Shape dynamics in 2 + 1 dimensions

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Abstract   Shape Dynamics is a formulation of General Relativity where refoliation invariance is traded for local spatial conformal invariance. In this paper we explicitly construct Shape Dynamics for a torus universe in 2 + 1 dimensions through a linking gauge theory that ensures dynamical equivalence with General Relativity. The Hamiltonian we obtain is formally a reduced phase space Hamiltonian. The construction of the Shape Dynamics Hamiltonian on higher genus surfaces is not explicitly possible, but we give an explicit expansion of the Shape Dynamics Hamiltonian for large CMC volume. The fact that all local constraints are linear in momenta allows us to quantize these explicitly under a certain assumption on the kinematic Hilbert space, and the quantization problem for Shape Dynamics turns out to be equivalent to reduced phase space quantization. We consider the large CMC-volume asymptotics of conformal transformations of the wave function. We then discuss the similarity of Shape Dynamics on the 2-torus with the explicitly constructible strong gravity Shape Dynamics Hamiltonian in higher dimensions.

Keywords   Canonical general relativity · General relativity in 2 + 1 dimensions · Shape dynamics · Dirac quantization

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1 Introduction

Shape Dynamics is constructed as a reformulation of General Relativity in which spacetime refoliation invariance is traded for spatial conformal invariance that preserves the total spatial volume [3,13,15] using a linking gauge theory. The development of Shape Dynamics was inspired by Dirac's work [10] on CMC (constant mean extrinsic curvature) gauge, York's method for solving the initial value problem [19,23,24] and Machian ideas developed by Barbour and collaborators [1,4]. One particular motivation for dealing with Shape Dynamics is the desire to have two distinct gauge theoretic descriptions of the same theory at ones disposal, one based on spacetime and covariance and the other based on spatial conformal invariance. This increases the number of available tools to attack problems. For example, an appealing feature of the Shape Dynamics description is that all local constraints turn out to be linear in momenta and the corresponding gauge transformation have a simple geometric interpretation.

The explicit construction of the Shape Dynamics Hamiltonian however requires the general solution of a partial differential equation, which is equivalent to partially solving the initial value problem of General Relativity using York's method. This is a serious complication, which introduces nonlocalities into the Hamiltonian and obstructs many straightforward investigations. To learn about Shape Dynamics it is therefore very valuable to consider exactly solvable nontrivial gravitational models. This provides the main motivation for this paper: We consider a nontrivial model in which Shape Dynamics can be constructed explicitly allowing us to study its generic features.

The probably best-known example of a nontrivial exactly solvable gravitational system is pure gravity on the torus in $2 + 1$ dimensions [8,9,17,18]. The technical reason for the simplifications in this model is two-fold: first, one is able to solve the initial value problem of ADM gravity explicitly on the 2-torus. This is important for the construction of classical Shape Dynamics and occurs only on the 2-torus and 2-sphere;[1] pure gravity on higher-genus surfaces is more intricate since we lack methods to solve the initial value problem in general. Second, the reduced phase space (after solving for initial data) is finite dimensional, which is a generic feature of pure gravity in $2+1$ dimensions. This is important for quantization, because a finite dimensional system admits generic quantum theories while nontrivial quantum systems with infinitely many degrees of freedom are sparse.

The plan for this paper is as follows: We start with the explicit construction of pure Shape Dynamics on the 2-torus in Sect. 2 and show its equivalence with General Relativity using the method of linking gauge theories. The trading of refoliation invariance for local spatial conformal invariance turns all local constraints into phase space functions that are linear in the momenta, while the remaining Shape Dynamics Hamiltonian turns out to formally coincide with the reduced phase space Hamiltonian, which at large CMC-volume becomes the conformal constraint that changes the total

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[1] The sphere is a degenerate case, since it admits only one canonical pair of degrees of freedom (the volume and the mean extrinsic curvature). This admits only the de Sitter solution which contains no interesting dynamics.

volume. In Sect. 3 we consider Shape Dynamics on a higher-genus surface, which does not admit straightforward solutions to the initial value problem. In this sense the problem of nonlocality for higher genus is closer in spirit to that encountered in higher dimensions. We attack this problem by constructing a perturbation expansion and we recover a fully conformal theory in the large volume regime. We find that the generically nonlocal Hamiltonian becomes the integral over a local density at large volume and turns again into the conformal constraint that changes the total volume. We then use the classical results to formally quantize Shape Dynamics on the torus in Sect. 4. Due to linearity of the local constraints, one can implement them at the quantum level and thus formally quantize the analogue of the Wheeler–DeWitt equation, i.e. infinitely many degrees of freedom. In this construction we refrain from giving a complete description of the Hilbert space, which would require to solve additional measure theoretic problems. In an excursion we discuss the large-CMC-volume asymptotics of Shape Dynamics wave functions, which resembles aspects of a quantum gravity/CFT correspondence. Before we conclude in Sect. 6 we discuss in Sect. 5 how the results of this paper compare to higher dimensions; in particular we compare the 2 + 1 torus to the strong gravity limit.

2 Equivalence of general relativity and shape dynamics

In this section we establish the equivalence between General Relativity and Shape Dynamics on the $(2 + 1)$-dimensional torus universe by explicitly constructing the linking theory relating the two. For simplicity we assume a positive cosmological constant Λ. We start with the general construction of the linking theory before focusing on the torus, which allows us, in contrast to higher dimensions or even on a higher-genus surface, to explicitly work out Shape Dynamics.

Our starting point is the ADM Hamiltonian on the usual ADM phase space Γ_{ADM}, expressed in terms of the metric $g_{ab}(x)$ and its canonically conjugate momentum density $\pi^{ab}(x)$:

$$
\begin{aligned}
H &= S(N) + H(\xi), \\
S(N) &= \int d^2x \, N \left(\frac{1}{\sqrt{|g|}} \pi^{ab} G_{abcd} \pi^{cd} - \sqrt{|g|} \, (R - 2\Lambda) \right), \\
H(\xi) &= \int d^2x \, \pi^{ab} \mathcal{L}_\xi g_{ab}.
\end{aligned}
\tag{1}
$$

Here $G_{abcd} = g_{ac} g_{bd} - g_{ab} g_{cd}$ is the 2-dimensional super-metric, and S and H denote the ADM scalar and diffeomorphism constraints.

The central idea behind Shape Dynamics is that we want to trade the local scalar ADM constraints $S(N)$, which are quadratic in momenta, for local constraints that are linear in momenta, because linear constraints admit a geometric interpretation as generators of transformations of the spatial metric. A priori there is an infinite set of such constraints possible, all of the form $T[g; x)_{ab} \pi^{ab}(x)$, where $T[g, x)_{ab}$ denotes a local symmetric tensor constructed from the metric and its derivatives. Arguably the simplest choice is $\pi(x) = g_{ab}(x) \pi^{ab}(x)$, the generator of spatial

conformal transformations. However, we do not want to trade all scalar constraints, but rather seek to retain one combination of the ADM scalar constraints to generate classical dynamics. The simplest choice to achieve that is to restrict oneself to those conformal transformations that preserve the total spatial volume generated by $\pi(x) - \langle\pi\rangle_g\sqrt{|g|}(x)$. This choice has the surprising feature that one can prove that symmetry trading is always possible, because the volume preserving conformal transformations turn out to generate York scaling.[2] We do not know of any other generator that can be shown to always allow for symmetry trading, but we also lack a uniqueness proof.

