Duality Transformation on the Lattice, $SU(2)$

Lattice-regularized partition function

$$
\begin{aligned}
\mathcal{Z}(\beta) &= \int \prod_{\rm links} dU_{\rm link}\; \exp\left(\sum_{\rm plaq} \beta\, \frac{\mathrm{Tr}\; U_{\rm plaq} + \mathrm{c.c.}}{2\; \mathrm{Tr}\; 1}\right) \\
&\to \int DA_\mu\; \exp\left(-\frac{1}{2g_d^2}\int d^dx\; \mathrm{Tr}\; F_{\mu\nu}^2\right), \\
\beta &= \frac{2N}{a^{4-d} g_d^2}
\end{aligned}
$$

the continuum limit is obtained at $a \to 0, \quad \beta \to \infty$ and

$$
\left\{
\begin{array}{ll}
g_2^2 = \frac{2N}{a^2\beta} = \text{fixed}, & d = 2, \\
g_3^2 = \frac{2N}{a\beta} = \text{fixed}, & d = 3, \\
\Lambda = \frac{1}{a}\, \exp\left(-\frac{12\beta\pi^2}{11N^2}\right) = \text{fixed}, & d = 4.
\end{array}
\right.
$$

Insert a unity for every plaquette:

$$1 = \prod_{\text{plaquettes}} \int dU_{\text{plaq}} \delta(U_{\text{plaq}}\,,\; U_1 U_2 U_3^\dagger U_4^\dagger)$$

$$\delta(U,V) = \sum_{J=0,\frac{1}{2},1,\frac{3}{2},\dots} (2J+1)\; D^J_{m_1 m_2}(U^\dagger) D^J_{m_2 m_1}(V),$$

$$V = U_1 U_2 U_3^\dagger U_4^\dagger .$$

Wigner D-functions

Wigner D-functions are eigenfunctions of the square of the angular momentum operator (written in terms of, say, three Euler angles α, β, γ),

$$\mathbf{J}^2 D^J_{mn}(\alpha,\beta,\gamma) = J(J+1) D^J_{mn}(\alpha,\beta,\gamma),$$
$$J = 0,\ \frac{1}{2},\ 1,\ \frac{3}{2}\ldots,\quad -J \le m,n \le +J,$$

and can be said to be eigenfunctions of a spherical top; they are $(2J+1)^2$-fold degenerate. The 'magnetic' quantum numbers m, n have the meaning of the projections of the angular momentum of a spherical top on the third axes in the 'body-fixed' and 'lab' frames. One can parameterize a 2×2 unitary matrix by Euler angles as

$$U = \exp(i\alpha\tau^3)\exp(i\beta\tau^2)\exp(i\gamma\tau^3).$$

There two sets of angular momenta operators,

$$S_0 = S_3 = -i\frac{\partial}{\partial\alpha},\quad S_{\pm 1} = \frac{1}{\sqrt{2}}(S_1 \pm iS_2) = \frac{i}{\sqrt{2}} e^{\pm i\alpha}\left(\mp \mathrm{ctg}\beta\frac{\partial}{\partial\alpha} + i\frac{\partial}{\partial\beta} \pm \frac{1}{\sin\beta}\frac{\partial}{\partial\gamma}\right),$$

$$T_3 = -i\frac{\partial}{\partial\gamma}, \quad T_1 = -i\frac{\partial}{\partial\gamma}, \quad T_{\pm 1} = \frac{i}{\sqrt{2}}e^{\mp i\gamma}\left(\mp \mathrm{ctg}\beta\frac{\partial}{\partial\gamma} + i\frac{\partial}{\partial\beta} \mp \frac{1}{\sin\beta}\frac{\partial}{\partial\alpha}\right)$$

$$\mathbf{S}^2 = \mathbf{T}^2 \equiv \mathbf{J}^2$$

$$= -\frac{\partial^2}{\partial\beta^2} - \mathrm{ctg}\beta\frac{\partial}{\partial\beta} - \frac{1}{\sin^2\beta}\left(\frac{\partial^2}{\partial\alpha^2} - 2\cos\beta\frac{\partial^2}{\partial\alpha\partial\gamma} + \frac{\partial^2}{\partial\gamma^2}\right)$$

such that

$$[S_i S_j] = i\epsilon^{ijk}S_k, \quad [T_i T_j] = i\epsilon^{ijk}T_k, \quad [S_i T_j] = 0.$$

