# Quantum Theory on the Lattice

- Classical vs. Quantum mechanics. Path integrals
- God's favorite idea
- Gauge theory on the lattice
- How masses come out of nothing. Asymptotic freedom
- How to compute 30 mln integrals
- Dual lattice. Adding angular momenta. 6j symbols
- Quantum gravity on the lattice
- Yang–Mills theory as quantum gravity with “æther”
- How to get $\$10^6$

## Feynman's formulation of Quantum Mechanics:

Let us consider a non-relativistic particle with mass $m$ in a one-dimensional potential well $V(x)$. Its Lagrangian is

$$L = \frac{m\dot{x}^2}{2} - V(x) \tag{1}$$

and its energy is $H = \frac{m\dot{x}^2}{2} + V(x)$. In Quantum mechanics one introduces the Hamiltonian

$$\mathcal{H} = -\frac{1}{2m}\frac{d^2}{dx^2} + V(x). \tag{2}$$

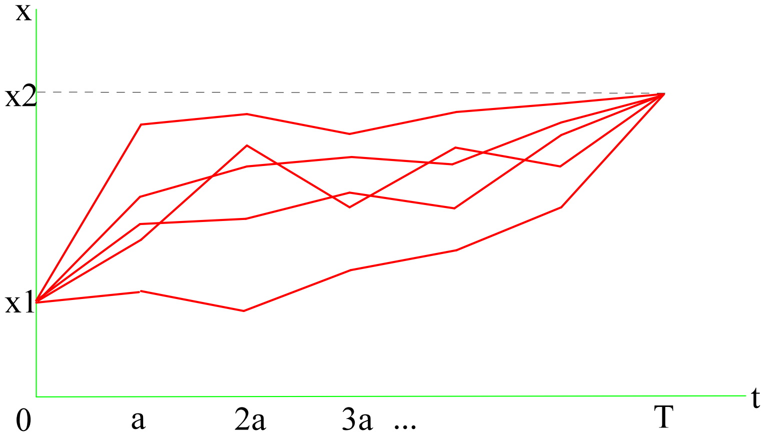
To find the (quantized) energy levels $E_n$ and the stationary wave functions $\psi_n(x)$ one solves the Schrödinger eqn:

$$\mathcal{H}\psi_n(x) = E_n\,\psi_n(x) \tag{3}$$

The evolution operator is $e^{-i\mathcal{H}T}$ where $T$ is time.

The probability amplitude that the particle goes from point $x_1$ to point $x_2$ during time $T$ is

$$\begin{aligned} A_{12} &= \langle x_2 | e^{-\frac{iHT}{\hbar}} | x_1 \rangle = \sum_n \psi_n^*(x_2)\, e^{-iE_nT}\, \psi_n(x_1) \\ &= \int_{x(0)=x_1}^{x(T)=x_2} Dx(t)\ \exp\ \frac{i}{\hbar} \int_0^T dt \left[ \frac{m\dot{x}^2(t)}{2} - V\left(x(t)\right) \right] \end{aligned}$$



Path integral over trajectories
with x(0)=x1, x(T)=x2

Discretized action

$$S = \sum_n a \left[ \frac{m}{2} \left( \frac{x(t_n) - x(t_{n-1})}{a} \right)^2 - V\left(x(t_n)\right) \right].$$

Path integral can be understood as the limit of an infinite number of ordinary integrations

– over the intermediate points $x_1 \dots x_N$:

$$A_{12} = \lim_{N \to \infty} \mathcal{N} \prod_{n=1}^{N} \int dx_n \, e^{\frac{i}{\hbar} S(x_1, \dots, x_N)}.$$