To perform this trading of symmetries we construct a linking theory following the "best matching procedure" outlined in [3]. To best match with respect to conformal transformations that preserve the total volume we consider the ADM phase space as a subspace of a larger phase space $\Gamma_{ext} = \Gamma_{ADM} \times \Gamma_\phi$, where Γ_ϕ is the phase space of a scalar field $\phi(x)$, whose canonically conjugate momentum density is denoted by $\pi_\phi(x)$. The phase space functions on Γ_{ADM} are naturally identified with those phase space functions on Γ_{ext} that are independent of (ϕ, π_ϕ). We can thus recover usual ADM gravity in this larger system by introducing an additional first-class constraint

$$Q(x) := \pi_\phi(x) \approx 0 \tag{2}$$

and add it smeared with a Lagrange multiplier $\rho(x)$ to the ADM Hamiltonian, which is now $H = S(N) + H(\xi) + Q(\rho)$. Let us now consider a canonical transformation from $(g_{ab}, \pi^{ab}, \phi, \pi_\phi)$ to $(G_{ab}, \Pi^{ab}, \Phi, \Pi_\phi)$ generated by the generating functional

$$F = \int d^2x \left(g_{ab}(x)e^{2\hat\phi(x)}\Pi^{ab}(x) + \phi(x)\Pi_\phi(x) \right). \tag{3}$$

Here $\hat\phi$ is defined in terms of ϕ by subtracting a spatial average, that has a non-trivial dependence on the metric,

$$\hat\phi(x) := \phi(x) - \frac{1}{2}\ln\left\langle e^{2\phi}\right\rangle_g, \tag{4}$$

where we use the shorthands $\langle f\rangle_g = V_g^{-1}\int d^2x\sqrt{|g|}f$ and $V_g = \int d^2x\sqrt{|g|}$. Notice that we constructed $\hat\phi$ such that the conformal factor $e^{2\hat\phi}$ preserves the total volume. The canonical transformation of the elementary variables, that is generated by (3), can be worked out explicitly:

$$\begin{aligned}
g_{ab}(x) &\to G_{ab}(x) = e^{2\hat\phi(x)}g_{ab}(x),\\
\pi^{ab}(x) &\to \Pi^{ab}(x) = e^{-2\hat\phi(x)}\left(\pi^{ab}(x) - \tfrac{1}{2}\sqrt{|g|}(x)g^{ab}(x)\langle\pi\rangle\left(1 - e^{2\hat\phi(x)}\right)\right),\\
\phi(x) &\to \Phi(x) = \phi(x),\\
\pi_\phi(x) &\to \Pi_\phi(x) = \pi_\phi(x) - 2\left(\pi(x) - \langle\pi\rangle\sqrt{|g|}(x)\right),
\end{aligned} \tag{5}$$

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[2] By York scaling we mean that the transverse part of π^{ab} and π scale with opposite conformal weights, which is important for having a unique solution to the Lichnerowicz–York equation appearing in York's method for solving the initial value problem.

using shorthand notation $\pi(x) = \pi^{ab}(x)g_{ab}(x)$ and $\langle\pi\rangle = V^{-1}\int d^2x\,\pi(x)$. This transformation leads us to the constraints of the linking theory:

$$H = S(N) + H(\xi) + Q(\rho),$$

$$S(N) = \int d^2x\, N\left[\frac{e^{-2\hat{\phi}}}{\sqrt{|g|}}\left(\pi^{ab}G_{abcd}\pi^{cd} - \frac{1}{2}\left(\pi - \langle\pi\rangle(1 - e^{6\hat{\phi}})\sqrt{|g|}\right)^2 + \frac{1}{2}\pi^2\right)\right.$$

$$\left. -\sqrt{|g|}\left(R[g] - 2\Delta\hat{\phi} - 2\Lambda e^{2\hat{\phi}}\right)\right], \tag{6}$$

$$H(\xi) = \int d^2x\, e^{-2\hat{\phi}}\left(\pi^{ab} - \frac{1}{2}\sqrt{|g|}g^{ab}\langle\pi\rangle\left(1 - e^{2\hat{\phi}}\right)\right)\left(\mathscr{L}_\xi e^{2\hat{\phi}}g\right)_{ab}(x),$$

$$Q(\rho) = \int d^2x\, \rho(x)\left(\pi_\phi(x) - 2\left(\pi(x) - \langle\pi\rangle\sqrt{|g|}(x)\right)\right),$$

where $S(N)$, $H(\xi)$ and $Q(\rho)$ are obtained by applying (5) to (1) and (2). One can check that after integrating by parts and using $Q = 0$ the constraint $H(\xi)$ turns into the usual form of the diffeomorphism constraint $H(\xi) = \int d^2x\left(\pi^{ab}\mathscr{L}_\xi g_{ab} + \pi_\phi\mathscr{L}_\xi\phi\right)$. We will use this form of the constraint below. The linking theory thus contains the usual diffeomorphism constraint, a conformal constraint that preserves the total 2-volume and a scalar constraint that arises as a modification of the ADM refoliation constraint.

2.1 Linking theory on the torus

We will now exploit some special properties of two dimensional metrics on the torus to simplify the constraints (6). First of all, it is a well-known fact that all metrics on the torus are conformally flat. The space of flat metrics modulo diffeomorphisms is finite dimensional and admits a convenient parametrization of the space of metrics on the torus. We follow [8,9] where possible.

To make this more explicit let us fix a global chart on the 2-torus $\mathbb{T}^2$, which allows us to uniquely identify any point $x \in \mathbb{T}^2$ with its coordinates $(x^1, x^2) \in [0, 1)^2$. In these coordinates we can decompose an arbitrary metric g_{ab} as

$$g_{ab}(x) = e^{2\lambda(x)}\left(f^*\bar{g}\right)_{ab}(x), \tag{7}$$

where λ is a conformal factor, f is a (small) diffeomorphism and $\bar{g}$ a flat reference metric. We can make this decomposition unique by requiring $\bar{g}$ to be of the form

$$\bar{g} = \frac{1}{\tau_2}\left(dx^1 \otimes dx^1 + \tau_1(dx^1 \otimes dx^2 + dx^2 \otimes dx^1) + (\tau_1^2 + \tau_2^2)dx^2 \otimes dx^2\right) \tag{8}$$

where $\tau = (\tau_1, \tau_2)$ denote the Teichmüller parameters. There is a slight redundancy left in the decomposition having to do with the fact that f is only determined up to an isometry of $\bar{g}$, i.e. up to translations in x^1 and x^2. If we require f to leave $(0, 0)$ invariant, we really obtain a one-to-one map between metrics on the torus and the data (λ, f, τ).

We can explicitly decompose the momentum density[3]

$$\pi^{ab} = e^{-2\lambda}\left(p^{ab} + \frac{1}{2}\pi\bar{g}^{ab} + \sqrt{|\bar{g}|}\left(\bar{D}^a Y^b + \bar{D}^b Y^a - \bar{g}^{ab}\bar{D}_c Y^c\right)\right), \qquad (9)$$

in terms of a trace π, a vector field Y and a transverse traceless tensor density (w.r.t. $\bar{g}$), which we can explicitly parametrize by

$$p = \frac{1}{2}\Bigg(\left((\tau_1^2 - \tau_2^2)p_2 - 2\tau_1\tau_2 p_1\right)\partial_1 \otimes \partial_1$$
$$+ (\tau_2 p_1 - \tau_1 p_2)(\partial_1 \otimes \partial_2 + \partial_2 \otimes \partial_1) + p_2\,\partial_2 \otimes \partial_2\Bigg). \qquad (10)$$

This decomposition is such that π is conjugate to 2λ and p_i is conjugate to τ_i.

