Gauge Theory on the Lattice

The Euclidean Yang–Mills partition function is

$$\mathcal{Z} = \int DA_\mu \, \exp\left(-\frac{1}{2g^2}\int d^dx \, \mathrm{Tr}\, F_{\mu\nu}F_{\mu\nu}\right)$$

where g^2 is the gauge coupling constant; $[g^2] = [m]^{4-d}$. In $4d$ it is dimensionless.

The very simply-written YM action in fact encodes enormously rich dynamics. First, it is believed that the theory leads to the confinement of color – a unique phenomenon having no analogs in the history of physics. Second, it is believed that the originally massless YM fields A_μ disappear from the physical spectrum. Instead, gauge-invariant or 'colorless' bound states must appear, which have a nonzero mass. Third, the whole rich realm of strong (or nuclear) interactions, from π's to ^{238}U is, in principle, deducible from Quantum Chromodynamics (=QCD) being nothing but YM theory based on the $SU(3)$ gauge group.

($\$1 \cdot 10^6$ prize..)

Averaging over the group *or* integration over the Haar measure

$$\int dU = \int d(UV) = \int d(VU).$$

Problem

Show that

$$\int dU = 1, \qquad \int dU\; U_i^\alpha U_\beta^{\dagger j} = \frac{1}{N}\delta_\beta^\alpha \delta_i^j$$

$$\int dU\; U_i^\alpha U_j^\beta = \begin{cases} \frac{1}{2}\,\epsilon_{ij}\,\epsilon^{\alpha\beta} & \text{for } SU(2) \\ 0 & \text{for } SU(N),\; N>2 \end{cases}$$

$$\int dU\; U_{i_1}^{\alpha_1} U_{i_2}^{\alpha_2} U_{\beta_1}^{\dagger j_1} U_{\beta_2}^{\dagger j_2}$$

$$= \frac{1}{N^2-1}\left[\delta_{\beta_1}^{\alpha_1}\delta_{\beta_2}^{\alpha_2}\left(\delta_{i_1}^{j_1}\delta_{i_2}^{j_2} - \frac{1}{N}\delta_{i_2}^{j_1}\delta_{i_1}^{j_2}\right) + \delta_{\beta_2}^{\alpha_1}\delta_{\beta_1}^{\alpha_2}\left(\delta_{i_2}^{j_1}\delta_{i_1}^{j_2} - \frac{1}{N}\delta_{i_1}^{j_1}\delta_{i_2}^{j_2}\right)\right].$$

YM theory on the lattice [K. Wilson (1974), A. Polyakov (1975)]

Lattice-regularized partition function

$$\begin{aligned}
\mathcal{Z}(\beta) &= \int \prod_{\text{links}} dU_{\text{link}}\ \exp\left(\sum_{\text{plaq}} \beta\, \frac{\mathrm{Tr}\ U_{\text{plaq}} + \text{c.c.}}{2\ \mathrm{Tr}\ 1}\right) \\
&\to \int DA_\mu\ \exp\left(-\frac{1}{2g_d^2}\int d^dx\ \mathrm{Tr}\ F_{\mu\nu}^2\right), \\
\beta &= \frac{2N}{a^{4-d}g_d^2}
\end{aligned}$$

the continuum limit is obtained at $a \to 0, \quad \beta \to \infty$ and

$$\left\{\begin{array}{rl}
g_2^2 = \frac{2N}{a^2\beta} = \text{fixed}, & d = 2, \\
g_3^2 = \frac{2N}{a\beta} = \text{fixed}, & d = 3, \\
\Lambda = \frac{1}{a}\ \exp\left(-\frac{12\beta\pi^2}{11N^2}\right) = \text{fixed}, & d = 4.
\end{array}\right.$$

Λ has the dimension on mass and gives the scale in the continuum theory. It is known from experiment. YM theory in $4d$ is **asymptotically free**, with the ‘running coupling constant’ given by

$$\frac{8\pi^2}{g^2(\mu)} = \frac{11\,N}{3}\,\ln\frac{\mu}{\Lambda}.$$

Asymptotic freedom: At small separations quarks interact **weakly**!

Wilson’s criteria of confinement: the area behaviour of the Wilson loop

$$\begin{aligned} W_{js} &= \prod_{\text{links}} \mathrm{Tr}_{js}(UUU.....U) \\ &\to \mathrm{Tr}\,\mathrm{P}\,\exp i\oint dx_\mu\,A^a_\mu\,t^a_{js} \end{aligned}$$

$$\langle W\rangle = \exp\left[-V(R)\,T\right] \qquad \text{at } T\to\infty.$$

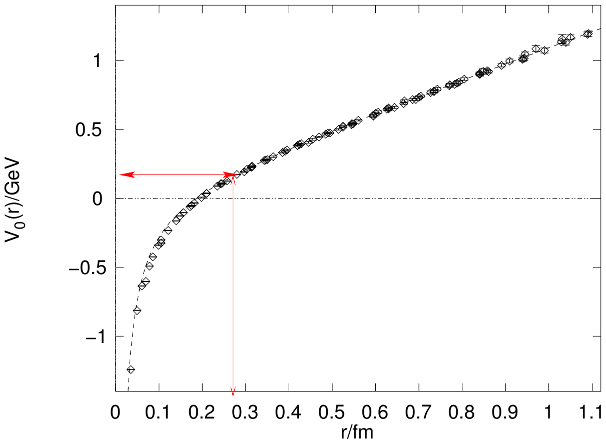
$V(R)$ is the potential between quark and antiquark at separation R.
The area law, $W \sim \exp(-\sigma\,\mathrm{Area})$, means

$$V(R) = \sigma\, R,$$

i.e. the linear rising potential. The 'string tension' must be

$$\begin{aligned}
\sigma &\simeq g_2^2 \quad \text{in} \quad d=2,\\
\sigma &\simeq g_3^4 \quad \text{in} \quad d=3,\\
\sigma &= c_\sigma\,\Lambda^2 \quad \text{in} \quad d=4,\\
\Lambda &= \frac{1}{a}\exp\left(-\frac{8\pi^2}{\frac{11}{3}N\,g^2}\right)
\end{aligned}$$

Transmutation of dimensions: Physical dimensional parameters are "made of" the ultraviolet cutoff!!



The potential energy of two infinitely-heavy quarks, as function of their separation, simulated on an $SU(2)$ lattice. The units come from setting $\sqrt{\sigma} = 420\,\mathrm{MeV}$ [G. Bali et al. (1995)].

Mass gap

Correlation function

$$< \mathrm{Tr}U(x\,\mathrm{Tr}U(y) > \sim \exp(-m\,|x-y|) \qquad \left(\simeq < F^a_{\mu\nu}F^a_{\mu\nu}(x)\,F^b_{\kappa\lambda}F^b_{\kappa\lambda}(y) > \right)$$

$$m = c_m\,\Lambda = \frac{1}{a}\exp\left(-\frac{8\pi^2}{\frac{11}{3}N\,g^2}\right).$$

Numerically, $m \simeq 1400\,\mathrm{MeV}$.