geq)$ and $(H_i^<)$ are distributed as (possibly defective) random walks on $\mathbb{Z}$ started from 0 with non-negative increments.

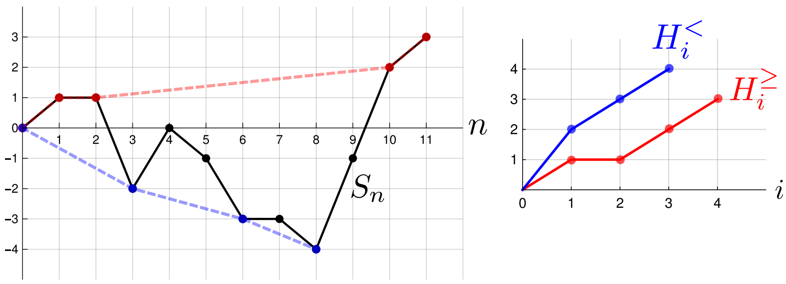


Figure 5: A walk (S_n) started from 0 with its weak ascending and strict descending ladder processes.

The walk (S_n) is said to *drift to* $\pm\infty$ respectively if $\lim_{n\to\infty} S_n = \pm\infty$ almost surely. If neither is the case then (S_n) is said to *oscillate*. It is easy to see that we have the following relations between the ladder processes and the walk (S_n):

$$\begin{aligned}
&(S_n) \text{ defective} &&\iff (H_i^{\geq}) \text{ defective}, \quad (H_i^{<}) \text{ defective}\\
&(S_n) \text{ proper and drifting to } \infty &&\iff (H_i^{\geq}) \text{ proper}, \quad (H_i^{<}) \text{ defective}\\
&(S_n) \text{ proper and drifting to } -\infty &&\iff (H_i^{\geq}) \text{ defective}, \quad (H_i^{<}) \text{ proper}\\
&(S_n) \text{ proper and oscillating} &&\iff (H_i^{\geq}) \text{ proper}, \quad (H_i^{<}) \text{ proper}
\end{aligned} \tag{5}$$

We introduce the characteristic function $\phi(\theta)$ of (S_n), and the probability generating functions $G^{\geq}(z)$ and $G^{<}(z)$ of the ladder processes by

$$\phi(\theta) := \mathbf{E}_0[e^{i\theta S_1}] = \sum_{k=-\infty}^{\infty} \nu(k)e^{ik\theta}, \qquad G^{\geq}(z) = \mathbf{E}_0[z^{H_1^{\geq}}], \qquad G^{<}(z) = \mathbf{E}_0[z^{H_1^{<}}]. \tag{6}$$

The following relation between these three function will play an important role in analyzing the perimeter process.

Proposition 2 (Wiener-Hopf factorization)**.** *The functions in (6) satisfy for $\theta \in \mathbb{R}$*

$$1 - \phi(\theta) = (1 - G^{\geq}(e^{i\theta}))(1 - G^{<}(e^{-i\theta})). \tag{7}$$

Proof. Let us denote by $\nu^{\geq}(k) := \mathbf{P}_0[H_1^{\geq} = k]$ and $\nu^{<}(k) := \mathbf{P}_0[H_1^{<} = k]$ the probability distributions of the first weak ascending and strict descending ladder heights. Let us compute $\nu^{\geq}(k)$ for $k \geq 0$ as follows. Either (S_n) performs an initial jump of size k, with probability $\nu(k)$, or it spends some time among the negative integers before jumping to $S_n = k$ at time $n \geq 2$. In the latter case, let $j \in \{2, \ldots, n-1\}$ be its last visit to $-m := \max\{S_i : 2 \leq i < n\} \leq -1$, i.e. its intermediate maximum. By decomposing the walk at j it is not hard to see that we have the relation

$$\nu^{\geq}(k) = \nu(k) + \sum_{m\in\mathbb{Z}} \nu^{<}(m)\nu^{\geq}(k+m). \qquad (k \geq 0)$$

Similarly, the probability $\nu^{<}(k)$ for $k > 0$ easily follows from decomposing the walk at its first visit to its minimum $m := \min\{S_i : 2 \leq i < n\} \geq 0$, leading to

$$\nu^{<}(k) = \nu(-k) + \sum_{m\in\mathbb{Z}} \nu^{<}(k+m)\nu^{\geq}(m). \qquad (k > 0)$$

It follows that

$$\begin{aligned}
\phi(\theta) &= \sum_{k\geq 0} \nu(k)e^{ik\theta} + \sum_{k>0} \nu(-k)e^{-ik\theta}\\
&= \sum_{k\geq 0} \nu^{\geq}(k)e^{ik\theta} + \sum_{k>0} \nu^{<}(k)e^{-ik\theta} - \sum_{\kappa,m\in\mathbb{Z}} \nu^{<}(m)\nu^{\geq}(k+m)e^{ik\theta}\\
&= G^{\geq}(e^{i\theta}) + G^{<}(e^{-i\theta}) - G^{<}(e^{-i\theta})G^{\geq}(e^{i\theta}) = 1 - (1 - G^{\geq}(e^{i\theta}))(1 - G^{<}(e^{-i\theta})).
\end{aligned}$$

□

2.4 An admissibility criterion

Let us return to the perimeter process (P_i) of a Boltzmann planar map with a marked vertex and perimeter 2ℓ, whose law for admissible $\mathbf{q}$ is given by (4). Our admissibility assumption implies that there exists a $c_\mathbf{q} > 0$ such that for all $k \in \mathbb{Z}$,

$$\lim_{p\to\infty} \frac{W_\bullet^{(p+k)}}{W_\bullet^{(p)}} = c_\mathbf{q}^k. \tag{8}$$

Actually, since $W_\bullet^{(p)}$ includes at the very least the number of rooted plane trees with p edges, of which there are $4^{p+o(1)}$, we know that $c_{\mathbf{q}} \geq 4$. The transition probabilities (4) are seen to converge as the perimeter becomes large, the limit being

$$\nu(k) = \nu_{\mathbf{q}}(k) := \lim_{p\to\infty} \mathbb{P}_\bullet^{(\ell)}(P_{i+1} = p + k | P_i = p) = c_{\mathbf{q}}^k \cdot \begin{cases} q_{k+1} & \text{for } k \geq 0 \\ 2W^{(-k-1)} & \text{for } k < 0. \end{cases} \tag{9}$$

By Fatou's lemma ν defines a possibly defective probability measure on $\mathbb{Z}$, putting us in the setting of the last section. We introduce the random walk (S_n) with law ν started at ℓ under the probability $\mathbf{P}_\ell$ accordingly.

In terms of the law ν we may rewrite the law of the perimeter process as

$$\mathbb{P}_\bullet^{(\ell)}(P_{i+1} = p + k | P_i = p) = \frac{h^\downarrow(p+k)}{h^\downarrow(p)} \nu(k) \qquad \text{for } p \geq 1, k \in \mathbb{Z}, \tag{10}$$

where we introduced the notation

$$h^\downarrow(\ell) := W_\bullet^{(\ell)} c_{\mathbf{q}}^{-\ell} \mathbf{1}_{\{\ell \geq 0\}}. \tag{11}$$

In particular $h^\downarrow$ is *ν-harmonic* on the positive integers:

$$\sum_{k\in\mathbb{Z}} h^\downarrow(\ell + k)\nu(k) = h^\downarrow(\ell) \qquad \text{for } \ell \geq 1. \tag{12}$$

According to Proposition 2 the characteristic function of S_1 admits the Wiener-Hopf factorization

$$1 - \phi(\theta) = (1 - G^{\geq}(e^{i\theta}))(1 - G^{<}(e^{-i\theta})). \tag{13}$$

We will now prove

Lemma 1. *If $\mathbf{q}$ is admissible then there exists a $\beta \in [0, 1]$ such that*

$$G^{<}(z) = 1 - \sqrt{1 - \beta z}.$$

The proof will be a simple consequence of the following probabilistic interpretation of the good old Tutte's decomposition:

Lemma 2. *If $\mathbf{q}$ is admissible then for $p \geq 1$ we have*

$$\nu(-k-1) = \frac{1}{2} \sum_{m=-\infty}^{\infty} \nu(m)\nu(-k-m-1)$$

Proof. Fix $\ell > k \geq 1$ and let $\mathfrak{m}_\bullet$ be a pointed Boltzmann planar map with perimeter 2ℓ. Let A be the event that the removal of the root edge of $\mathfrak{m}_\bullet$ leaves a pointed map to the left of the root edge with half-perimeter $\ell - k - 1$. By peeling the root edge we immediately find using Proposition 1 that

$$\mathbb{P}_\bullet^{(\ell)}(A) = \mathbb{P}_\bullet^{(\ell)}(G_{*,k}) = C\,\nu(-k-1), \qquad C := \frac{1}{2}\frac{h^\downarrow(\ell-k-1)}{h^\downarrow(\ell)}.$$