Writing the linking theory constraints (6) in terms of λ, f, τ, π, Y and p we get

$$S(N) = \int d^2x N \Bigg[\frac{e^{-2(\hat{\phi}+\lambda)}}{\sqrt{|\bar{g}|}}\left((p^{ab} + (PY^{ab})\sqrt{|\bar{g}|})\bar{g}_{ac}\bar{g}_{bd}(p^{cd} + (PY^{cd})\sqrt{|\bar{g}|})\right.$$

$$-\frac{1}{2}\left(\pi - \langle\pi\rangle\sqrt{|\bar{g}|}\left(1 - e^{2(\hat{\phi}+\lambda)}\right)\right)^2\Bigg) + 2\sqrt{|\bar{g}|}\left(\bar{\Delta}(\hat{\phi}+\lambda) + \Lambda e^{2(\hat{\phi}+\lambda)}\right)\Bigg]$$

$$H(\xi) = \int d^2x \xi^a \left(\sqrt{|\bar{g}|}\bar{\Delta}Y_a + \frac{1}{2}e^{2\lambda}\bar{D}_a\left(e^{-2\lambda}\pi\right) + \pi_\phi\phi_{,a}\right)$$

$$Q(\rho) = \int d^2x \rho(x)\left(\pi_\phi(x) - 2\left(\pi(x) - \langle\pi\rangle\sqrt{|\bar{g}|}(x)e^{2\lambda(x)}\right)\right), \qquad (11)$$

where we used the shorthand $PY_{ab} = \bar{D}_a Y_b + \bar{D}_b Y_a - \bar{g}_{ab}\bar{g}^{cd}\bar{D}_c Y_d$.

To complete the definition of a linking theory, we specify two sets of gauge-fixing conditions,

$$\phi(x) = 0 \quad \text{for GR and} \quad \pi_\phi(x) = 0 \quad \text{for SD}, \qquad (12)$$

which we will now use to reconstruct General Relativity and Shape Dynamics respectively.

2.2 Recovering general relativity

To recover standard ADM gravity on the torus let us impose the gauge-fixing condition $\phi(x) = 0$ to the linking theory. To perform the phase space reduction from the extended phase space to the ADM phase space, we need to fix the Lagrange multipliers such that the gauge-fixing is propagated. Since the momentum density π_ϕ occurs only in the constraints Q, we have to solve

$$0 = \{Q(\rho), \phi(x)\} = \rho(x) \qquad (13)$$

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[3] All indices here are raised with the reference metric $\bar{g}$.

for the Lagrange-multiplier ρ, which implies $\rho = 0$, hence the constraints $Q(x)$ are gauge-fixed and in fact drop out of the Hamiltonian, which becomes independent of π_ϕ. Hence, one can perform the phase space reduction by setting $\phi \equiv 0$, $\rho \equiv 0$ and π_ϕ arbitrary[4] in (6). The Hamiltonian on the ADM phase space thus reads

$$H = S(N) + H(\xi), \tag{14}$$

where $S(N)$ and $H(\xi)$ are precisely the scalar—resp. diffeomorphism constraint of General Relativity in the ADM formulation as given in (1). We note that we explicitly retained refoliation invariance.

2.3 Recovering shape dynamics

To recover Shape Dynamics we employ the gauge-fixing condition $\pi_\phi(x) = 0$. We will see that the decomposed form (11) of the constraints allows us to find the explicit Shape Dynamics Hamiltonian through a phase space reduction $(\phi, \pi_\phi) \to (\phi_0, 0)$. To find this map we can use $\pi_\phi \equiv 0$, so the Q constraints become

$$Q(\rho) = \int d^2x \rho(x) \left(\pi(x) - \langle \pi \rangle \sqrt{|g|(x)} \right), \tag{15}$$

which implies that $\pi(x)$ is a covariant constant. Using this and $\pi_\phi = 0$, we find that the diffeomorphism constraint implies that

$$PY_{ab}(x) = 0, \tag{16}$$

which implies that the scalar constraint is independent of $Y_a(x)$. Using these simplifications in the scalar constraints, we find

$$S(N) = \int d^2x \sqrt{|\bar{g}|} N \left(e^{-2(\hat{\phi}+\lambda)} \frac{\bar{g}_{ac}\bar{g}_{bd} p^{ab} p^{cd}}{|\bar{g}|} - e^{2(\hat{\phi}+\lambda)} \frac{\langle \pi \rangle^2 - 4\Lambda}{2} + 2\bar{\Delta}(\hat{\phi} + \lambda) \right). \tag{17}$$

Using (8) and (10) we find that

$$\frac{\bar{g}_{ac}\bar{g}_{bd} p^{ab} p^{cd}}{|\bar{g}|} = \frac{\tau_2^2}{2}(p_1^2 + p_2^2) \tag{18}$$

is a spatial constant in the chosen chart.

We see that the constraints $S(N)$ would be solved if we were able to choose $e^{4(\hat{\phi}+\lambda)} = \frac{2\bar{g}_{ac}\bar{g}_{bd} p^{ab} p^{cd}}{|\bar{g}|(\langle \pi \rangle^2 - 4\Lambda)}$. However, this is in general obstructed by the volume-preservation condition $\int d^2x \sqrt{|\bar{g}|} e^{2(\hat{\phi}+\lambda)} = V$. This means that the constraints generating

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[4] Had the constraints $Q(\rho)$ not dropped out after gauge fixing ρ, we would have had to solve $Q \equiv 0$ for π_ϕ to complete the phase space reduction.

the refoliations are not completely gauge fixed by the condition $\pi_\phi(x) = 0$. Indeed it turns out that among the infinitely many constraints $S(N)$ one remains first class, which after phase space reduction becomes our Shape Dynamics Hamiltonian H_{SD}. More concretely, there exists a lapse N_0 such that $S(N_0)$ Poisson commutes with π_ϕ, i.e. it satisfies the lapse fixing equation

$$\{S(N_0), \pi_\phi(x)\} = F_{N_0}(x) - e^{2(\hat{\phi}+\lambda)}\sqrt{|\bar{g}|}\langle F_{N_0}\rangle = 0$$

$$\text{where } F_N = N\left(-2e^{-2(\hat{\phi}+\lambda)}\bar{g}_{ac}\bar{g}_{bd}\,p^{ab}p^{cd} - |\bar{g}|e^{2(\hat{\phi}+\lambda)}(\langle\pi\rangle^2 - 4\Lambda)\right) + \sqrt{|\bar{g}|}\bar{\Delta}N.$$

(19)

If one imposes on N_0 a normalization condition $\int d^2x\sqrt{|\bar{g}|}e^{2(\hat{\phi}+\lambda)}N_0 = V$, then (19) has a unique solution. We want to project out this first-class part $S(N_0)$ from the full set of constraints $S(x)$ to end up with a purely second-class set of constraints $\widetilde{S}(x)$ that we can solve. We can perform the projection in different ways, but a particularly convenient way of doing this is by defining

$$\widetilde{S}(x) := S(x) - \frac{S(N_0)}{V}\sqrt{|\bar{g}|(x)}e^{2(\hat{\phi}(x)+\lambda(x))},$$

(20)

which automatically satisfies $\widetilde{S}(N_0) = 0$. Identifying $H_{\text{SD}} = S(N_0)$, we arrive at the modified Lichnerowicz–York equations

$$0 = \widetilde{S}(x) = \sqrt{|\bar{g}|}\left(e^{-2(\hat{\phi}+\lambda)}\frac{\bar{g}_{ac}\bar{g}_{bd}\,p^{ab}p^{cd}}{|\bar{g}|}\right.$$
$$\left. -\frac{1}{2}e^{2(\hat{\phi}+\lambda)}\left(\langle\pi\rangle^2 - 4\Lambda + 2\frac{H_{\text{SD}}}{V}\right) - 2\bar{\Delta}(\hat{\phi}+\lambda)\right),$$

(21)

$$V = \int d^2x\sqrt{|\bar{g}|}e^{2(\hat{\phi}+\lambda)},$$

which we need to solve for $\hat{\phi}$ and H_{SD}. A solution is found by taking $\hat{\phi} + \lambda$ to be spatially constant. More precisely, from the second equation it follows that

$$\hat{\phi} = -\lambda + \frac{1}{2}\ln V.$$

(22)

Now H_{SD} can be easily determined from the first equation in (21),

$$H_{\text{SD}} = \frac{1}{V}\frac{\bar{g}_{ac}\bar{g}_{bd}\,p^{ab}p^{cd}}{|\bar{g}|} - \frac{V}{2}\left(\langle\pi\rangle^2 - 4\Lambda\right) = \frac{\tau_2^2}{2V}(p_1^2 + p_2^2) - \frac{V}{2}\left(\langle\pi\rangle^2 - 4\Lambda\right).$$

(23)

Notice that to find $H_{\text{SD}} = S(N_0)$ we did not have to solve the lapse fixing equation explicitly. In this case we can solve (19) straightforwardly using the fact that $\hat{\phi} + \lambda$ is constant and the result is simply $N_0 = 1$. In general however the lapse fixing equation

is quite complicated and we are lucky that we don't actually have to solve it to derive the Shape Dynamics Hamiltonian (as we will again see in Sect. 3). As a matter of fact, as a constraint $\widetilde{S} = 0$ is completely equivalent to

$$S(x) - \langle S \rangle \sqrt{|\bar{g}|(x)} e^{2(\hat{\phi}(x) + \lambda(x))} = 0, \tag{24}$$

which does not refer to a lapse at all.

