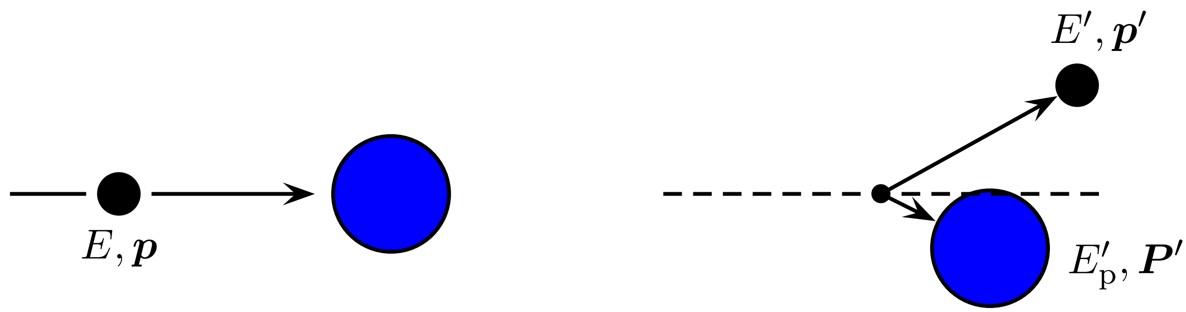
PARTICLE SHAPES FROM SCATTERING EXPERIMENTS



Classical scattering experiment: **Rutherford (1912)**:

• Scattering of low energy α particle off atoms:
 $\Rightarrow$ Atoms have small, heavy nucleus ($r \leq 10^{-14}$ m)

Advantageous to use electrons as projectile in stead of α's:

❶ Point-like particle;

❷ Interaction via virtual photon exchange:
- well understood process (QED);
- coupling constant small ($\alpha = \frac{1}{137}$) $\Rightarrow$ higher order diagrams play minor role; Born level calculation suffices.

SCATTERING KINEMATICS

Structures under study are small: $r_{\text{nucleon}} \simeq 0.8$ fm

$\Rightarrow$ Need electron beam with small wavelength

$\Rightarrow$ High momentum ($p \geq 200\text{MeV}/c$) $\Rightarrow$ highly relativistic electrons

Highly relativistic particles $\Rightarrow$ use 4-vector notation:

$$\begin{aligned} x &= (x_0,\, x_1,\, x_2,\, x_3) = (tc,\, \boldsymbol{x}) \\ p &= (p_0,\, p_1,\, p_2,\, p_3) = (E/c,\, \boldsymbol{p}) \end{aligned}$$

Special relativity reminder:

- Scalar product of two 4-vectors is *Lorentz invariant.*
- In particular, square of 4-vector is *Lorentz invariant.*
- Square of 4-momenta is *invariant mass*:

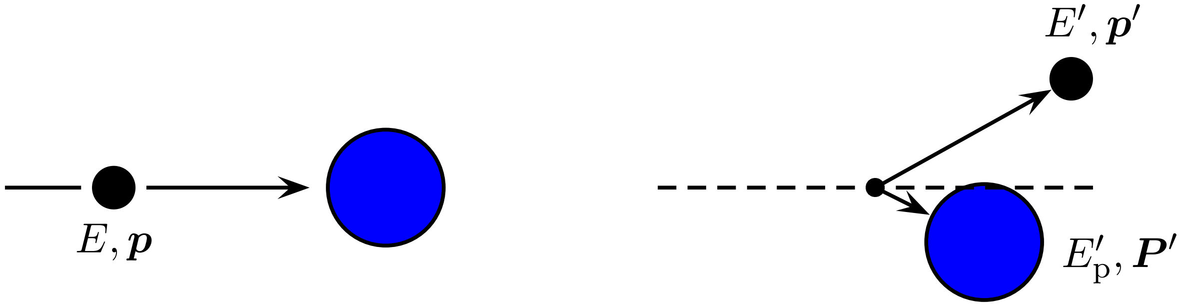
$$p^2 = E^2/c^2 - \boldsymbol{p}^2 = m^2c^2$$