On the other hand we may choose to first peel the edge directly to the right of root edge, leading to a half-perimeter P_1, and only afterwards to peel the root edge (see figure below). Then A occurs iff an event $\mathrm{G}_{*,k'}$ is followed by $\mathrm{G}_{*,k-k'-1}$ for $0 \leq k' < k$ or $\mathrm{C}_{k'}$ is followed by $\mathrm{G}_{*,k+k'-1}$ for $k' \geq 1$. One easily finds

$$\mathbb{P}_\bullet^{(\ell)}(A \text{ and } P_1 = \ell + m) = \frac{C}{2} \begin{cases} \nu(m)\nu(-k-m-1) + \nu(-k-m-1)\nu(m) & \text{for } m \geq 0 \\ \nu(m)\nu(-k-m-1) & \text{for } -k+1 \leq m \leq -1 \\ 0 & \text{for } m \leq -k. \end{cases}$$

Summing over m gives desired result. ◻

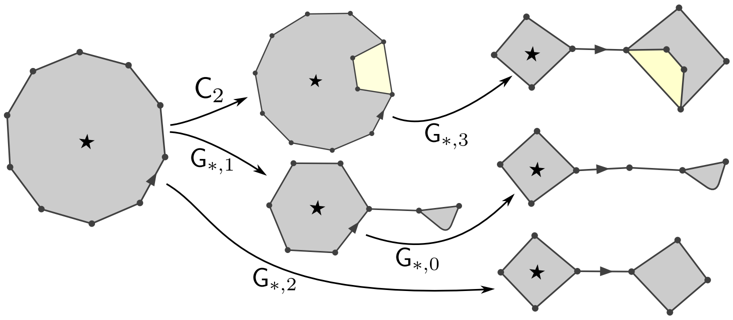


Figure 6

Proof of Lemma 1. In terms of the characteristic function $\phi(\theta)$ Lemma 2 translates into

$$[e^{ik\theta}](1-\phi(\theta))^2 = 0 \qquad \text{for } k \le -2,$$

where we use the notation $[e^{ik\theta}]f(\theta) = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)e^{-ik\theta}$. By the Wiener-Hopf factorization this implies

$$[e^{ik\theta}](1-G^{\ge}(e^{i\theta}))^2(1-G^{<}(e^{-i\theta}))^2 = 0 \qquad \text{for } k \le -2.$$

This can only occur when $(1-G^{<}(e^{-i\theta}))^2 = 1-\beta e^{-i\theta}$ for some $\beta \in [0,1]$, which implies the desired formula for $G^{<}(z)$. □

Notice that $(H_i^{<})$ is proper iff $G^{<}(1) = 1$, i.e. $\beta = 1$. If $\beta < 1$ then by (5) the walk (S_i) is either defective or it drifts so ∞. We will now see that this cannot happen.

Proposition 3. *If* $\mathbf{q}$ *is admissible then* $G^{<}(z)$ *and* $h^{\downarrow}(l)$ *are given by the universal formulas*

$$G^{<}(z) = 1-\sqrt{1-z} \qquad \text{and} \qquad h^{\downarrow}(l) = 4^{-l}\binom{2l}{l}\mathbf{1}_{\{l\ge 0\}}.$$

Proof. If $\mathbf{q}$ is admissible, the Boltzmann planar map with a marked vertex and half-perimeter $\ell \ge 1$ is a.s. finite and therefore the perimeter process (P_i) will a.s. hit zero in finite time. Hence

$$\begin{aligned} 1 &= \sum_{n=1}^{\infty} \mathbb{P}_{\bullet}^{(\ell)}(P_1 > 0, P_2 > 0, \ldots, P_{n-1} > 0, P_n = 0) \\ &= \frac{h^{\downarrow}(0)}{h^{\downarrow}(\ell)} \sum_{n=1}^{\infty} \mathbf{P}_{\ell}(S_1 > 0, S_2 > 0, \ldots, S_{n-1} > 0, S_n = 0). \end{aligned}$$

Since $h^{\downarrow}(0) = 1$, we observe that $h^{\downarrow}(\ell)$ is the probability under $\mathbf{P}_{\ell}$ that the walk (S_n) hits $\{0,1,2,\ldots\}$ at 0 (before being killed). But also

$$\begin{aligned} h^{\downarrow}(\ell) &= \mathbf{P}_0((H_i^{<}) \text{ visits } \ell \text{ before being killed}) \\ &= [z^{\ell}]\frac{1}{1-G^{<}(z)} = [z^{\ell}]\frac{1}{\sqrt{1-\beta z}} = 4^{-l}\binom{2l}{l}\beta^{l}. \end{aligned} \tag{14}$$

However (8) and (11) imply that $h^{\downarrow}(\ell+1)/h^{\downarrow}(\ell) \to 1$ as $\ell \to \infty$, and therefore we must have $\beta = 1$. □

In particular, it follows from (5) that the walk (S_i) is proper and either oscillates or drifts to $-\infty$.

Remark 3. *Using* $G^{\ge}(e^{i\theta}) = 1-(1-\phi(\theta))/(1-G^{<}(e^{-i\theta}))$ *we deduce the explicit expression*

$$G^{\ge}(z) = \sum_{p,k\ge 0} h^{\downarrow}(k)\,\nu(k+p)\,z^p = \sum_{p,k\ge 0} h^{\downarrow}(k)\,q_{k+p+1}\,c_{\mathbf{q}}^{k+p}\,z^p$$

for the p.g.f. of the ascending ladder process.

Perhaps surprisingly Proposition 3 has a converse, in the sense that any random walk sharing this strict descending ladder process arises from some admissible weight sequence $\mathbf{q}$.

Theorem 1. *If the first strict descending ladder step of a random walk (Y_n) with step distribution μ on $\mathbb{Z}$ has p.g.f. $z \mapsto 1 - \sqrt{1-z}$ then the sequence $\mathbf{q}$ given by*

$$q_k := \mu(k-1)\,(\mu(-1)/2)^{k-1} \tag{15}$$

is admissible and $\mu = \nu_{\mathbf{q}}$.

Proof. First we deduce from the Wiener-Hopf factorization that $\mu(k) > 0$ for all $k < 0$. In particular, the sequence $\mathbf{q}$ is well-defined.

Let us fix $\ell > 0$ and a peel algorithm $\mathcal{A}$. We will now iteratively construct a random peeling exploration (without target) as in Definition 3 starting with the unique two-face map $\mathfrak{e}_0$ with a hole and root face both of degree 2ℓ. Given $\mathfrak{e}_i$ and the peel edge $e = \mathcal{A}(\mathfrak{e}_i)$, we specify that conditionally on the degree $2p$ of the hole to which e is incident the events C_k and G_{k_1,k_2} occur independently of $\mathfrak{e}_i$ with probabilities

$$\mathbb{P}(\mathrm{C}_k|p) = \frac{\mu(k-1)\mu(-p-k)}{\mu(-p-1)}, \qquad \mathbb{P}(\mathrm{G}_{k_1,k_2}|p) = \frac{\mu(-k_1-1)\mu(-k_2-1)}{2\mu(-p-1)}. \tag{16}$$

These probabilities indeed sum to 1 by a similar argument as before: the law of the descending ladder process implies that the characteristic function $\psi(\theta)$ of Y_1 satisfies $[e^{ik\theta}](1-\psi(\theta))^2 = 0$ for $k \leq -2$.

We claim that this peeling process terminates almost surely after a finite number of steps, which means that there exists a $n > 0$ such that $\mathfrak{e}_n$ has no holes. If we denote the latter map by $\mathfrak{m} = \mathfrak{e}_n$ then it is clear that $(\mathfrak{e}_i)$ is the peeling exploration of $\mathfrak{m}$ with peeling algorithm $\mathcal{A}$ in the sense of Definition 3.

In order to justify our claim, we introduce the function g which associates to a planar map $\mathfrak{e}$ with holes the quantity

$$g(\mathfrak{e}) = |\mathfrak{e}| + \sum_{\text{holes } h \text{ of } \mathfrak{e}} f^{\downarrow}(\deg(h)/2) \quad \text{with} \quad f^{\downarrow}(p) := \frac{\mu(-1)h^{\downarrow}(p)}{\mu(-p-1)}, \tag{17}$$

where $|\mathfrak{e}|$ is the number of inner vertices of $\mathfrak{e}$, i.e. the vertices that are not incident to a hole. One may check that $(g(\mathfrak{e}_i))_i$ determines a (positive) martingale with respect to the filtration generated by the sequence $(\mathfrak{e}_i)$ above. Indeed, if the peel edge $\mathcal{A}(\mathfrak{e}_i)$ is incident to a hole h of degree $2p$ then

$$\begin{aligned}
\mathbb{E}^{(\ell)}[g(\mathfrak{e}_{i+1}) - g(\mathfrak{e}_i)|\mathfrak{e}_i] &\overset{(17)}{=} -f^{\downarrow}(p) + \mathbb{E}^{(\ell)}\left[\,|\mathfrak{e}_{i+1}| - |\mathfrak{e}_i| + \sum_{\substack{\text{holes } h' \text{ originating}\\ \text{from } h}} f^{\downarrow}(\deg(h')/2)\,\middle|\,\mathfrak{e}_i\right]\\
&= -f^{\downarrow}(p) + \sum_{k\geq 1}\mathbb{P}(\mathrm{C}_k|p) f^{\downarrow}(p+k-1) + \sum_{\substack{k_1+k_2=p-1\\ k_1,k_2\geq 0}} \mathbb{P}(\mathrm{G}_{k_1,k_2}|p)(f^{\downarrow}(k_1)+f^{\downarrow}(k_2))\\
&\overset{(16)}{=} -f^{\downarrow}(p) + \sum_{k\geq 1}\frac{\mu(k-1)\mu(-1)h^{\downarrow}(p+k-1)}{\mu(-p-1)} + 2\sum_{k_1=0}^{p-1}\frac{\mu(k_1-p)\mu(-1)h^{\downarrow}(k_1)}{2\mu(-p-1)}\\
&= \frac{\mu(-1)}{\mu(-p-1)}\left(-h^{\downarrow}(p) + \sum_{k\in\mathbb{Z}}\mu(k)h^{\downarrow}(p+k)\right),
\end{aligned}$$

but the expression in parentheses vanishes because $h^{\downarrow}$ is μ-harmonic on $\mathbb{Z}_{>0}$.