The Shape Dynamics Hamiltonian H_{SD} (23) is exactly the reduced phase space Hamiltonian constraint. The more familiar Hamiltonian H_{York} generating evolution in York time $\langle \pi \rangle$ (see e.g. [8] section 3.3) is obtained by noting that the variable canonically conjugate to $\langle \pi \rangle$ is V and therefore by solving $H_{\text{SD}} = 0$,

$$H_{\text{York}} = V = \tau_2 \frac{\sqrt{p_1^2 + p_2^2}}{\sqrt{\langle \pi \rangle^2 - 4\Lambda}}. \tag{25}$$

We can now perform explicitly the phase space reduction of the linking theory and describe Shape Dynamics on the ADM phase space through its total Hamiltonian and first-class constraints

$$\begin{aligned}
H &= \mathcal{N} H_{\text{SD}} + H(\xi) + C(\rho) \\
H_{\text{SD}} &= \frac{\tau_2^2}{2V}(p_1^2 + p_2^2) - \frac{V}{2}\left(\langle \pi \rangle^2 - 4\Lambda\right) \\
H(\xi) &= \int d^2 x \pi^{ab} \mathscr{L}_\xi g_{ab} \\
C(\rho) &= \int d^2 x \rho \left(\pi - \langle \pi \rangle \sqrt{|g|}\right).
\end{aligned} \tag{26}$$

The gauge symmetries are indeed spatial diffeomorphisms, conformal transformations that preserve the total volume and global time reparametrizations. Despite the different set of symmetries, the equivalence with standard General Relativity is obvious: the Shape Dynamics Hamiltonian coincides on the reduced phase with the CMC Hamiltonian, while the constraints C provide the CMC gauge-fixing conditions.

Although we know the Shape Dynamics Hamiltonian explicitly on the torus, it is instructive to observe that the Shape Dynamics Hamiltonian constraint H can be expanded in powers of the inverse volume, because it shows two properties that we can investigate in more complicated models. This expansion is a systematic approximation to Shape Dynamics that is a good approximation in an asymptotic large volume regime, i.e. where $V \to \infty$ while keeping the other degrees of freedom finite. In this regime we find two important features of Shape Dynamics:

1. **Asymptotic locality:** The leading order of the Hamiltonian, which becomes exact in the limit $V \to \infty$, is $\langle \pi \rangle^2 - 4\Lambda + \mathscr{O}(V^{-2}) \approx 0$. As a constraint, this is equivalent to

$$V\left(\langle \pi \rangle - 2\sqrt{\Lambda}\right) = \int d^2 x \left(\pi(x) - 2\sqrt{\Lambda}\sqrt{|g|}(x)\right) \approx 0 \quad \text{for } V \to \infty \tag{27}$$

which is diffeomorphism invariant as the integral over a *local* density and by inspection invariant under conformal transformations that preserve the total volume.

2. **Full conformal invariance:** Since the Shape Dynamics Hamiltonian constraint is asymptotically equivalent to $\langle\pi\rangle - const. \approx 0$, we can add it to the conformal constraints C to obtain in the large volume limit $C(x) + H_{SD} = \pi(x) - const. \approx 0$, which generates full conformal transformations, i.e., including those that change the total spatial volume. Notice that this requires us to interpret the Shape Dynamics Hamiltonian as a constraint, rather than a generator of physical dynamics.

Let us have a quick look at the **2-sphere**: the linking theory and phase space reduction can be performed following the same steps as on the torus with two small modifications. (1) there are no Teichmüller parameters on the sphere, so there is only one canonical pair of physical degrees of freedom (V and $\langle\pi\rangle$); (2) the total spatial curvature does not vanish, but is 8π. Shape Dynamics on the sphere thus takes the form of (26), except for the Hamiltonian, which is $H_{SD} = -\frac{V}{2}(\langle\pi\rangle^2 - 4\Lambda) - 8\pi$.

3 Higher-genus surfaces

On a higher-genus surface, we can still use the decomposition analogous to (7) and (9), but the explicit construction of the Shape Dynamics Hamiltonian constraint on the torus rested on the explicit solvability of the modified Lichnerowicz–York equation (21). The Lichnerowicz–York equation on a higher-genus surface (or in higher dimensions) is not explicitly solvable. We thus restrict our construction of Shape Dynamics to an approximation scheme and consider an expansion that becomes exact in the large-volume limit.[5] Again, we follow [8,9] when possible.

3.1 Preparations

In genus $g \geq 2$ we can perform a decomposition analogous to (7) and (9),

$$
\begin{aligned}
g_{ab}(x) &= e^{2\lambda(x)}(f^*\bar{g})_{ab}(x), \\
\pi^{ab}(x) &= e^{-2\lambda}\left(p^{ab}(x) + \frac{1}{2}\bar{g}^{ab}(x)\pi(x) + \sqrt{|\bar{g}|}(x)\bar{g}^{ab}(x)\bar{g}^{cd}(x)PY_{cd}(x)\right),
\end{aligned}
\tag{28}
$$

where now we take the reference metric $\bar{g}$ to be of unit volume $\int d^2x\sqrt{|\bar{g}|} = 1$ and constant scalar curvature $\bar{R}$. According to the Gauss–Bonnet theorem, $\bar{R}$ is given by

$$
\bar{R} = -8\pi(g-1).
\tag{29}
$$

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[5] See [2] for an analogous expansion.

Modulo diffeomorphisms the space of such metrics corresponds to the genus g Teich-müller space, which has dimension $6g - 6$. Unfortunately no simple explicit para-metrization for $\bar{g}$ is known, so we will keep the parametrization implicit.

We can again write the linking theory using the decomposition (28). The only dif-ference compared to the constraints (11) for the torus is the subtraction from $S(N)$ of a spatial curvature term.