- Rearrange to get the important relationship:

$$\boxed{E^2 - \boldsymbol{p}^2c^2 = m^2c^4}$$

- In ultrarelativistic limit ($E \gg mc^2$): $E \simeq |\boldsymbol{p}|c$

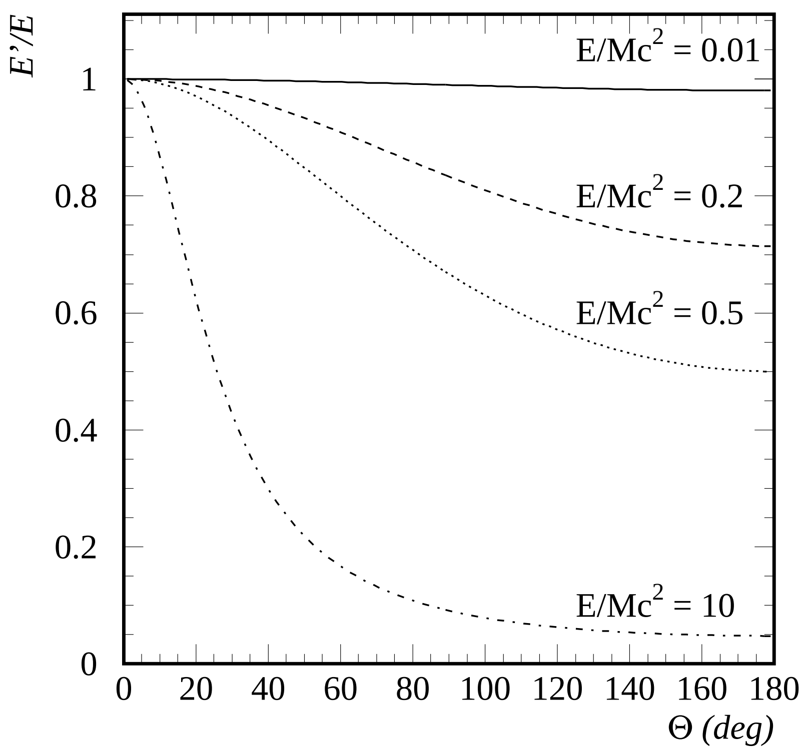
Electron scatters elastically off nucleus at rest:



$$p = (E/c, \boldsymbol{p}), \quad p' = (E'/c, \boldsymbol{p}'), \quad P = (Mc, \boldsymbol{0}), \quad P' = (E'_{\mathrm{p}}/c, \boldsymbol{P}')$$

From conservation of 4-momentum (energy and momentum):
One-to-one relationship between scattering angle, θ, and energy of scattered particle, E':

$$\boxed{E' = \frac{E}{1 + (E/Mc^2)(1 - \cos\theta)}}$$



RUTHERHORD CROSS SECTION

<u>Classical</u> Rutherford cross section for scattering of particle of charge ze off nucleus of charge Ze (target recoil neglected):

$$\boxed{\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{\text{Rutherford}} = \frac{(zZe^2)^2}{(4\pi\epsilon_0)^2(4E_{\text{kin}})^2\sin^4\frac{\theta}{2}}}$$

(Similar formula for "scattering" of comet in Suns gravitational field)

NON-RELATIVISTIC QUANTUM MECHANICS

Assumptions:

- heavy target $\Rightarrow$ neglect recoil
- Ze small, so that $Z\alpha \ll 1$ $\Rightarrow$ Born approximation good

Describe incomming and outgoing (scattered) electron by plane waves:

$$\psi_{\mathrm{i}} = \frac{1}{\sqrt{V}}\mathrm{e}^{i\boldsymbol{p}\boldsymbol{x}/\hbar} \qquad \psi_{\mathrm{f}} = \frac{1}{\sqrt{V}}\mathrm{e}^{i\boldsymbol{p}'\boldsymbol{x}/\hbar}$$