One may check that there exists a $c > 0$ such that $2\mathbb{P}(\mathrm{G}_{0,p-1}|p) = \mu(-1)\mu(-p)/\mu(-p-1) > c$ for all $p \geq 1$. This means that at each step of the exploration, before the last hole disappears, the number of inner vertices increases with a probability at least c. Hence we have for any $n \geq 1$ that

$$f^{\downarrow}(\ell) = \mathbb{E}^{(\ell)}[g(\mathfrak{e}_n)] \geq \mathbb{E}^{(\ell)}[|\mathfrak{e}_n|] = \sum_{i=1}^{n} \mathbb{E}^{(\ell)}[|\mathfrak{e}_i| - |\mathfrak{e}_{i-1}|]$$
$$\geq c \sum_{i=1}^{n} \mathbb{P}[\mathfrak{e}_i \text{ has at least one hole}] \geq n\, c\, \mathbb{P}[\mathfrak{e}_n \text{ has at least one hole}].$$

But that means that the probability that the peeling algorithm has not terminated after n steps is $\mathcal{O}(n^{-1})$, hence the number of steps is almost surely finite.

Next using the rigidity of the peeling operations one may determine the probability of obtaining a particular planar map $\mathfrak{m} \in \mathcal{M}^{(\ell)}$ in this way by taking the product of the probabilities in (16) for the required steps. When doing this all factors $\mu(-p-1)$ with $p \geq 1$ cancel except for an overall $1/\mu(-\ell-1)$, and one is left with a probability

$$\mathbb{P}(\mathfrak{m}) = \frac{2}{\mu(-\ell-1)} \left(\frac{\mu(-1)}{2}\right)^{|\mathsf{Vertices}(\mathfrak{m})|} \prod_{f \in \mathsf{Faces}(\mathfrak{m}) \setminus \{f_r\}} \mu(\deg(f)/2 - 1)$$
$$= \frac{2}{\mu(-\ell-1)} \left(\frac{\mu(-1)}{2}\right)^{\ell+1} \prod_{f \in \mathsf{Faces}(\mathfrak{m}) \setminus \{f_r\}} \mu(\deg(f)/2 - 1) \left(\frac{\mu(-1)}{2}\right)^{\deg(f)/2-1},$$

where we used Euler's formula to obtain the second equality. This probability indeed reproduces the distribution $\mathbb{P}^{(\ell)}(\mathfrak{m})$ of (1) when using the identification (15). In particular, we read off that

$$W^{(\ell)}(\mathbf{q}) = \frac{1}{2}\mu(-l-1)(\mu(-1)/2)^{-\ell-1} < \infty. \tag{18}$$

From the considerations above it follows that the expected number of vertices of $\mathfrak{m}$ is given by $f^{\downarrow}(\ell) = W_{\bullet}^{(\ell)}(\mathbf{q})/W^{(\ell)}(\mathbf{q})$, and therefore

$$W_{\bullet}^{(\ell)}(\mathbf{q}) = f^{\downarrow}(\ell)W^{(\ell)}(\mathbf{q}) = h^{\downarrow}(\ell)(\mu(-1)/2)^{-\ell}$$

is finite as well. Since we also have $W_{\bullet}^{(\ell+1)}/W_{\bullet}^{(\ell)} \to 2/\mu(-1) = c_{\mathbf{q}}$ as $\ell \to \infty$, $\mathbf{q}$ is admissible. The fact that $\nu_{\mathbf{q}}(k) = \mu(k)$ for $k \in \mathbb{Z}$ follows directly from combining (9) with (15) and (18). □

Remark 4 (Non-bipartite maps)**.** *For simplicity we have restricted ourselves here to the bipartite setting, but it is straightforward to extend the mentioned results to non-bipartite planar maps. In that case, one keeps track of the perimeter $(\hat{P}_i)$ (instead of the half-perimeter) and its corresponding random walk $(\hat{S}_i)$ with step distribution $\hat{\nu}$. Lemma 2 then becomes*

$$\hat{\nu}(-k-2) = \frac{1}{2} \sum_{m=-\infty}^{\infty} \hat{\nu}(m)\hat{\nu}(-k-m-2) \qquad \textit{for} \qquad k \geq 1,$$

which implies that the p.g.f. $\hat{G}^{<}(z)$ of the first strict descending ladder step must be of the form

$$(1 - \hat{G}^{<}(z))^2 = (1 - \beta z)(1 + rz),$$

with $-\beta < r \leq \beta \leq 1$. Then $\hat{h}^{\downarrow}(p) = [z^p]1/\sqrt{(1-\beta z)(1+rz)}$ and its exponential growth determines that $\beta = 1$. The difference with the bipartite case is thus that there is a one-parameter family, parametrized by $r \in (-1, 1)$, of possible descending ladder processes. Apart from that the results, including Theorem 1, go through without change.

2.5 Boltzmann planar maps with a marked face

We denote by $\mathcal{M}_{\square}^{(\ell,m)}$ the set of rooted planar map with a distinguished face of degree $2m$ different from the root face, which has degree $2p$. To introduce a Boltzmann planar map with a marked face, we adapt the weight $w_{\mathbf{q}}$ to skip the marked face (denoted f_{m}), i.e.

$$w_{\mathbf{q}}(\mathfrak{m}_{\square}) := \prod_{f \in \mathrm{Faces}(\mathfrak{m}_{\square})\setminus\{f_{\mathrm{r}}, f_{\mathrm{m}}\}} q_{\deg(f)/2}.$$

If $W_{\square}^{(\ell,m)}(\mathbf{q}) := w_{\mathbf{q}}(\mathcal{M}_{\square}^{(\ell,m)})$ is finite, the distribution

$$\mathbb{P}_{\square}^{(\ell,m)}(\mathfrak{m}_{\square}) = \frac{w_{\mathbf{q}}(\mathfrak{m}_{\square})}{W_{\square}^{(\ell,m)}(\mathbf{q})}$$

defines the *Boltzmann planar map* $\mathfrak{m}_{\square}$ *with a marked face* of degree $2m$ and perimeter 2ℓ.

Proposition 4. *If* $\mathbf{q}$ *is admissible then* $W_{\square}^{(\ell,m)}(\mathbf{q}) < \infty$ *for all* $\ell, m \geq 1$ *and is given by*

$$W_{\square}^{(\ell,m)}(\mathbf{q}) = \frac{1}{2} H_m(\ell)\, c_{\mathbf{q}}^{\ell+m} \qquad \text{with} \qquad H_m(\ell) = \frac{\ell}{\ell+m} h^{\downarrow}(\ell)\, h^{\downarrow}(m). \tag{19}$$

The fact that $2m \cdot W_{\square}^{(\ell,m)}$ is symmetric in ℓ and m should not come as a surprise: if we distinguish one of the $2m$ oriented edges that have the marked face on their right-hand side, then the marked face and the root face play an equivalent role in the map.

Proof. Let us fix $\ell, m \geq 1$ and a peel algorithm $\mathcal{A}$. Then $\mathcal{A}$ induces a filled-in peeling exploration on any planar map with a marked face but also on pointed planar maps. If we denote by $(P_i(\mathfrak{m}_\bullet))$ the induced perimeter process on a pointed map $\mathfrak{m}_\bullet$, then we claim there exists a 2-to-1 mapping

$$\Psi : \left\{\mathfrak{m}_\bullet \in \mathcal{M}_\bullet^{(\ell+m)} : P_i(\mathfrak{m}_\bullet) \in \{0, m+1, m+2, \ldots\} \text{ for all } i\right\} \to \mathcal{M}_{\square}^{(\ell,m)}$$

that preserves the weight in the sense that $w_{\mathbf{q}}(\Psi(\mathfrak{m}_\bullet)) = w_{\mathbf{q}}(\mathfrak{m}_\bullet)$. This would imply that $W_{\square}^{(\ell,m)} \leq W_\bullet^{(\ell+m)} < \infty$ and that, conditionally on $P_i(\mathfrak{m}_\bullet) \in \{0, m+1, m+2, \ldots\}$ for all i, $\Psi(\mathfrak{m}_\bullet)$ under $\mathbb{P}_\bullet^{(\ell+m)}$ is distributed as $\mathfrak{m}_{\square}$ under $\mathbb{P}_{\square}^{(\ell,m)}$.