When we impose the gauge fixing $\pi_\phi = 0$, we obtain the analogue of (17):

$$S(N) = \int d^2x \sqrt{|\bar{g}|} N \left(e^{-2(\hat{\phi}+\lambda)} \frac{\bar{g}_{ac}\bar{g}_{bd} p^{ab} p^{cd}}{|\bar{g}|} - e^{2(\hat{\phi}+\lambda)} \frac{\langle \pi \rangle^2 - 4\Lambda}{2} + 2\bar{\Delta}(\hat{\phi}+\lambda) - \bar{R} \right). \tag{30}$$

Reusing our discussion for the torus, we construct the second-class part $\widetilde{S}$ of S accord-ing to (24),

$$\widetilde{S}(x) = S(x) - \langle S \rangle \sqrt{|\bar{g}|(x)} e^{2(\hat{\phi}(x)+\lambda(x))}. \tag{31}$$

Identifying the remaining first-class constraint $\langle S \rangle$ with H_{SD}/V, we obtain the modi-fied Lichnerowicz–York equations for genus $g \geq 2$,

$$0 = \widetilde{S}(x) = \sqrt{|\bar{g}|} \left(e^{-2(\hat{\phi}+\lambda)} \frac{\bar{g}_{ac}\bar{g}_{bd} p^{ab} p^{cd}}{|\bar{g}|} \right.$$
$$\left. - \frac{1}{2} e^{2(\hat{\phi}+\lambda)} \left(\langle \pi \rangle^2 - 4\Lambda + 2\frac{H_{SD}}{V} \right) + 2\bar{\Delta}(\hat{\phi}+\lambda) - \bar{R} \right), \quad (32)$$
$$V = \int d^2x \sqrt{|\bar{g}|} e^{2(\hat{\phi}+\lambda)}.$$

To simplify the notation let us define $\mu = \hat{\phi} + \lambda - \frac{1}{2} \ln V$ and $p^2 := \frac{\bar{g}_{ac}\bar{g}_{bd} p^{ab} p^{cd}}{|\bar{g}|}$. Then (32) can be written

$$\frac{1}{V} p^2 e^{-2\mu} - \frac{1}{2} V \left(\langle \pi \rangle^2 - 4\Lambda + \frac{2H_{SD}}{V} \right) e^{2\mu} + 2\bar{\Delta}\mu - \bar{R} = 0 \quad \text{and} \quad \langle e^{2\mu} \rangle_{\bar{g}} = 1. \tag{33}$$

In the following we will drop the subscript $\bar{g}$ and keep in mind that averages $\langle \cdot \rangle$ are taken with respect to $\bar{g}$ (except for $\langle \pi \rangle$).

Equation (33) is nearly identical to the standard Lichnerowicz–York equation in $2 + 1$ dimensions. The only difference is that we have a restriction on μ and to com-pensate this we have an additional constant H_{SD} to solve for. The existence of a unique solution for μ and H_{SD} (as a function of $\bar{g}_{ab}$, p^{ab}, V and $\langle \pi \rangle$) is a direct consequence of the existence and uniqueness properties of the usual Lichnerowicz–York equation.

The key simplification that allowed us to explicitly construct Shape Dynamics on the torus is that there one can choose the constant curvature metric $\bar{g}_{ab}$ such that $\bar{g}_{ac}\bar{g}_{bd} p^{ab} p^{cd}/|\bar{g}|$ is spatially constant (as is apparent from (7)). For genus 2 and

higher the LY equation is much harder to solve. However, we can already deduce some properties of H_{SD} by integrating expression (33),

$$H_{SD} = -\frac{V}{2}(\langle\pi\rangle^2 - 4\Lambda) - \bar{R} + \frac{1}{V}\langle p^2 e^{-2\mu}\rangle. \tag{34}$$

We have chosen our second-class constraints (31) in such a way that the solution μ will not depend on $\langle\pi\rangle$ (or Λ), and therefore the same holds for the last term in (34). Hence, our choice is special in that it produces a Hamiltonian that is quadratic in the momentum $\langle\pi\rangle$ conjugate to V.

3.2 Large-volume asymptotic expansion

Although we can not solve (33) explicitly, our modified LY equation does allow for an interesting perturbative expansion. Indeed notice that the volume V appears explicitly in (33) and should be treated as a parameter when solving the equation. Therefore we can try to find solutions μ and H_{SD} expanded in powers of $1/V$ and construct the Shape Dynamics in the infinite volume limit.[6] To do this we make the ansatz

$$e^{2\mu} = \sum_{k=0}^{\infty} \Omega_k V^{-k} \quad \text{and} \quad H_{SD} = \sum_{k=-1}^{\infty} H_k V^{-k}. \tag{35}$$

Indeed, from carefully looking at (33) it follows that higher powers of V can not occur. From the normalization $\langle e^{2\mu}\rangle = 1$ we get the restrictions $\langle\Omega_0\rangle = 1$ and $\langle\Omega_k\rangle = 0$ for $k > 0$.

The leading order of (33) is proportional to V and fixes $H_{-1} = -\frac{1}{2}(\langle\pi\rangle^2 - 4\Lambda)$. At order V^0 the equation then reads

$$-\Omega_0 H_0 - \bar{R} + 2\bar{\Lambda}\ln(\Omega_0) = 0, \tag{36}$$

which is clearly solved by $H_0 = -\bar{R}$ and $\Omega_0 = 1$. The LY equation now becomes

$$\frac{1}{V}\frac{1}{1 + \sum_k \frac{\Omega_k}{V^k}}p^2 - \left(1 + \sum_k \frac{\Omega_k}{V^k}\right)\left(-\bar{R} + \sum_k \frac{H_k}{V^k}\right)$$

$$-\bar{R} + \bar{\Lambda}\ln\left(1 + \sum_k \frac{\Omega_k}{V^k}\right) = 0. \tag{37}$$

If we define the polynomials A_k and B_k in Ω_1 through Ω_k by[7]

---

[6] Notice that this can not easily be done in the reduced phase space approach, in which also the volume itself has to be solved for in terms of $\bar{g}$ and the momenta.

[7] The first few polynomials are $A_0 = 1$, $A_1 = -\Omega_1$, $A_2 = -\Omega_2 + \Omega_1^2$, $A_3 = -\Omega_3 + 2\Omega_1\Omega_2 - \Omega_1^3$ and $B_0 = 0, B_1 = -\Omega_1^2/2$, $B_2 = \Omega_1^3/3 - \Omega_1\Omega_2$.

$$\frac{1}{1 + \sum_k \frac{\Omega_k}{V^k}} = \sum_k \frac{A_k[\Omega]}{V^k} \quad \text{and} \quad \ln\left(1 + \sum_k \frac{\Omega_k}{V^k}\right) - \sum_k \frac{\Omega_k}{V^k} = \frac{1}{V} \sum_k \frac{B_k[\Omega]}{V^k}$$

(38)

then from (34) or from integrating (37) it follows that for $k \geq 1$

$$H_k = \left\langle p^2 A_{k-1}[\Omega] \right\rangle.$$

(39)

The order V^{-k} equation in (37) then allows us to solve Ω_k explicitly in terms of Ω_0 through Ω_{k-1},

$$\Omega_k = (\bar{\Delta} + \bar{R})^{-1} \left(-A_{k-1}[\Omega] p^2 - \bar{\Delta} B_{k-1}[\Omega] + \sum_{l=1}^k H_l \Omega_{k-l} \right),$$

(40)

where the operator $\bar{\Delta} + \bar{R}$ is negative definite and therefore has a well-defined inverse.

We have therefore obtained a general algorithm to solve the modified LY equation order by order through the recurrence relation (40) together with (39). We have calculated the first few H_k explicitly leading to a Hamiltonian

$$H_{\text{SD}} = -\frac{V}{2}(\langle \pi \rangle^2 - 4\Lambda) - \bar{R} + \frac{1}{V}\langle p^2 \rangle + \frac{1}{V^2}\left\langle (p^2 - \langle p^2 \rangle)(\bar{\Delta} + \bar{R})^{-1}(p^2 - \langle p^2 \rangle) \right\rangle$$
$$+ \frac{1}{2V^3}\left[\bar{R}\left\langle ((\bar{\Delta} + \bar{R})^{-1}(p^2 - \langle p^2 \rangle))^3 \right\rangle \right.$$
$$\left. + 3\left\langle (p^2 + \langle p^2 \rangle)((\bar{\Delta} + \bar{R})^{-1}(p^2 - \langle p^2 \rangle))^2 \right\rangle \right] + \cdots.$$

(41)

In general H_k will be a function homogeneous in p^2 of order k.[8]

The expansion has features similar to a tree level (Feynman diagram) expansion with propagator $(\bar{\Delta} + \bar{R})^{-1}$ and source term p^2/V. To make this connection more explicit, let us view the modified LY equation (33) as the Euler–Lagrange equation of some action S_{LY}. Such an action can be easily constructed,

$$S_{LY}[\mu, H] = \int d^2x \sqrt{\bar{g}} \left(\mu \bar{\Delta} \mu - \bar{R}\mu - \left(H - \frac{\bar{R}}{2} \right)(e^{2\mu} - 1) - \frac{1}{2}\frac{e^{-2\mu}}{V}p^2 \right),$$

(42)

where μ is now viewed as an unrestricted function since the Lagrange multiplier $H = \frac{1}{2}(H_{\text{SD}} + \frac{V}{2}(\langle \pi \rangle^2 - 4\Lambda) + \bar{R})$ enforces the constraint $\langle e^{2\mu} \rangle = 1$ on variation.