If $S \gg \hbar$ quantum mechanics becomes classical: small fluctuations of trajectories lead to a large variation of the action, and phase factors for close trajectories $e^{iS/\hbar}$ annihilate each other. Only those trajectories contribute to the path integral, whose small variation doesn't change the action, $\delta S = 0$. But this condition is the Euler–Lagrange equation of motion:

$$-\frac{d}{dt}\frac{dL}{d\dot{x}} + \frac{dL}{dx} = 0\,!$$

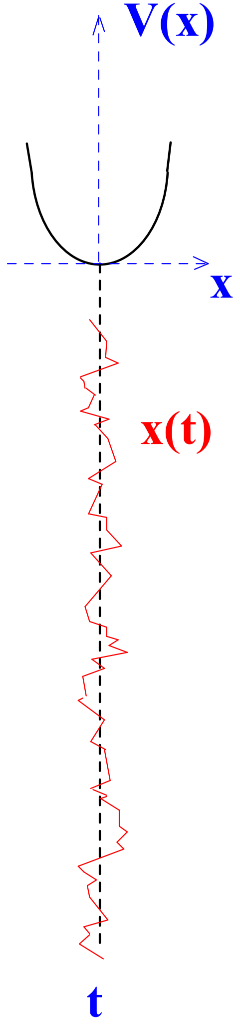
In classical theory one differentiates
In quantum theory one integrates

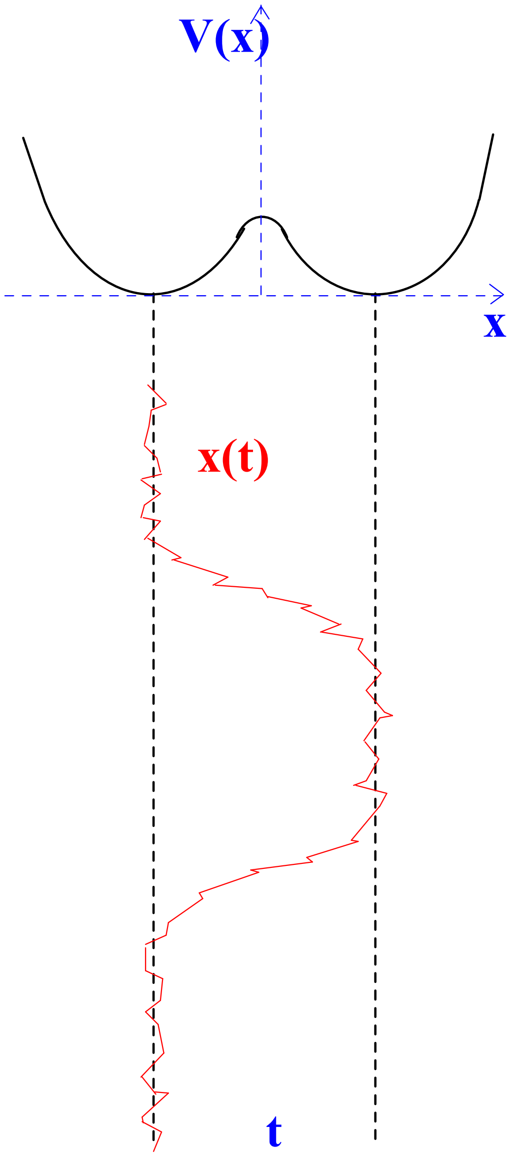
To cut out the ground state: take $t \longrightarrow -i\tau,\ T \longrightarrow -iT,\ (T \longrightarrow \infty)$. Put $x_1 = x_2$ and integrate over it. This is called the partition function: (henceforth $\hbar \longrightarrow 1$ )

$$\begin{aligned} \mathcal{Z} &= \sum_n e^{-E_n T} \\ &= \int Dx(\tau)\ \exp\left\{-\int_0^T d\tau\ \left[\frac{m}{2}\left(\frac{dx}{d\tau}\right)^2 + V\left(x(\tau)\right)\right]\right\}. \end{aligned}$$

In statistical mechanics one implies that $T = \frac{1}{kT^{\rm o}} = \beta$ where $T^{\rm o}$ is temperature. In particle physics T is the observation (Euclidean) time. If one is interested only in the ground (or vacuum) state $E_0$ one cuts it out by taking temperature $T^{\rm o} \longrightarrow 0$ or observation time $T \longrightarrow \infty$.

How do typical trajectories $x(\tau)$ look like?