This mapping Ψ is obtained by comparing the sequences of events $(\mathsf{C}_k, \mathsf{G}_{*,k}, \mathsf{G}_{k,*})$ in the filled-in peeling of both $\mathfrak{m}_\bullet \in \mathcal{M}_\bullet^{(\ell+m)}$ and $\mathfrak{m}_\bullet \in \mathcal{M}_\bullet^{(\ell+m)}$. To be precise we set $\Psi(\mathfrak{m}_\bullet) = \mathfrak{m}_{\square}$ provided that at each peeling step $\mathfrak{e}_i \to \mathfrak{e}_{i+1}$ (except the last one when $\mathfrak{e}_{i+1} = \mathfrak{m}_\bullet$ respectively $\mathfrak{e}_{i+1} = \mathfrak{m}_{\square}$) the same event occurs and that at each step the same map is used to fill in a possible hole. Notice that then $P_i(\mathfrak{m}_{\square}) = P_i(\mathfrak{m}_\bullet) - m$ for all $i \geq 0$. If the perimeters before the last step are respectively $P_i(\mathfrak{m}_\bullet) = p$ and $P_i(\mathfrak{m}_{\square}) = p - m > 0$, then the last events necessarily correspond to $\mathsf{G}_{*,p-1}$ or $\mathsf{G}_{p-1,*}$ for $\mathfrak{m}_\bullet$ and M_{p-1} for $\mathfrak{m}_{\square}$ (both leaving a hole of degree $p-1$). Rigidity then implies that this indeed gives a 2-to-1 mapping (the 2 coming from the two options $\mathsf{G}_{*,p-1}$ or $\mathsf{G}_{p-1,*}$ in the last step) provided we restrict to pointed map for which the perimeter process satisfies $P_i(\mathfrak{m}_\bullet) > m$. Since Ψ preserves the set of (non-root, non-marked) faces it also preserves the weight $w_{\mathbf{q}}$.

It remains to compute $W_{\square}^{(\ell,m)}(\mathbf{q})$. We have

$$\begin{aligned}
W_{\square}^{(\ell,m)}(\mathbf{q}) &= \sum_{\mathfrak{m}_{\square} \in \mathcal{M}_{\square}^{(\ell,k)}} w_{\mathbf{q}}(\mathfrak{m}_{\square}) = \frac{1}{2} \sum_{\mathfrak{m}_\bullet \in \mathcal{M}_\bullet^{(\ell+k)}} w_{\mathbf{q}}(\mathfrak{m}_\bullet) \mathbf{1}_{\{P_i(\mathfrak{m}_\bullet) \in \{0, m+1, m+2, \ldots\} \text{ for all } i\}} \\
&= \frac{1}{2} W_\bullet^{(\ell+m)}\, \mathbb{P}_\bullet^{(\ell+m)}[P_i(\mathfrak{m}_\bullet) \in \{0, m+1, m+2, \ldots\} \text{ for all } i] \\
&= \frac{W_\bullet^{(\ell+m)}}{2h^{\downarrow}(\ell+m)}\, \mathbf{P}_{\ell+m}\left[(S_i) \text{ hits } \{m, m-1, m-2, \ldots\} \text{ at } 0\right] \\
&= \frac{1}{2} c_{\mathbf{q}}^{\ell+m}\, \mathbf{P}_\ell\left[(S_i) \text{ hits } \mathbb{Z}_{\leq 0} \text{ at } -m\right].
\end{aligned}$$

The last probability depends only on the strict descending ladder process $(H_i^<)$, and is therefore independent of $\mathbf{q}$. We leave it as an exercise to show that this probability is indeed given by $H_m(\ell)$ (Hint: for $\ell, k \geq 0$ we have $\mathbf{P}_0[(H_i^<) \text{ visits } \ell+k] = \sum_{m=0}^k \mathbf{P}_0[(H_i^<) \text{ hits } \{\ell, \ell+1, \ldots\} \text{ at } \ell+m]\mathbf{P}_0[(H_i^<) \text{ visits } \ell-m]$, which implies that $H_m(\ell)$ has to satisfy $h^\downarrow(\ell+k) = \sum_{m=0}^k H_m(\ell) h^\downarrow(\ell-m)$. Then use generating functionology to find the unique solution.) □

A direct consequence of this proposition is that the law of the perimeter process (P_i) associated to the filled-in peeling exploration of a Boltzmann planar map with a marked face of degree $2m$ is given by

$$\mathbb{P}_{\square}^{(\ell,m)}(P_{i+1} = p+k | P_i = p) = \frac{H_m(p+k)}{H_m(p)} \nu(k) \qquad (p > 0, p+k > 0) \tag{20}$$

while the process terminates (event M_{p+m-1}) with probability $\nu(-p-m)/H_m(p)$. In particular, it agrees (up to first hitting of $\mathbb{Z}_{\leq 0}$) with the law of the random walk (S_i) under the conditional probability $\mathbf{P}_\ell(\,\cdot\,|(S_i) \text{ hits } \mathbb{Z}_{\leq 0} \text{ at } -m)$.

2.6 Infinite Boltzmann planar maps

To study growth properties of, say, geodesic balls in random planar maps it is very convenient to first consider a limit in which the planar maps become infinitely large. One way to achieve this is by conditioning a Boltzmann planar map to have a fixed number n of vertices or faces and then to take the limit $n \to \infty$ (in the appropriate local topology). This can certainly be done using enumerative methods, but from the peeling exploration point of view such conditioning is not particularly natural. For fixed $\mathbf{q}$, the only free parameter we have introduced so far in our Boltzmann planar maps is the half-perimeter ℓ. It would be natural to look for a local limit by taking $\ell \to \infty$. However, keeping the root where it is, this will lead to a map with a root face of infinite degree. Instead we wish to examine our Boltzmann planar map in a neighbourhood of another root edge uniformly sampled in the map. This can naturally be done by considering a Boltzmann planar map of half-perimeter ℓ and a marked face of half-degree 1, which we collapse to a marked edge in the usual way. Equivalently, we may consider a Boltzmann planar map with a half-perimeter of 1 and a marked face of half-degree m, and consider the limit $m \to \infty$.

Observe that for all $p \geq 1$,

$$\frac{H_m(p+k)}{H_m(p)} \xrightarrow{m\to\infty} \frac{h^\uparrow(p+k)}{h^\uparrow(p)} \qquad \text{with} \qquad h^\uparrow(p) := 2p \cdot 4^{-p}\binom{2p}{p}.$$

Proposition 5. *If* $\mathbf{q}$ *is admissible then the following are equivalent:*

(i) for all $\ell \geq 1$*, the probability that the marked face of degree* $2m$ *of a Boltzmann planar map with perimeter* 2ℓ *is incident to the root edge approaches* 0 *as* $m \to \infty$*;*

(ii) $\sum_{k\in\mathbb{Z}} h^\uparrow(p+k)\nu(k) = h^\uparrow(p)$ *for all* $p \geq 1$

(iii) $\sum_{k\geq 0} h^\uparrow(k+1)\nu(k) = 1$.

(iv) (S_i) *oscillates;*

(v) $\nu_{\mathbf{q}}(-k) \cdot k^{3/2} \to 0$ *as* $k \to \infty$.

When any of these properties hold we call $\mathbf{q}$ *critical.*

Proof. Using that

$$\frac{H_m(p+k)}{H_m(p)} = \frac{p+m}{p+k+m}\frac{h^\uparrow(p+k)}{h^\uparrow(p)} \leq p\frac{h^\uparrow(p+k)}{h^\uparrow(p)},$$

dominated convergence implies that the probability that the marked face is encountered at the first peeling step satisfies

$$1 - \mathbb{P}_{\square}^{(\ell,m)}(P_1 > 0) \overset{(20)}{=} 1 - \sum_{k=-\ell+1}^{\infty} \frac{\ell+m}{\ell+k+m}\frac{h^{\uparrow}(\ell+k)}{h^{\uparrow}(\ell)}\nu(k) \xrightarrow{m\to\infty} 1 - \sum_{k\in\mathbb{Z}} \frac{h^{\uparrow}(\ell+k)}{h^{\uparrow}(\ell)}\nu(k).$$

This proves the equivalence of (i) and (ii) for any half-perimeter ℓ. Moreover, the equivalence with (v) follows from the fact that the left-hand side is equal to

$$\nu(-\ell-m)/H_m(\ell) \sim \nu(-\ell-m)\cdot(\ell+m)^{3/2} \qquad \text{as } m\to\infty.$$

Clearly (ii) implies (iii) by setting $p=1$. Conversely, using that $h^{\downarrow}$ is ν-harmonic we have for $p \geq 1$ that

$$\sum_{k\in\mathbb{Z}}(h^{\uparrow}(k+p+1) - h^{\uparrow}(k+p))\nu(k) = \sum_{k\in\mathbb{Z}} h^{\downarrow}(k+p)\nu(k) = h^{\downarrow}(p) = h^{\uparrow}(p+1) - h^{\uparrow}(p).$$

Hence (iii) also implies (ii) by induction on p.