According to Fermi's Golden rule cross section is now

$$\frac{\sigma v}{V} = W = \frac{2\pi}{\hbar}\left|\langle\psi_{\mathrm{f}}|\mathcal{H}_{\text{int}}|\psi_{\mathrm{i}}\rangle\right|^2\frac{\mathrm{d}n}{\mathrm{d}E}$$

Density of states:

$$\mathrm{d}n(|\boldsymbol{p}'|) = \frac{4\pi|\boldsymbol{p}'|^2\mathrm{d}|\boldsymbol{p}'|\cdot V}{(2\pi\hbar)^3}.$$

Cross section for scattering into solid angle element $\mathrm{d}\Omega$:

$$\mathrm{d}\sigma\cdot v\cdot\frac{1}{V} = \frac{2\pi}{\hbar}\left|\langle\psi_{\mathrm{f}}|\mathcal{H}_{\text{int}}|\psi_{\mathrm{i}}\rangle\right|^2\frac{V|\boldsymbol{p}'|^2\mathrm{d}|\boldsymbol{p}'|}{(2\pi\hbar)^3\mathrm{d}E}\,\mathrm{d}\Omega$$

For high energies ($v \to c$, $|\boldsymbol{p}| \to E/c$):

$$\boxed{\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{V^2E'^2}{(2\pi)^2(\hbar c)^4}\left|\langle\psi_{\mathrm{f}}|\mathcal{H}_{\text{int}}|\psi_{\mathrm{i}}\rangle\right|^2}$$

MATRIX ELEMENT

Interaction operator for charge e in electric field ϕ is $\mathcal{H}_{\text{int}} = e\phi$.

$$\begin{aligned}
\langle\psi_{\rm f}|\mathcal{H}_{\text{int}}|\psi_{\rm i}\rangle &= \frac{e}{V}\int \mathrm{e}^{-i\boldsymbol{p}'\boldsymbol{x}/\hbar}\phi(\boldsymbol{x})\mathrm{e}^{i\boldsymbol{p}\boldsymbol{x}/\hbar}\mathrm{d}^3x \\
&= \frac{e}{V}\int \phi(\boldsymbol{x})\mathrm{e}^{i\boldsymbol{q}\boldsymbol{x}/\hbar}\mathrm{d}^3x,
\end{aligned}$$

where the *momentum transfer* is defined by

$$\boldsymbol{q} \equiv \boldsymbol{p} - \boldsymbol{p}'.$$

By applying Green's theorem, which says

$$\int \left(u\nabla^2 v - v\nabla^2 u\right) \mathrm{d}^3x = 0,$$

and inserting

$$\mathrm{e}^{i\boldsymbol{q}\boldsymbol{x}/\hbar} = \frac{-\hbar^2}{|\boldsymbol{q}|^2}\,\nabla^2\,\mathrm{e}^{i\boldsymbol{q}\boldsymbol{x}/\hbar}$$

the matrix element now reads

$$\begin{aligned}
\langle\psi_{\rm f}|\mathcal{H}_{\text{int}}|\psi_{\rm i}\rangle &= \frac{-e\hbar^2}{V|\boldsymbol{q}|^2}\int \left[\nabla^2\phi(\boldsymbol{x})\right]\mathrm{e}^{i\boldsymbol{q}\boldsymbol{x}/\hbar}\mathrm{d}^3x \\
&= \frac{e\hbar^2}{\epsilon_0 V|\boldsymbol{q}|^2}\int \rho(\boldsymbol{x})\mathrm{e}^{i\boldsymbol{q}\boldsymbol{x}/\hbar}\mathrm{d}^3x,
\end{aligned}$$

where $\rho(\boldsymbol{x})$ is the (static) charge density related to the field through Poisson's equation:

$$\nabla^2\phi(\boldsymbol{x}) = \frac{-\rho(\boldsymbol{x})}{\epsilon_0}.$$

Insert the *charge distribution function* f defined by $\rho(\boldsymbol{x}) = Zef(\boldsymbol{x})$. Normalized function $f(\boldsymbol{x})$ describes spatial distribution of charge.