---

[8] As the H_k are actually functions on the cotangent bundle to Teichmüller space, one might ask how natural they are from the perspective of Teichmüller spaces. As a partial answer we notice that $H_1 = \langle p^2 \rangle$ is related to the canonical Weil–Petersson metric [20,21], while H_2 is closely related to its curvature [22].

We can rewrite (42) by singling out the quadratic part in μ and H,

$$
S_{LY}[\mu, H] = \int d^2x \sqrt{\bar{g}} \left(\mu(\bar{\Delta} + \bar{R})\mu - 2\mu H + \bar{R}\left(\frac{2}{3}\mu^3 + \frac{1}{3}\mu^4 + \cdots\right) \right.
$$
$$
\left. - H(2\mu^2 + \cdots) - \frac{1}{2}\frac{e^{-2\mu}}{V}p^2 \right). \tag{43}
$$

The Feynman rules can be read off and we can find H (and therefore H_{SD}) by computing its tree-level one-point function. Notice that the action (42) which we use to derive the Hamiltonian is similar to that of two dimensional Liouville gravity [12]. More precisely, it is of the form of a Liouville action plus a perturbation given by a source term proportional to p^2/V.

Two remarks are in order:

1. We observe from (34) that in the limit $V \to \infty$ the Shape Dynamics Hamiltonian again approaches a form that is equivalent to (27). The Hamiltonian is thus asymptotically local and provides the volume-changing generator of conformal transformations, so full conformal invariance is asymptotically attained.
2. The first three terms in the large-volume expansion (41) sum up to an expression equivalent to the temporal gauge Hamiltonian $S(N \equiv 1)$, which is *local*.

4 Dirac quantization in metric variables

To expose the difference between Shape Dynamics and General Relativity, we consider the Dirac quantization of pure gravity on the torus in $2 + 1$ dimensions in metric variables, usually referred to as Wheeler–DeWitt quantization. For the sake of completeness we first follow [8] and revisit the problems associated with the nonlocality arising from the solution of the diffeomorphism constraint in the Wheeler–DeWitt approach. Subsequently we show how these problems are solved by trading local Hamiltonian constraints for local conformal constraints, which allows us to perform a Dirac quantization program for Shape Dynamics.

4.1 Dirac quantization of general relativity on the $2 + 1$ torus

Contrary to first order variables, one can not readily quantize General Relativity in metric variables even in $2 + 1$ dimensions on the torus and sphere. The reason is well explained in [8], which we follow here. Using the standard decomposition of the metric and momenta, we can solve the diffeomorphism—constraints for the transverse part of the momenta

$$
\bar{Y}_i = -\frac{1}{2}\left(\bar{\Delta} + \frac{k}{2}\right)^{-1}\left(e^{2\lambda}\bar{\nabla}_i\left(e^{-2\lambda}\pi\right)\right), \tag{44}
$$

where $k = 0$ for the torus. To perform a Wheeler–DeWitt quantization reference [8] chooses a polarization for which the configuration operators are given by functionals

of the spatial metric and formally considers a Schrödinger representation on wave functions $\psi[f, \lambda; \tau]$, which reduces to $\psi[\lambda; \tau]$. Assuming that the inner product is constructed from a divergence-free measure, we can quantize the momenta by replacing $\pi \to -\frac{i}{2} \frac{\delta}{\delta \lambda}$ and $p^{ab} \to -i \frac{\partial}{\partial \hat{g}_{ab}}$. The diffeomorphism constraint on the torus still acts nontrivially on the conformal factor, and its solution can be quantized as

$$\bar{Y}_i[\hat{\pi}] = \frac{i}{4} \bar{\Delta}^{-1} \left(e^{2\lambda} \bar{\nabla}_i \left(e^{-2\lambda} \frac{\delta}{\delta \lambda} \right) \right). \tag{45}$$

This expression is plugged into the Hamiltonian and leads to nonlocal terms in the Wheeler–DeWitt equation that are not practically manageable and lead to notorious difficulties in the Wheeler–DeWitt approach [7]. To make a connection with reduced phase space quantization reference [8] assumes a solution $\psi_o[\lambda; \tau]$ to the Wheeler–DeWitt equation and restricts it to constant York time T through

$$\hat{\psi}_o(T, \tau) := \int \mathscr{D}\lambda e^{iT \int d^2x e^{2\lambda}} \psi_o[\lambda; \tau] \tag{46}$$

and inserts this into the Wheeler–DeWitt equation (with vanishing cosmological constant). In terms of the Teichmüller Laplacian $\Delta_o := -\tau_2^2 \left(\partial_{\tau_1}^2 + \partial_{\tau_2}^2 \right)$ this yields

$$\left(\left(T \frac{\partial}{\partial T} \right)^2 + \Delta_o \right) \hat{\psi}_o(T, \tau) = \int D\lambda e^{iT \int d^2x e^{2\lambda}} \left(T^2 \left(e^{4\lambda} - V^2 \right) + 4e^{2\lambda} \bar{\Delta}\lambda \right) \psi_o[\lambda; \tau], \tag{47}$$

where the RHS vanishes if ψ_o has support only on spatially constant conformal factors, while the LHS is equivalent to a reduced phase space quantization.

4.2 Dirac quantization of shape dynamics on the 2 + 1 torus

We now follow essentially the same strategy as in the previous subsection but for Shape Dynamics. We choose a polarization where functionals of the metric are configuration variables and formally consider a Schrödinger representation on functionals $\psi[\lambda, f; \tau]$, such that functionals of the metric are represented by multiplication operators. We would like to specify a Hilbert space by defining it as the space of square integrable functionals with respect to a measure $D\lambda \, Df \, d^2\tau$, but it is notoriously difficult to construct an explicit measure Df on the diffeomorphism group, such that the Hilbert space is separable and supports the diffeomorphism generators as essentially self-adjoint operators. We will thus refrain from such a construction and rather *assume* that there exists a measure Df such that the operators $U_{f_o}\psi[\lambda, f; \tau] := \psi[f_o^*\lambda, f_o \circ f; \tau]$ is unitary. We will also assume that there is a measure $D\lambda$ such that $i \int d^2x \rho(x) \frac{\delta}{\delta \lambda(x)}$ extends to an essentially self-adjoint operator for all smooth smearing functions $\rho(x)$.

We seek a representation of the local constraints of Shape Dynamics, whose non-vanishing Poisson brackets are:

$$\{H(\xi), C(\rho)\} = C(\mathscr{L}_\xi \rho) \quad \text{and} \quad \{H(\xi_1), H(\xi_2)\} = H([\xi_1, \xi_2]). \tag{48}$$

We start with the local conformal constraint $\pi(x) - \frac{\sqrt{|\bar{g}|}e^{2\lambda(x)}}{V}\int d^2y\,\pi(y) = 0$, which, taking into account the aforementioned assumptions about the measure $D\lambda\,Df\,d^2\tau$, can be readily quantized as

$$-\frac{i}{2}\left(\frac{\delta}{\delta\lambda(x)} - \frac{\sqrt{|\bar{g}|}e^{2\lambda(x)}}{V}\int d^2y\,\frac{\delta}{\delta\lambda(y)}\right)\psi[\lambda, f; \tau] = 0, \tag{49}$$

where we work in a chart where the components of $\bar{g}_{ab}$ are constant. The solution to this constraint is that ψ depends only on the homogeneous mode of $\lambda(x)$. We can thus write the general solution to the local conformal constraints as a wave function of $\psi[f; V, \tau]$, where V denotes the spatial volume.