It remains to show the equivalence of (iii) and (iv). Recall that (S_i) oscillates iff $(H_i^{\geq})$ is proper, i.e. $1 = \sum_{\ell=0}^{\infty} \mathbf{P}_0(H_1^{\geq} = \ell)$. Let us compute the latter probability as follows,

$$\begin{aligned}
\mathbf{P}_0(H_1^{\geq} = \ell) &= \sum_{n\geq 1} \mathbf{P}_0(S_1,\ldots S_{n-1} < 0, S_n = \ell) \\
&= \nu(\ell) + \sum_{k\geq 1}\sum_{n\geq 2} \mathbf{P}_0(S_1,\ldots S_{n-2} < 0, S_{n-1} = -k)\cdot \nu(\ell+k) \\
&= \nu(\ell) + \sum_{k\geq 1} \mathbf{P}_0((H_i^{<}) \text{ visits } k)\cdot\nu(\ell+k) \\
&\overset{(14)}{=} \sum_{k\in\mathbb{Z}} h^{\downarrow}(k)\nu(\ell+k).
\end{aligned}$$

Hence, (S_i) oscillates iff

$$\sum_{l\geq 0}\sum_{k\in\mathbb{Z}} h^{\downarrow}(k)\nu(\ell+k) = \sum_{k\geq 0}\left(\sum_{\ell=0}^{k} h^{\downarrow}(\ell)\right)\nu(k) = \sum_{k\in\mathbb{Z}} h^{\uparrow}(k+1)\nu(k) = 1,$$

which finishes the proof. □

If $\mathfrak{m}$ is a (rooted) planar map we denote by $[\mathfrak{m}]_r$, $r \geq 0$, the *ball of radius* r in $\mathfrak{m}$, given by the set of vertices at graph distance at most r from the root vertex together with all edges joining those vertices. One can put a distance on the space $\mathcal{M}$ of planar maps, by setting

$$d_{\mathrm{loc}}(\mathfrak{m},\mathfrak{m}') = \frac{1}{1+\sup\{r\geq 0 : [\mathfrak{m}]_r = [\mathfrak{m}']_r\}}$$

for any two planar maps $\mathfrak{m},\mathfrak{m}' \in \mathcal{M}$. In words: the larger the neighbourhood of the root on which $\mathfrak{m}$ and $\mathfrak{m}'$ agree, the closer they are with respect to this distance. The space $\mathcal{M}$ with this distance is not complete: its completion $\overline{\mathcal{M}}$ contains both finite maps and infinite ones, which by construction can be specified as a sequence $(\mathfrak{m}_i)$ of maps such that $[\mathfrak{m}_i]_r = [\mathfrak{m}_j]_r$ for all $0 \leq r \leq i \leq j$. The distance d_{loc} induces a topology on $\overline{\mathcal{M}}$, which we call the *local topology*. For a sequence of random finite maps, we can therefore ask whether it possesses a distributional limit, the *local limit*, with respect to this topology.

In our case we wish to study the local limit of Boltzmann planar maps with a marked face of growing degree. Judging by Proposition 5(i) we should better restrict to critical $\mathbf{q}$, otherwise the limit will contain a face of infinite degree.

Theorem 2. *Suppose* $\mathbf{q}$ *is admissible and critical, and* $\ell \geq 1$ *fixed. There exists a probability distribution* $\mathbb{P}_\infty^{(\ell)}$ *on infinite planar maps* $\mathfrak{m}_\infty \in \overline{\mathcal{M}}$*, such that almost surely* $\mathfrak{m}_\infty$ *is one-ended and all its faces and vertices have finite degree, and such that* $\mathfrak{m}_\square$ *under* $\mathbb{P}_\square^{(\ell,m)}$ *converges in distribution in the local topology to* $\mathfrak{m}_\infty$ *under* $\mathbb{P}_\infty^{(\ell)}$ *as* $m \to \infty$.

Sketch of proof. We fix $\ell \geq 1$ and a peeling algorithm $\mathcal{A}$ (to be specified below), and consider the filled-in peeling exploration $(\mathfrak{e}_i)_{i\geq 0}$ of $\mathfrak{m}_\square$ under $\mathbb{P}_\square^{(\ell,m)}$. As a Markov chain $(\mathfrak{e}_i)_{i\geq 0}$ converges in law to a Markov chain $(\mathfrak{e}_i^\infty)_{i\geq 0}$ as $m \to \infty$, because the same is true for the transition probabilities as we have seen above. Moreover, from Proposition 5(i) it follows that for all $i \geq 0$ almost surely $\mathfrak{e}_i^\infty$ has a single hole, i.e. the event M_k does not occur.

We want to define $\mathfrak{m}_\infty \in \overline{\mathcal{M}}$ by specifying a sequence of maps that encompass balls of increasing radius. To ensure this without too much work we choose our peeling algorithm $\mathcal{A}$ in such a way that it always selects a peel edge that is incident to a vertex with minimal distance to the root vertex (as measured by the graph distance). Then we claim that for $r > 0$ one can always find an $i_r > 0$ such that the minimal distance from the vertices incident to the hole of $\mathfrak{e}_{i_r}$ is larger than r. This follows from the fact that at each step the probability that the vertex at minimal distance becomes an inner vertex (in the event of $\mathsf{G}_{*,k}$) is uniformly bounded from below. The sequence $(\mathfrak{e}_{i_r})_r$ then defines a probability measure $\mathbb{P}_\infty^{(\ell)}$ on $\mathfrak{m}_\infty \in \overline{\mathcal{M}}$, which by the properties of $(\mathfrak{e}_i)$ is easily seen to be one-ended and to have faces and vertices of finite degree.

Finally, to show that $\mathfrak{m}_\square$ converges to $\mathfrak{m}_\infty$ in the local topology it suffices to check that for any $i > 0$ and any planar map $\mathfrak{e}$ with holes

$$\mathbb{P}_\square^{(\ell,m)}(\mathfrak{e}_i = \mathfrak{e}) \xrightarrow{m\to\infty} \mathbb{P}_\infty^{(\ell)}(\mathfrak{e}_i^\infty = \mathfrak{e}).$$

But this is implied by the convergence of the Markov chain $(\mathfrak{e}_i)_{i\geq 0}$ to $(\mathfrak{e}_i^\infty)_{i\geq 0}$ as $m \to \infty$. ◻

The proof also implies that the perimeter process of $\mathfrak{m}_\infty$ under $\mathbb{P}_\infty^{(\ell)}$ for the peeling exploration induced by $\mathcal{A}$ is the large-m limit of that of $\mathfrak{m}_\square$ under $\mathbb{P}_\square^{(\ell,m)}$, i.e.

$$\mathbb{P}_\infty^{(\ell)}(P_{i+1} = p + k | P_i = p) = \frac{h^\uparrow(p+k)}{h^\uparrow(p)} \nu(k). \tag{21}$$

This corresponds exactly to the law of (S_i) under $\mathbf{P}_\ell$ conditioned to stay positive (i.e. $\mathbf{P}_\ell(\,\cdot\,|S_1,\ldots,S_n > 0)$ as $n \to \infty$), which we will sometimes denote by $(S_i^\uparrow)$.

Remark 5. *Let us check that our criteria for admissibility and criticality agree with those of [8], which states that* $\mathbf{q}$ *is admissible iff there exists a positive solution* $x > 0$ *to* $f_{\mathbf{q}}(x) = 1 - 1/x$ *where*

$$f_{\mathbf{q}}(x) := \sum_{k=1}^\infty q_k \binom{2k-1}{k-1} x^{k-1} = 2\sum_{k=0}^\infty q_{k+1}\, h^\downarrow(k+1)\,(4x)^k.$$

First of all we notice that if $h^\downarrow$ *is* $\nu_{\mathbf{q}}$*-harmonic on* $\mathbb{Z}_{>0}$ *then such a solution is given by* $x = c_{\mathbf{q}}/4 = 1/(2\nu(-1))$ *since* $1 = 2h^\downarrow(1) = 2\sum_{k=-1}^\infty \nu(k)h^\downarrow(k+1) = 2\nu(-1) + 2\sum_{k=0}^\infty q_{k+1}h^\downarrow(k+1)c_{\mathbf{q}}^k$*. Conversely one may check that if* $x > 0$ *is the smallest solution to* $f_{\mathbf{q}}(x) = 1 - 1/x$ *and one sets* $\mu(k) = q_{k+1}(4x)^k$ *for* $k \geq 0$ *and* $\mu(-1) = 1/(2x)$*, then* μ *can be extended to a unique measure on* $\mathbb{Z}$ *for which* $h^\downarrow$ *is* μ*-harmonic by recursively setting* $\mu(-\ell) = h^\downarrow(\ell) - \sum_{k=1-\ell}^\infty \nu(k)h^\downarrow(\ell+k)$ *for* $\ell = 2, 3, \ldots$*. Finally the criticality condition* $\sum_{k=0}^\infty h^\uparrow(k+1)\nu(k) = 1$ *of Proposition 5(iii) is easily seen to be equivalent to* $x^2 f_{\mathbf{q}}'(x) = 1$ *when* $x = c_{\mathbf{q}}/4$ *is the smallest solution to* $f_{\mathbf{q}}(x) = 1 - 1/x$*, which is the criticality criterion from [8].*