We have finally:

$$\boxed{\langle\psi_{\rm f}|\mathcal{H}_{\text{int}}|\psi_{\rm i}\rangle = \frac{Z\cdot 4\pi\alpha\hbar^3 c}{|\boldsymbol{q}|^2\cdot V}\int f(\boldsymbol{x})\,\mathrm{e}^{i\boldsymbol{q}\boldsymbol{x}/\hbar}\,\mathrm{d}^3x.}$$

Matrix element contains Fourier integral of the charge function $f(\boldsymbol{x})$:

$$\boxed{F(\boldsymbol{q}) = \int f(\boldsymbol{x})\, \mathrm{e}^{i\boldsymbol{q}\boldsymbol{x}/\hbar}\, \mathrm{d}^3x}$$

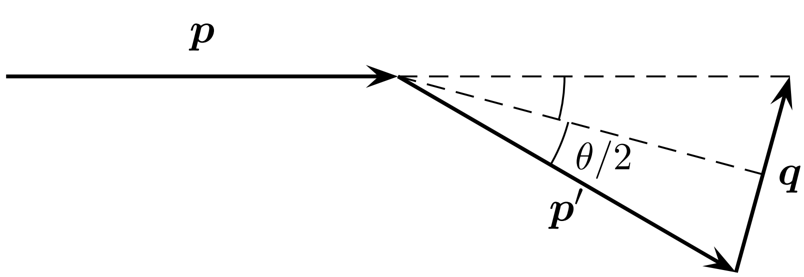
Form factor $F(\boldsymbol{q})$ contains all information about the spatial distribution of the charge of the object under study.

Rutherford scattering: neglect spatial extension of charge: $f(\boldsymbol{x}) = \delta(\boldsymbol{x})$
Thus, $F(\boldsymbol{q}) = 1$ and one finds:

$$\boxed{\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{\text{Rutherford}} = \frac{4Z^2\alpha^2(\hbar c)^2 E'^2}{|\boldsymbol{q}c|^4}.}$$

$1/\boldsymbol{q}^4$-dependence of electromagnetic cross section:
$\Rightarrow$ very low event rates at large momentum transfers (large angles)

With no recoil ($E = E'$, $|\boldsymbol{p}| = |\boldsymbol{p}'|$):



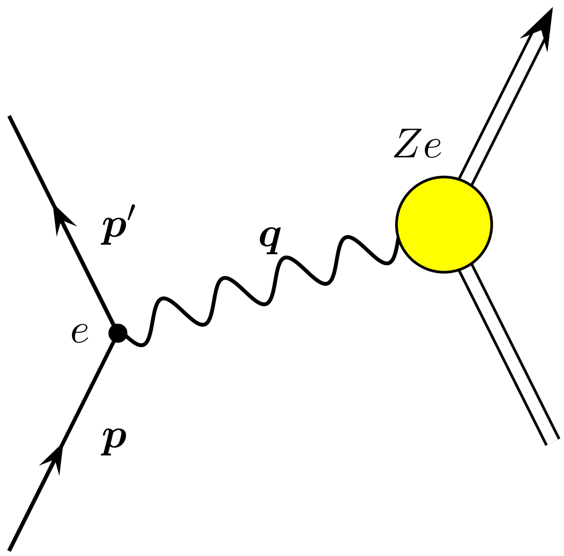
$$|\boldsymbol{q}| = 2|\boldsymbol{p}|\sin\frac{\theta}{2}$$

Relativistic Rutherford formula:

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{\text{Rutherford}} = \frac{Z^2\alpha^2(\hbar c)^2}{4\sin^4\frac{\theta}{2}}$$

RUTHERFORD SCATTERING AND FIELD THEORY

The electromagnetic interaction of the electron with the charge distribution is mediated by the exchange of a virtual photon:



Photon couples with strength e to electron and Ze to nucleus:
$\Rightarrow$ factor Ze^2 to matrix element and $(Ze^2)^2$ in cross section

Photon wavelength

$$\lambda = \frac{h}{|\boldsymbol{q}|} = \frac{h}{|\boldsymbol{p}|} \cdot \frac{1}{2\sin\frac{\theta}{2}}$$

If $\lambda \gg$ size of target particle:
$\Rightarrow$ internal structures cannot be resolved
$\Rightarrow$ target particle may be considered pointlike

Photon propagator term in amplitude:

$$\frac{1}{Q^2 + M^2c^2} \quad \longrightarrow \quad \frac{1}{Q^2},$$

since photon is massless ($M = 0$).
Contribution to cross section: $1/Q^4$

MOTT CROSS SECTION

Include effects due to electron spin:

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)^*_{\text{Mott}} = \left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{\text{Rutherford}} \cdot \left(1 - \beta^2 \sin^2\frac{\theta}{2}\right), \quad \text{with } \beta = \frac{v}{c}.$$

In ultra-relativistic limit ($\beta \to 1$):

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)^*_{\text{Mott}} = \left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{\text{Rutherford}} \cdot \cos^2\frac{\theta}{2}$$

$\Rightarrow$ Mott falls off more rapidly at large angles than Rutherford.

Mott differs from Rutherford due to **helicity conservation:**
For highly relativistic particles, the projection of their spin $\boldsymbol{s}$ on the direction of their motion $\boldsymbol{p}/|\boldsymbol{p}|$ is a conserved quantity.

Define *helicity* by:

$$\boxed{h = \frac{\boldsymbol{s} \cdot \boldsymbol{p}}{|\boldsymbol{s}| \cdot |\boldsymbol{p}|}}$$

Spin and momentum in same direction: $h = +1$
Spin and momentum in opposite directions: $h = -1$

Helicity conserved: scattering through 180° prohibited.

NUCLEON FORM FACTORS

RECOIL

For nucleons, cannot ignore target recoil:

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{\mathrm{Mott}} = \left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)^{*}_{\mathrm{Mott}} \cdot \frac{E'}{E}.$$

Energy transfer from electron is now significant: must use Lorentz invariant 4-momentum transfer to describe scattering:

$$\begin{aligned} q^2 &= (p-p')^2 = 2m_{\mathrm{e}}^2c^2 - 2\left(EE'/c^2 - |\boldsymbol{p}||\boldsymbol{p}'|\cos\theta\right) \\ &\simeq \frac{-4EE'}{c^2}\sin^2\frac{\theta}{2} \end{aligned}$$

Often used in order to work with positive quantities:

$$Q^2 = -q^2$$

MAGNETIC MOMENT

Must take into account:

1) interaction between electron charge and nucleon charge;

2) interaction between electron current and nucleon magnetic moment.

Magnetic moment of <u>pointlike</u> spin-$\frac{1}{2}$ particle of mass M and charge e:

$$\boxed{\mu = g \cdot \frac{e}{2M} \cdot \frac{\hbar}{2}}$$

Gyromagnetic ratio $g = 2$ results from relativistic QM.

The magnetic interaction is associated with “spin flip” of the nucleon.
From conservation of helicity and total angular momentum:

- scattering at 0° prohibited
- scattering at 180° preferred

$\Rightarrow$ Extra factor $\sin^2 \frac{\theta}{2}$ in cross section.
Use that $\sin^2 \frac{\theta}{2} = \cos^2 \frac{\theta}{2} \cdot \tan^2 \frac{\theta}{2}$ to write cross section in form:

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right) = \left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{\mathrm{Mott}} \cdot \left[1 + 2\tau \tan^2 \frac{\theta}{2}\right]$$

where

$$\tau = \frac{Q^2}{4M^2c^2}.$$

Magnetic interaction: increased probability for large scattering angles.