We now turn to the spatial diffeomorphism constraint. Exponentiating the spatial diffeomorphism constraint to finite diffeomorphisms implies that for each small diffeomorphism f_o there is a unitary operator acting as the pull-back under a diffeomorphism:

$$U_{f_o}\psi : [\lambda, f; \tau] \mapsto \psi[f_o^*\lambda, f_o \circ f; \tau), \tag{50}$$

where we assumed above that U_{f_o} is unitary. The pull-back action $f_o^*\lambda$ on the conformal factor is the source of the nonlocal terms that we encountered in the action of the diffeomorphisms in the previous subsection. This action is however trivial on the space of solutions to the local conformal constraint, since $f_o^*V = V$. We can thus easily impose the diffeomorphism constraint

$$U_{f_o}\psi[f; V, \tau] = \psi[f_o \circ f; V, \tau) \equiv \psi[f, V, \tau) \tag{51}$$

for all diffeomorphisms f_o, which implies for solutions to the local conformal constraint that $\psi[f; V, \tau)$ is independent of f. We thus find that the solution space to the local constraints of Shape Dynamics consists of Schrödinger wave functions $\psi(V, \tau)$. We would have ended up with an induced measure $d\mu(V, \tau)$ for these functions had we specified an explicit measure at the beginning, but due to the formal nature of our discussion, we do not have such a result.

To proceed, we assume from now on that the wave functions $\psi(V, \tau)$ are elements of the Hilbert space $\mathscr{H}_o$ used in reduced phase space quantization [17]. We now consider the Shape Dynamics Hamiltonian $H_{SD} = \tau_2^2\left(p_1^2 + p_2^2\right) - V\left(\langle\pi\rangle^2 - 4\Lambda\right)$, which can be quantized on the factor $\mathscr{H}_o$ that remains after solving the linear constraints by replacing $p_i \to -i\frac{\partial}{\partial\tau^i}$ and $\langle\pi\rangle \to -i\frac{\partial}{\partial V}$. This leads to the quantum Shape Dynamics Hamiltonian

$$H = -\tau_2^2\left(\partial_{\tau_1}^2 + \partial_{\tau_2}^2\right) + V^2\left(\partial_V^2 + 4\Lambda\right). \tag{52}$$

This is the covariant reduced phase space Hamiltonian [17]. We thus confirmed the expectation of the last subsection that Dirac quantization of Shape Dynamics should be equivalent to reduced phase space quantization.

Excursion: Large volume behavior of the quantum theory

We saw in the classical theory that full conformal invariance is attained in the large-volume regime. Let us now investigate this question in the quantum theory. This is particularly interesting in light of a large-CMC-volume/conformal field theory correspondence. For a first investigation we neglect the issue of modular invariance and simply investigate the asymptotic volume dependence of solutions $\psi(V, \tau_1, \tau_2)$ to the Shape Dynamics Wheeler–DeWitt equation

$$\left(-\tau_2^2 \left(\partial_{\tau_1}^2 + \partial_{\tau_2}^2\right) + V^2 \left(\partial_V^2 + 4\Lambda\right)\right) \psi(V, \tau_1, \tau_2) = 0. \tag{53}$$

We can use a separation ansatz $\psi(V, \tau_1, \tau_2) = v(V)m(\tau_1, \tau_2)$ and introduce separation constants α. This implies $\tau_2^2 \left(\partial_{\tau_1}^2 + \partial_{\tau_2}^2\right) m(\tau_1, \tau_2) = \alpha m(\tau_1, \tau_2)$ and

$$v''(V) + \left(4\Lambda - \frac{\alpha}{V^2}\right) v(V) = 0, \tag{54}$$

which is solved by

$$v_\pm(V) = \sqrt{V\pi\sqrt{\Lambda}} e^{\pm i \frac{\pi}{2} \left(\frac{1}{2} + \sqrt{\alpha + 1/4}\right)} \left(J_{\sqrt{\alpha+1/4}}(2\sqrt{\Lambda}V) \pm i Y_{\sqrt{\alpha+1/4}}(2\sqrt{\Lambda}V)\right) \tag{55}$$

in terms of Bessel functions J_ν, Y_ν. We are interested in the limit $V \to \infty$, where the term $\frac{\alpha}{V^2}$ vanishes and the two linearly independent asymptotic solutions are

$$v_\pm(V) = e^{\pm i 2\sqrt{\Lambda}V} \left(1 + \mathcal{O}\left(\frac{1}{V}\right)\right) \quad \text{for } V \to \infty, \tag{56}$$

which is at leading order *independent* of the separation constant α and hence asymptotically true for all solutions to the Shape Dynamics Wheeler–DeWitt equation. The general asymptotic scaling under global conformal transformations $C = -iV\partial_V$ is thus

$$Cv_\pm(V) = \pm 2\sqrt{\Lambda}Vv_\pm(V) \quad \text{for } V \to \infty. \tag{57}$$

The difference between the Shape Dynamics wave function and the reduced phase space quantization wave function is that the Shape Dynamics wave function is formally (when the assumptions in the previous section hold) a function of the full metric $\psi[g]$ that is constrained to be independent of the local degrees of freedom (by diffeomorphism—and local conformal invariance) and is thus completely specified by its dependence on V, τ_1, τ_2, while the reduced phase space quantization wave function

is a function $\psi(V, \tau_1, \tau_2)$ on reduced phase space only. We thus have the asymptotic scaling of a generic Shape Dynamics wave function under global conformal transformations

$$C\psi_\pm[g] = \pm 2\sqrt{\Lambda} V \psi_\pm[g] \text{ for } V \to \infty, \tag{58}$$

which can be combined with manifest invariance of $\psi[g]$ under volume preserving conformal transformations $C_{VPCT}(x)\psi_\pm[g] = 0$ to yield the asymptotic scaling of $\psi[g]$ under arbitrary conformal transformations

$$C(x)\psi_\pm[g] = \pm 2\sqrt{\Lambda}\sqrt{|g|(x)}\psi_\pm[g] \text{ for } V \to \infty. \tag{59}$$

Notice the explicit deviations from this scaling at order $1/V$, which can be derived by straightforward series expansion of $v_\pm(V)$. Compare this with [11] where a similar result is discussed in the context of the AdS/CFT-correspondence and [14] for a large CMC-volume/CFT discussion.

5 Comparison with shape dynamics in higher dimensions

Pure gravity in $2 + 1$ dimensions does not possess local degrees of freedom, which is the technical reason for many explicit constructions that can *not* be performed in higher dimensions. However, having an example that can be worked out explicitly can hint towards strategies for the treatment of higher dimensions. It is the purpose of this section to explore one hint for higher dimensions that can be drawn from Shape Dynamics on a torus in $2 + 1$ dimensions. A generalization in $3 + 1$ dimensions of the simplifications occurring on the $2+1$ torus universe is the strong gravity or BKL-limit, where spatial derivatives in the ADM Hamilton constraints can be neglected, so

$$S_{ADM}^{BKL}(x) = \frac{\pi^{ab}(x)G_{abcd}(x)\pi^{cd}(x)}{\sqrt{|g|(x)}} + 2\Lambda\sqrt{|g|(x)}. \tag{60}$$

In this case we can algebraically solve the analogue of the Lichnerowicz–York equation and $\langle e^{6\hat\phi} \rangle = 1$ for the Shape Dynamics BKL Hamiltonian in $3 + 1$ dimensions

$$H_{SD}^{BKL} = V\left(\left\langle \sqrt{\sigma^{ab}g_{ac}g_{bd}\sigma^{cd}} \right\rangle_g^2 - \frac{1}{6}\langle\pi\rangle_g^2 + 2\Lambda\right), \tag{61}$$

where $\sigma^{ab} = \pi^{ab} - \frac{1}{3}\pi g^{ab}$. We observe that the ingredients V, $\langle\sqrt{Tr(\sigma.\sigma)}\rangle$ and $\langle\pi\rangle$ are each invariant under volume preserving conformal transformations as well as invariant under spatial diffeomorphisms at the expense of *nonlocality*. The absence of spatial derivatives is what allows us to explicitly construct the BKL-Shape Dynamics Hamiltonian in higher dimensions, i.e. for a similar reason for which the Shape