3 Scaling limit of the perimeter process

From the last section it should be clear that the perimeter process (P_i) associated to the exploration of a Boltzmann planar map $\mathfrak{m}_\star$ with target, contains the crucial information about its geometry. Indeed, conditionally on the (P_i) the map $\mathfrak{m}_\star$ can be obtained by iteratively performing one of the peeling operations: at the i'th step if $P_{i+1} \geq P_i$ then C_k occurs with $k = P_{i+1} - P_i + 1$ and otherwise $\mathrm{G}_{*,k}$ or $\mathrm{G}_{k,*}$ with equal probability when $k = P_i - P_{i+1} - 1$ and the hole is filled in with an independent Boltzmann planar map of perimeter k. For each of the targets $(\bullet, \square, \infty)$ we have seen that the perimeter process has the law of an h-transform of the random walk (S_n) killed upon hitting $\mathbb{Z}_{\leq 0}$, corresponding to conditioning the walk either to hit $\mathbb{Z}_{\leq 0}$ at a specific point or to avoid $\mathbb{Z}_{\leq 0}$.

To understand the asymptotics of the perimeter process a good starting point is to determine the asymptotics of S_n as $n \to \infty$, which in turn depends on the tail behaviour of the measure ν.

Proposition 6. *If* $\mathbf{q}$ *is admissible and*

$$\left\{\begin{array}{l} \textit{subcritical} \\ \textit{critical and } \sum_{k=1}^\infty k^{3/2}\nu(k) < \infty \\ \textit{critical and } \nu(k) \sim c \cdot k^{-a} \textit{ for } a \in (\frac{3}{2}, \frac{5}{2}) \end{array}\right\} \textit{ then } \nu(-k) \sim c' \cdot \left\{\begin{array}{l} k^{-3/2} \\ k^{-5/2} \\ k^{-a} \end{array}\right\} \textit{ and } \frac{\mathbf{P}_0(S_1 > k)}{\mathbf{P}_0(S_1 < -k)} \to \left\{\begin{array}{l} 0 \\ 0 \\ \cos(\pi a) \end{array}\right\}$$

Proof. To do. □

When $\mathbf{q}$ is critical and $\sum_{k=1}^\infty k^{3/2}\nu(k) < \infty$ then we will say that $\mathbf{q}$ is *generic critical*, which encompasses all critical weight sequences $\mathbf{q}$ with finite support and those for which $q_k c_{\mathbf{q}}^k$ falls off exponentially. If $\nu(k) \sim k^{-a}$ for $a \in (\frac{3}{2}, \frac{5}{2})$ we say $\mathbf{q}$ is *non-generic critical with parameter* a. In each of the cases of Proposition 6 we introduce the asymptotics

$$\nu(-k) \sim \mathrm{p}_{\mathbf{q}}\, k^{-a}.$$

Remark 6. *Proposition 6 does not assert the existence of such* $\mathbf{q}$ *for all values of* $a \in [3/2, 5/2]$. *However, one such family of sequences may easily be realized by specifying that the weak ascending ladder process is given by the p.g.f.* $G^{\geq}(z) = 1 - c(1-z)^{a-3/2}$. *If one further fixes* c *such that* $\nu(k) = 0$, *namely* $c = \pi\Gamma(a - 1/2)/\Gamma(a)$, *then one obtains the measure*

$$\nu(k) = \frac{\sqrt{\pi}}{2|\Gamma(3/2 - a)|} \frac{\Gamma(3/2 - a + k)}{\Gamma(3/2 + k)} \mathbf{1}_{k \neq 0}, \qquad (k \in \mathbb{Z})$$

with corresponding admissible sequence $\mathbf{q}$ *given by* $q_k = \nu(k-1)(\nu(-1)/2)^{k-1}$. *As* $a \to 5/2$ *this gives* $q_k = \frac{1}{12}\mathbf{1}_{k=2}$, *corresponding exactly to critical quadrangulations.*

Proposition 7. *In each of the cases of Proposition 6 we have the following convergence in distribution under* $\mathbf{P}_0$ *in the Skorokhod topology,*

$$\left(\frac{S_{\lfloor nt \rfloor}}{n^{1/(a-1)}}\right)_{t \geq 0} \xrightarrow[n\to\infty]{\text{(d)}} \left(\Upsilon_a(\mathrm{p}_{\mathbf{q}}\, t)\right)_{t \geq 0},$$

where $\Upsilon_a(t)$ *is the stable Lévy process of index* $a - 1$ *and Lévy measure*

$$\Pi(\mathrm{d}x) = \cos(\pi a) \frac{\mathrm{d}x}{x^a} \mathbf{1}_{\{x>0\}} + \frac{\mathrm{d}x}{|x|^a} \mathbf{1}_{\{x<0\}}.$$

Proof. It is classical that the convergence is granted as soon as it holds at $t = 1$, for which we may rely on a generalized central limit theorem (see e.g. [1, Theorem 8.3.1]). According to the latter together with the asymptotics of Proposition 6, we have the convergence $n^{-1/(a-1)} S_n - b_n \to \Upsilon_a(\mathrm{p}_{\mathbf{q}})$ in distribution for a suitable sequence b_n. It only remains to show that $b_n \to 0$. For $a < 2$ this is true in general since no centering is required. For $a > 2$ the fact that (S_i) oscillates implies that it has zero mean and so does $\Upsilon_a(p_{\mathbf{q}})$, hence $b_n \to 0$. The case $a = 2$ is more tricky, since it does require centering although is has infinite first moments. For this case we refer to [4, Proposition 2]. □

If $\mathbf{q}$ is critical (which excludes $a = 3/2$) then the perimeter process (P_i) under $\mathbb{P}_\infty^{(\ell)}$, i.e. the infinite Boltzmann map of perimeter 2ℓ, is given by conditioning (S_i) under $\mathbf{P}_\ell$ to stay positive. According to the invariance principle of Caravenna & Chaumont [5] this implies that it converges to a stochastic process $\Upsilon_a^\uparrow$, which is the stable process Υ_a conditioned to stay positive and started from 0, i.e.

$$\left(\frac{S^\uparrow_{\lfloor nt \rfloor}}{n^{1/(a-1)}}\right)_{t\geq 0} \xrightarrow[n\to\infty]{\text{(d)}} \left(\Upsilon_a^\uparrow(p_{\mathbf{q}}\, t)\right)_{t\geq 0}. \tag{22}$$

In particular we get our first glimpse of the non-trivial growth properties of infinite Boltzmann maps: after n peeling steps the exploration frontier has a size of order $n^{1/(a-1)}$.

4 Geometry

We wish to study distances in infinite planar maps using a filled-in peeling exploration. To this end we should select a peeling algorithm $\mathcal{A}$ that explores the map by increasing distance and such that at any time the points on the exploration frontier are all roughly at the same distance. This is hard to achieve when one aims at the usual graph distance especially when the map contains faces of large degree, but becomes straightforward when dealing with distances on the dual map.

In this lecture we will concentrate on a certain random distance, the *first passage percolation distance*, which is particularly simple to analyze. If $\mathfrak{m}_\infty$ is an infinite planar map, we denote by $\mathfrak{m}_\infty^\dagger$ its dual, i.e. the map in which the roles of the faces and vertices are interchanged (see Figure 7). To each edge $e \in \mathsf{Edges}(\mathfrak{m}^\dagger)$ we associate an independent random variable x_e with an exponential distribution of mean 1, i.e. with density $e^{-x}\mathrm{d}x$. Then we may associate to $\mathfrak{m}^\dagger$ a continuous length metric space $\mathbb{X}(\mathfrak{m}_\infty^\dagger, (x_e))$ by viewing each edge as an interval $[0, x_e]$ with the standard Euclidean metric and performing appropriate identifications at the endpoints. This space has a natural origin corresponding to the vertex of $\mathfrak{m}_\infty^\dagger$ dual to the root face of $\mathfrak{m}_\infty$ (the open dot in the top-right figure of Figure 7), and each point of $\mathbb{X}(\mathfrak{m}_\infty^\dagger, (x_e))$ has a well-defined distance to this origin. In particular, for $t \in [0, \infty)$ fixed, we may define the submap $\mathrm{Ball}_t^{\mathrm{fpp}}(\mathfrak{m}_\infty) \subset \mathfrak{m}_\infty$ by cutting open all the edges of $\mathfrak{m}_\infty$ for which the dual edge has at least one point in $\mathbb{X}(\mathfrak{m}_\infty^\dagger, (x_e))$ at distance at least t (i.e. all the edges that are not completely blue in the top-right of figure 7), and keeping only the connected component containing the root face. Moreover, using the filling-in operation of Section 1.3 we introduce the *hull of the fpp-ball of radius t* by

$$\overline{\mathrm{Ball}}_t^{\mathrm{fpp}}(\mathfrak{m}_\infty) := \mathsf{Fill}(\mathrm{Ball}_t^{\mathrm{fpp}}(\mathfrak{m}_\infty), \mathfrak{m}_\infty) \subset \mathfrak{m}_\infty.$$

The main reason why we define our fpp-ball in this slightly contrived way is the close connection with the filled-in peeling exploration associated to the *uniform* peel algorithm $\mathcal{A}$, i.e. where $\mathcal{A}(\mathfrak{e})$ is a uniformly random element of $\mathsf{Active}(\mathfrak{e})$.