NUCLEON ANOMALOUS MAGNETIC MOMENT

Measured nucleon magnetic moments differ from pointlike expectation:

$$\text{Proton:} \qquad \mu_{\mathrm{p}} = \frac{g_{\mathrm{p}}}{2}\mu_{\mathrm{N}} = +2.79\,\mu_{\mathrm{N}}$$

$$\text{Neutron:} \qquad \mu_{\mathrm{n}} = \frac{g_{\mathrm{n}}}{2}\mu_{\mathrm{N}} = -1.91\,\mu_{\mathrm{N}}$$

where μ_{N} is nuclear magneton

$$\mu_{\mathrm{N}} = \frac{e\hbar}{2M_{\mathrm{p}}}$$

Determination of charge and current distribution of nucleons:
Scattering cross section given by Rosenbluth formula:

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right) = \left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{\mathrm{Mott}} \cdot \left[\frac{G_{\mathrm{E}}^2(Q^2) + \tau G_{\mathrm{M}}^2(Q^2)}{1+\tau} + 2\tau G_{\mathrm{M}}^2(Q^2)\tan^2\frac{\theta}{2}\right].$$

Separate *electric and magnetic form factors*, $G_{\mathrm{E}}(Q^2)$ and $G_{\mathrm{M}}(Q^2)$, by measuring cross section for same Q^2 but different values of $\tan^2\frac{\theta}{2}$
(Fig 6.1)
Important results in $Q^2 \to 0$ limit:

$$G_{\mathrm{E}}^{\mathrm{p}}(Q^2=0) = 1 \qquad G_{\mathrm{E}}^{\mathrm{n}}(Q^2=0) = 0$$

$$G_{\mathrm{M}}^{\mathrm{p}}(Q^2=0) = 2.79 \qquad G_{\mathrm{M}}^{\mathrm{n}}(Q^2=0) = -1.91$$

Measure form factors as function of Q^2:
Both for proton and neutron, fall off $\sim$ described by *dipole fit* (Fig 6.2)

$$G_{\mathrm{E}}^{\mathrm{p}}(Q^2) = \frac{G_{\mathrm{M}}^{\mathrm{p}}(Q^2)}{2.79} = \frac{G_{\mathrm{M}}^{\mathrm{n}}(Q^2)}{-1.91} = G^{\mathrm{dipole}}(Q^2),$$

where

$$G^{\mathrm{dipole}}(Q^2) = \left(1 + \frac{Q^2}{0.71\ (\mathrm{GeV}/c)^2}\right)^{-2}$$

Form factors are Fourier transform of spatial distributions.
Dipole fit $\Rightarrow$ spatial charge distribution is falling exponential

$$\rho(r) = \rho(0)\,\mathrm{e}^{-ar} \qquad \text{with } a = 4.27 \text{ fm}^{-1}$$

Nucleon "radius"

$$\sqrt{\langle r^2 \rangle_{\text{dipole}}} = 0.81 \text{ fm}.$$

Neutron: electrically neutral from outside, but study of form factor shows that it has electrically charged constituents:

$$\langle r^2 \rangle = -0.113 \pm 0.005 \text{ fm}^2.$$

CHARGE RADII OF PIONS AND KAONS

Pions and kaons are spin-0 particles: electric but no magnetic form factor

Form factors measured in scattering of pion/kaon beam off electrons in hydrogen target.
From form factors:

$$\sqrt{\langle r^2 \rangle_\pi} = 0.67 \pm 0.02 \text{ fm}$$
$$\sqrt{\langle r^2 \rangle_{\mathrm{K}}} = 0.58 \pm 0.04 \text{ fm}$$