Dynamics Hamiltonian could be explicitly constructed for the spherical- and torus-universe in $2 + 1$ dimensions.[9]

In an effort to include spatial derivatives, we could decompose a 3-dimensional metric analogously into a conformal factor, diffeomorphism and reference metric to the decomposition in 2 dimensions. For this we observe that locally one can specify a diffeomorphism class of a metrics by giving three independent curvature invariants e.g. $(\phi_1, \phi_2, \phi_3) = (R, R_{ab}R^{ab}, \frac{|R|}{|g|})$ and integration constants τ. Using the Yamabe problem on a compact manifold without boundary, one can then impose $R(x) = \langle R \rangle$ as a gauge fixing condition for the conformal factor and thus locally find reference metrics $g_{ab}[\langle R \rangle, \phi_2, \phi_3, \tau; x]$. Despite this being a *purely* formal construction, since in contrast with the 2-torus and sphere the construction of reference metrics is not feasible as it requires the inversion of a complicated system of coupled partial differential equations, one finds that one can still not solve for the Shape Dynamics Hamiltonian in all phase space, because of the position dependence of $\frac{\sigma^{ab}\sigma_{ab}}{|g|}(x)$. Let us therefore consider the restricted phase space

$$\Gamma_r = \left\{ (g, \pi) \in \Gamma_{ADM} : \frac{\sigma^{ab}\sigma_{ab}}{|g|}(x) = \left\langle \frac{\sigma^{ab}\sigma_{ab}}{|g|} \right\rangle \text{ and } R(x) = \langle R \rangle \right\}. \quad (62)$$

Since we can use $R(x) = \langle R \rangle =: R_o$ as a gauge fixing for the conformal gauge symmetry on a compact manifold without boundary, we see that the condition $\frac{\sigma^{ab}\sigma_{ab}}{|g|}(x) = \langle \frac{\sigma^{ab}\sigma_{ab}}{|g|} \rangle$ constrains one local physical degree of freedom, so Γ_r contains only 3/4 of the physical degrees of freedom of Shape Dynamics. The fact that the solution to the modified Lichnerowicz–York equation is homogeneous if all coefficient functions are homogeneous and that the volume preservation condition for the conformal factor implies that a homogeneous conformal factor is $\phi(x) = \frac{\ln(V)}{6}$ lets us find

$$H_{SD}|_{\Gamma_r} = \int d^3x \left(\frac{\pi^{ab} G_{abcd} \pi^{cd}}{\sqrt{|g|}} - (R - 2\Lambda)\sqrt{|g|(x)} \right), \quad (63)$$

i.e. the ADM-Hamiltonian with homogeneous lapse. Observing that on Γ_r we have $\int d^3x \frac{\pi^{ab} G_{abcd} \pi^{cd}}{\sqrt{|g|}} = V \langle \sqrt{\sigma^{ab} g_{ac} g_{bd} \sigma^{cd}} \rangle_g^2 - \frac{V}{6} \langle \pi \rangle^2$ and $\int d^3x \sqrt{|g|} R = V R_o$, where R_o denotes the Yamabe constant expressed as a nonlocal functional obtained by solving $R[e^{4\hat{\lambda}} g; x] = \langle R \rangle$ for $\lambda_o[g; x]$ and $R_o[g; x] := R[e^{4\hat{\lambda}[g;x]} g; x]$, we can immediately extend this to the conformal orbit Γ_c of Γ_r to yield

$$H_{SD}|_{\Gamma_c} = V \left(\left\langle \sqrt{\sigma^{ab} g_{ac} g_{bd} \sigma^{cd}} \right\rangle_g^2 - \frac{1}{6} \langle \pi \rangle^2 - \langle R_o \rangle + 2\Lambda \right), \quad (64)$$

which *differs* from the Shape Dynamics Hamiltonian for 1/4 of the physical degrees of freedom. In summary, we see that one has to resort to approximation schemes to

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[9] The Lichnerowicz–York equation on the torus is algebraically solvable by applying the maximum principle, while the BKL Hamiltonian is algebraic to begin with.

solve for the classical Shape Dynamics Hamiltonian that is equivalent to General Relativity, which is an obstruction to the direct construction of a quantum theory of Shape Dynamics.

One may understand this as a hint to use the BKL-Hamiltonian as a starting point for Shape Dynamics and introduce spatial derivatives as perturbations in the classical theory, which is conjectured to be a good approximation near singularities [5,6,16].

6 Conclusions

The true value of pure gravity on a torus in $2 + 1$ dimensions is that it is a nontrivial yet completely solvable model that exhibits many of the features of more complicated gravitational systems. It has thus established itself as a valuable testing ground for new gravitational theories such as Shape Dynamics that one can use to learn about the new theory. The main difficulty in constructing Shape Dynamics is to obtain explicit expressions for the Shape Dynamics Hamiltonian, which is generically nonlocal, on the full ADM phase space, so we are mainly interested in obtaining an explicit Shape Dynamics Hamiltonian. In these investigations we observed

1. The explicit (rather than formal) solvability of the initial value problem on the 2-torus (and 2-sphere) in CMC gauge is the technical reason for the explicit constructability of the Shape Dynamics Hamiltonian on these topologies. We find that the Shape Dynamics Hamiltonian formally coincides with the reduced phase space Hamiltonian and that this Hamiltonian is invariant under diffeomorphisms and conformal transformations that preserve the total volume. The difference between the two is however that the Shape Dynamics Hamiltonian is a function of the full ADM phase space that happens to functionally depend only on the image of reduced phase space under the canonical embedding, while the reduced phase space Hamiltonian is a phase space function on reduced phase space itself.
2. Although we cannot explicitly construct the Shape Dynamics Hamiltonian on higher-genus Riemann surfaces, we can construct it perturbatively. In particular we use an expansion that becomes good in a large-volume regime. The leading orders of this expansion then turn out to coincide with the temporal gauge Hamiltonian.
3. The Hamiltonian is in general a nonlocal phase space function. However, it becomes a local phase space function in the large-volume limit; in particular one finds that the leading order in a large-volume expansion turns the Hamiltonian into the conformal constraint that changes the total volume, so full conformal invariance is attained in this limit.
4. Since all local constraints are linear in momenta, one can formally quantize these as vector fields on configuration space assuming that there are measures such that these vector fields are divergence free. Then gauge invariance implies that the wave function is invariant under the flow generated by these vector fields, which in turn implies that the wave function depends only on reduced configuration space, which is finite dimensional. The Hamiltonian depends only on operators that preserve the reduced phase space, and thus Dirac quantization of the field theory is effectively reduced to reduced phase space quantization.

5. The construction of Shape Dynamics on the torus relied on the explicit solvability of the modified Lichnerowicz–York equation, which is absent in higher dimensions. However, a certain limit exists in $3 + 1$ dimensions, namely the strong gravity BKL limit, that is analogous to the torus case in the sense that the modified Lichnerowicz–York equation can be explicitly solved algebraically. One could attempt a derivative expansion starting with strong gravity, but it is not known whether such a program is feasible.

Lastly, let us remark that Shape Dynamics is a natural setting to discuss a quantum "large-CMC-volume/CFT" correspondence, which we were able to investigate explicitly in an excursion at the end of Sect. 4, where we found an explicit asymptotic scaling of solutions to the Wheeler–DeWitt equation under conformal transformations, which is very similar to the correspondence explored in [11].

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