Proposition 8. *Let $\mathfrak{m}_\infty$ be a fixed infinite planar map. The fpp-ball $\overline{\mathrm{Ball}}_t^{\mathrm{fpp}}(\mathfrak{m}_\infty)$ as function of $t \in [0, \infty)$ jumps at times $0 = T_0 < T_1 < T_2 < \cdots$ and*

- *the law of $\left(\overline{\mathrm{Ball}}_{T_i}^{\mathrm{fpp}}(\mathfrak{m}_\infty)\right)_i$ is that of the uniform peeling exploration of $\mathfrak{m}_\infty$;*

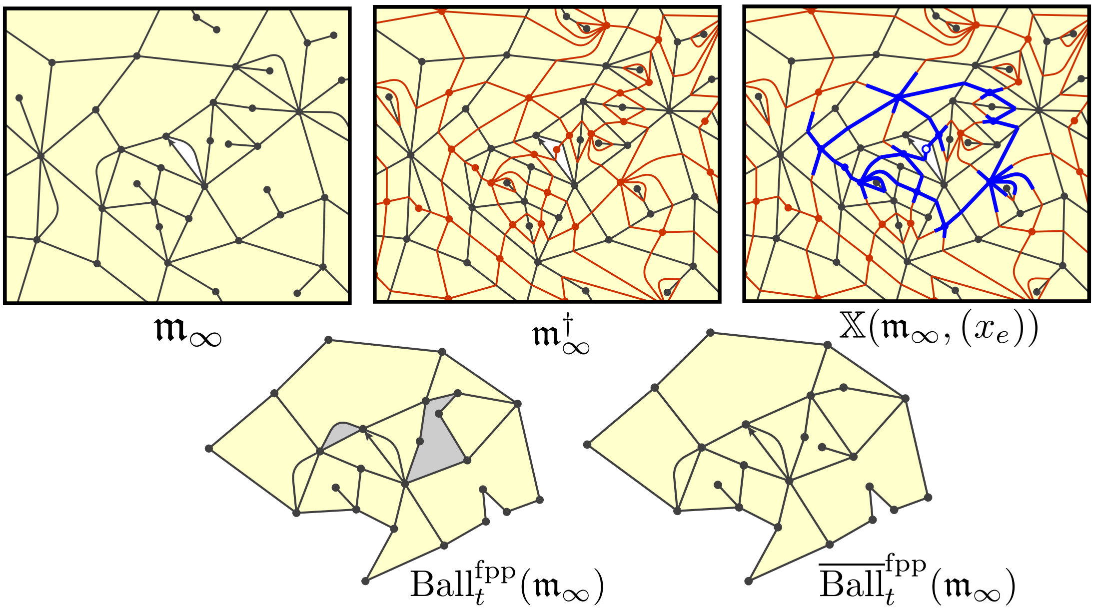


Figure 7: Top row (left to right): an infinite map $\mathfrak{m}_\infty$; the map together with its dual $\mathfrak{m}^\dagger$; the subset (blue) of points in the continuous length space associated to $\mathfrak{m}^\dagger$ which are at distance less than t from the root. Bottom row: the ball $\mathrm{Ball}_t^{\mathrm{fpp}}(\mathfrak{m}_\infty) \subset \mathfrak{m}_\infty$ of radius t; its hull $\overline{\mathrm{Ball}}_t^{\mathrm{fpp}}(\mathfrak{m}_\infty) \subset \mathfrak{m}_\infty$.

- *conditionally on* $\left(\overline{\mathrm{Ball}}_{T_i}^{\mathrm{fpp}}(\mathfrak{m}_\infty)\right)_i$ *the differences* $\Delta T_i = T_{i+1} - T_i$ *are independent and distributed as exponential random variables with mean* $1/(2P_i)$.

Sketch of proof. The basic idea is the following: for $i \geq 0$, conditionally on $\overline{\mathrm{Ball}}_{T_i}^{\mathrm{fpp}}(\mathfrak{m}_\infty) = \mathfrak{e}_i$, the next time $T_{i+1} > T_i$ that the fpp-ball jumps is when one of the edges dual to $\mathrm{Active}(\mathfrak{e})$ is fully explored (is colored completely blue in Figure 7). For simplicity we disregard the fact that some edges in $\mathrm{Active}(\mathfrak{e})$ are identified in $\mathfrak{m}_\infty$. Then, because of the memorylessness of the exponential distribution, at time T_i the size of the unexplored part of each of those edges is independent and exponentially distributed (with mean 1). It follows that the edge with the smallest unexplored part is uniform among the duals of $\mathrm{Active}(\mathfrak{e})$ and the size of its unexplored part is distributed as the minimum of $2P_i = |\mathrm{Active}(\mathfrak{e})|$ exponential variables, which again is an exponential variable but with mean $1/(2P_i)$. It can be seen that the identification of edges in $\mathrm{Active}(\mathfrak{e})$ does not spoil this analysis, see [3, Proposition 2.3] or [6, Proposition 33] for details. □

In particular we observe that, conditionally on the perimeter process (P_i), the corresponding process (T_i) of *fpp-distances* is distributed as

$$T_n = \sum_{i=0}^{n-1} \frac{\mathbf{e}_i}{2P_i} \qquad (n \geq 0),$$

where $(\mathbf{e}_i)$ is a sequence of independent exponential random variables of mean 1.

Now let us take $\mathfrak{m}_\infty$ to be an infinite Boltzmann map with perimeter 2 and critical weight sequence $\mathbf{q}$. Then we can easily estimate the expectation value $\mathbb{E}_\infty^{(1)}[T_n]$ using that (P_i) under $\mathbb{P}_\infty^{(1)}$ is distributed as $(S_i^\uparrow)$ under $\mathbf{P}_1$.

Lemma 3. *If* $\mathbf{q}$ *is critical then*

$$\mathbb{E}_\infty^{(1)}[\Delta T_n] = \sum_{k=n+1}^{\infty} \frac{1}{k}\mathbf{P}_1(S_k = 0) \qquad \textit{and} \qquad \mathbb{E}_\infty^{(1)}[T_n] \xrightarrow{n\to\infty} \sum_{k=1}^{\infty} \mathbf{P}_1(S_k = 0).$$

Proof. For $n \geq 0$ we have

$$\begin{aligned}
\mathbb{E}_{\infty}^{(1)}[\Delta T_n] &= \mathbb{E}_{\infty}^{(1)}\left[\frac{\mathbf{e}_n}{2P_n}\right] = \mathbf{E}_1\left[\frac{1}{2S_n^{\uparrow}}\right] = \sum_{k=1}^{\infty} \frac{1}{2k}\mathbf{P}_1(S_n^{\uparrow} = k) \\
&= \sum_{k=1}^{\infty} \frac{h^{\uparrow}(k)}{2k\, h^{\uparrow}(1)}\mathbf{P}_1(S_1 > 0, \ldots, S_{n-1} > 0, S_n = k) \\
&= \sum_{k=1}^{\infty} h^{\downarrow}(k)\mathbf{P}_1(S_1 > 0, \ldots, S_{n-1} > 0, S_n = k) \\
&= \sum_{k=1}^{\infty} \mathbf{P}_k((S_i) \text{ hits } \mathbb{Z}_{\leq 0} \text{ at } 0)\, \mathbf{P}_1(S_1 > 0, \ldots, S_{n-1} > 0, S_n = k) \\
&= \mathbf{P}_1((S_i) \text{ hits } \mathbb{Z}_{\leq 0} \text{ at } 0 \text{ strictly after time } n).
\end{aligned}$$

Now we make use of a well-known cycle lemma: consider an arbitrary walk of k steps on $\mathbb{Z}$ starting at 1 and ending at 0, then among the k walks obtained by cyclically permuting its increments there is exactly one walk that stays positive up to the last step. This implies that

$$\mathbf{P}_1(S_k = 0) = k\, \mathbf{P}_1(S_1 > 0, \ldots, S_{k-1} > 0, S_k = 0).$$

Hence,

$$\mathbb{E}_{\infty}^{(1)}[\Delta T_n] = \sum_{k=n+1}^{\infty} \mathbf{P}_1(S_1 > 0, \ldots, S_{k-1} > 0, S_k = 0) = \sum_{k=n+1}^{\infty} \frac{1}{k}\mathbf{P}_1(S_k = 0).$$

This proves the first claim, while the second follows from summing over n (from 0 to ∞). □

The second result says that the fpp-distance $T_\infty := \lim_{n\to\infty} T_n$ to a "point at infinity" has expectation value equal to the expected number of visits of (S_n) to 0, which is finite if and only if (S_n) is transient. For the weight sequences of Proposition 6 this happens precisely when $a < 2$. So for such weight sequences the geometry (as defined by fpp-distance) of the infinite Boltzmann map is quite degenerate: for any $t > 0$ the fpp-ball $\overline{\mathrm{Ball}}_t^{\mathrm{fpp}}(\mathfrak{m}_\infty)$ has a positive probability of being infinite!