V(x)
x
x(t)
t


V(x)
x
x(t)
t

## Text-book example: harmonic oscillator,

$$V(x) = \tfrac{m}{2}\omega_0^2\, x^2.$$

Since $x(t+T) = x(t)$ we can decompose arbitrary trajectory (over which we have to integrate!) in periodic functions,

$$x(t) = \sum_{n=-\infty}^{\infty} c_n \frac{1}{\sqrt{T}} \exp\left(2\pi i\, n \frac{t}{T}\right), \qquad c_{-n} = c_n^*$$

and the path integral will be understood as a product of integrals over all Fourier coefficients,

$$\int Dx(t) = \prod_n \int \frac{dc_n}{\sqrt{2\pi}}.$$

We shall need the ortho-normalization condition,

$$\int_0^T dt\, \frac{1}{\sqrt{T}}\, \exp\left(2\pi i\, m\frac{t}{T}\right)\, \frac{1}{\sqrt{T}}\, \exp\left(-2\pi i\, n\frac{t}{T}\right) = \delta_{m,n}.$$

We get in the exponent of the path integral

$$\frac{m}{2}\int dt\,(\dot{x}^2 + \omega_0^2\, x^2) = \frac{m}{2}\sum_{n=-\infty}^{\infty} c_n \left[\left(\frac{2\pi n}{T}\right)^2 + \omega_0^2\right] c_n$$

so that

$$\prod_n \int \frac{dc_n}{\sqrt{2\pi}}\, \exp\left(-\sum_n c_n\, \lambda_n\, c_n\right) \sim \prod_n \frac{1}{\sqrt{\lambda_n}}\,.$$

Harmonic oscillator's partition function is

$$\mathcal{Z} = \prod_n \sqrt{\frac{1}{\left(\frac{2\pi n}{T}\right)^2 + \omega_0^2}} = \frac{1}{\omega_0} \prod_{n=1}^{\infty} \frac{1}{\left(\frac{2\pi n}{T}\right)^2 + \omega_0^2}$$

$$= \frac{1}{\omega_0} \frac{\frac{\omega_0 T}{2}}{\sinh \frac{\omega_0 T}{2}} = \frac{T}{e^{\frac{\omega_0 T}{2}} - e^{-\frac{\omega_0 T}{2}}}$$

$$= T\, e^{-\frac{\omega_0 T}{2}} \left(1 + e^{-\omega_o T} + e^{-2\omega_0 T} + e^{-3\omega_0 T} + \ldots\right)$$

$$= (T) \sum_n e^{-E_n T} \overset{T \to \infty}{=} e^{-\frac{\omega_0 T}{2}}, \qquad E_n = \omega_0 \left(\frac{1}{2} + n\right)$$

[Actually, an infinite $\omega_0$-independent product $\prod_{n=1}^{\infty} \left(\frac{2\pi n}{T}\right)^2$ as due to the integration measure has been inserted to make the ratio finite].

[don't forget that in statistical mechanics $\omega_0 \longrightarrow \hbar\omega_0$, $T \longrightarrow \frac{1}{kT^o}$]

Important lesson:

Path Integral $=$ Product of integrals over Fourier coefficients

Quantum mechanics can be called $0+1$-dimensional field theory. In true field theory the 'coordinates' are fields depending on real space coordinates $\mathbf{x}$.

Typical path integral in Quantum Field Theory:

$$\mathcal{Z} = \int D\phi(\mathbf{x}, t)\; \exp\left(-S[\phi, \partial\phi, \ldots]\right)\; .$$

For example,

$$S[\phi, \partial\phi, ...] = \int_0^T dt \int d^dx \left(\frac{1}{2}\partial_0\phi\partial_0\phi + \frac{1}{2}\partial_i\phi\partial_i\phi + \frac{\lambda}{4}\phi^4\right) .$$

It is a field theory in $d+1$ dimensions.

We shall study many such examples ...

References

R.P. Feynman and A.R. Hibbs, Quantum Mechanics and Path Integrals, McGraw-Hill, N.Y.
(Russian translation at http://irodov.nm.ru/books.htm )

Problem

Read chapters 2 and 3 of that book. Find misprints in eqs. (3.62) and (3.64).