On the other hand, if $a > 2$ then (22) together with a *local limit theorem* (see e.g. [7, Theorem 4.2.1])

$$\mathbf{P}_1(S_k = 0) \sim c \cdot k^{-1/(a-1)}$$

implies that

$$\mathbb{E}_{\infty}^{(1)}[T_n] = \sum_{\ell=0}^{n-1} \sum_{k=\ell+1}^{\infty} \frac{1}{k}\mathbf{P}_1(S_k = 0) \sim C \cdot n^{\frac{a-2}{a-1}} \qquad \text{as } n \to \infty.$$

So far, we have not discussed the volume (in the sense of the number of vertices) explored in the peeling exploration. Luckily there is a simple way to heuristically derive the scaling with n. For large and fixed n, conditionally on P_n, with large probability the process (P_i) has made at least one negative jump of size $k \approx P_n$ before time n. On this event the explored map $\mathfrak{e}_n$ contains at least the vertices of the Boltzmann planar map of perimeter $2k$ filling in the hole created by the jump. We have already seen that the expected number $\mathbb{E}^{(k)}[|\mathfrak{m}|]$ of vertices is given by

$$\mathbb{E}^{(k)}[|\mathfrak{m}|] = \frac{W_\bullet^{(k)}}{W^{(k)}} = f^{\downarrow}(k) = \frac{\nu(-1)h^{\downarrow}(k)}{\nu(-k-1)} \sim c \cdot k^{a-1/2}.$$

Hence, we should expect the number of vertices $|\mathfrak{e}_n|$ to be of order $(P_n)^{a-1/2} \approx n^{(a-1/2)/(a-1)}$.

To summarize, after n uniform peeling steps the distance T_n, the perimeter P_n, and the volume $|\mathfrak{e}_n|$ scale as $n \to \infty$ as

$$T_n \approx n^{\frac{a-2}{a-1}}, \qquad P_n \approx n^{\frac{1}{a-1}}, \qquad |\mathfrak{e}_n| \approx n^{\frac{a-1/2}{a-1}}.$$

A further time-change then yields

$$|\overline{\mathrm{Ball}}_t^{\mathrm{fpp}}(\mathfrak{m}_\infty)| \approx t^{\frac{a-1/2}{a-2}} \qquad \text{and} \qquad |\partial\overline{\mathrm{Ball}}_t^{\mathrm{fpp}}(\mathfrak{m}_\infty)| \approx t^{\frac{1}{a-2}}$$

for the volume and perimeter of the fpp-ball of radius t. For precise statements and the law of the scaling limit, see [3, Section 4.1].

A Planar map editor

Perhaps the best way to get acquainted with the peeling exploration of planar maps is to try some examples and to observe the effects of the various peeling operations. Unfortunately, as the reader may have noticed, even with relatively small maps performing a gluing operation (G_{k_1,k_2}) by hand already becomes tedious and error-prone. Luckily there is an automated solution in the form of a *planar map editor* which can be accessed in the browser at:[1]

`http://www.nbi.dk/~budd/planarmap/examples/editor.html`

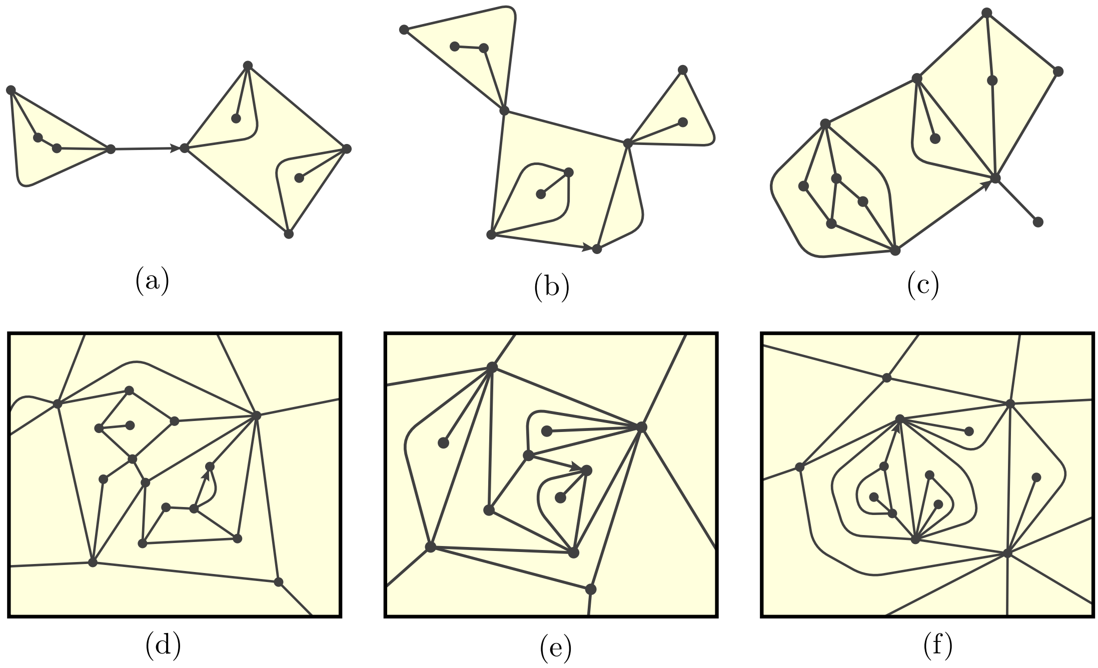
To perform a peeling exploration, say, of a finite planar map of perimeter 8 with the root face on the outside:

- Select a corner of the displayed edge (which is the starting configuration) and press «8» to produce a face of degree 8.
- For convenience you can mark a root edge by selecting a corner and pressing «M».
- If you like you can give the hole (the bounded face) a different color by selecting it, opening up the *SELECTION* menu, and choosing a color where it says *Fill*.
- To perform an operation C_k with, say, $k = 2$ to explore a face of degree $2k = 4$: choose a peel edge by selecting a corner of the hole, then press «4».
- To perform a gluing operation G_{k_1,k_2}: select two corners of the hole (hold «shift» to do so), then press «G».

Of course, one may also treat the outer face as the hole, which is particularly convenient when exploring an infinite planar map, and the operations C_k and G_{k_1,k_2} work similarly. The only thing one should keep in mind is that the gluing operation depends on the order in which the two corners are selected: gluing is always done clockwise, in the sense that the active edges that sit clockwise in between the first and second corner become part of the "finite" hole.

[1] The planar map editor is not just a fun tool to practice peeling, but can also be used to produce quality vector figures of planar maps (like all the ones appearing in these lecture notes).

Exercise Try to recreate the following planar maps by a peeling exploration, i.e. by only using the digit keys and «G» (and «Ctrl-Z» to undo). For (a), (b) and (c), start with a hole of degree 8 as described above. For (d), (e) and (f), start with a single edge, the root edge, and view the outer face as a hole of degree 2. To increase the challenge let a colleague be your peel algorithm $\mathcal{A}$, i.e. someone telling you which edge should be peeled next.



References

[1] Bingham, N. H., Goldie, C. M., and Teugels, J. L. *Regular variation*, vol. 27 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1989.

[2] Budd, T. The peeling process of infinite Boltzmann planar maps. *Electron. J. Combin. 23*, 1 (2016), Paper 1.28, 37.

[3] Budd, T., and Curien, N. Geometry of infinite planar maps with high degrees. To appear in *Electron. J. Probab.* Preprint available at `arXiv:1602.01328` (Feb. 2016).

[4] Budd, T., Curien, N., and Marzouk, C. Infinite random planar maps related to Cauchy processes. *arXiv:1704.05297 [math]* (Apr. 2017). arXiv: 1704.05297.

[5] Caravenna, F., and Chaumont, L. An invariance principle for random walk bridges conditioned to stay positive. *Electron. J. Probab. 18* (2013), no. 60, 32.

[6] Curien, N. Peeling random planar maps (lecture notes). Available at `https://www.math.u-psud.fr/~curien/cours/peccot.pdf`, 2017.

[7] Ibragimov, I. A., and Yu. *Independent and Stationary Sequences of Random Variables.* Wolters-Noordhoff, Gröningen, 1971.

[8] Marckert, J.-F., and Miermont, G. Invariance principles for random bipartite planar maps. *Ann. Probab. 35*, 5 (2007), 1642–1705.