It is convenient to use the unitary matrix U as a formal argument of the D-functions. Their main properties are:

- Multiplication law (summation over repeated indices understood):

$$D^J_{kl}(U_1 U_2) = D^J_{km}(U_1) D^J_{ml}(U_2).$$

- Unitarity ("*" denotes complex conjugate):

$$D^J_{kl}(U^\dagger) = \left(D^J_{lk}(U)\right)^*.$$

- Phase condition:

$$\left(D^J_{lk}(U)\right)^* = (-1)^{l-k} D^J_{-l,-k}(U), \quad D^J_{kl}(1) = \delta^{(2J+1)}_{kl}.$$

- Orthogonality and normalization:

$$\int dU \, D^{J_1}_{kl}(U^\dagger) D^{J_2}_{mn}(U) = \frac{1}{2J_1+1} \delta_{J_1 J_2} \, \delta_{kn} \, \delta_{lm}.$$

Integration here is over the Haar measure:

$$\int dU ... = \int d(SU) ... = \int d(US) ...; \qquad \int dU = 1.$$

- Completeness (the δ-function is understood in the Haar measure sense):

$$\delta(U,V)=\sum_J (2J+1)D^J_{kl}(U^\dagger)D^J_{lk}(V).$$

- Matrix element:

$$\int dU\,D^{J_1}_{a_1b_1}(U)D^{J_2}_{a_2b_2}(U)D^{J_3}_{a_3b_3}(U)$$
$$=\left(\begin{array}{ccc} J_1 & J_2 & J_3 \\ a_1 & a_2 & a_3 \end{array}\right)\left(\begin{array}{ccc} J_1 & J_2 & J_3 \\ b_1 & b_2 & b_3 \end{array}\right),$$

where $(\ldots)$ denote $3jm$ symbols.

- Decomposition of a direct product of irreducible representations (=irreps):

$$D^{J_1}_{a_1b_1}(U)D^{J_2}_{a_2b_2}(U)=\sum_J (2J+1)$$
$$\cdot\quad\left(\begin{array}{ccc} J & J_1 & J_2 \\ -c & a_1 & a_2 \end{array}\right)\left(\begin{array}{ccc} J & J_1 & J_2 \\ -d & b_1 & b_2 \end{array}\right)(-1)^{d-c}D^J_{cd}(U).$$

A "practical" definition of the $6j$ symbol $\{\ldots\}$ is via a contraction over projections in **three** $3jm$ symbols:

$$\sum_{klm}(-1)^{j_4-k+j_5-l+j_6-m}\begin{pmatrix} j_5 & j_1 & j_6 \\ l & p & -m \end{pmatrix}\begin{pmatrix} j_6 & j_2 & j_4 \\ m & q & -k \end{pmatrix}$$

$$\times\begin{pmatrix} j_4 & j_3 & j_5 \\ k & r & -l \end{pmatrix} = \begin{pmatrix} j_1 & j_2 & j_3 \\ -p & -q & -r \end{pmatrix}\begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{Bmatrix}.$$

The summation over projections k, l, m is such that $p = m - l$, $q = k - m$ and $r = l - k$ are kept fixed.

Another definition of the $6j$ symbol is via the full contraction of projections in **four** $3jm$ symbols:

$$\sum_{klmnop}(-1)^{j_4+n+j_5+o+j_6+p}\begin{pmatrix} j_1 & j_2 & j_3 \\ k & l & m \end{pmatrix}\begin{pmatrix} j_1 & j_5 & j_6 \\ k & o & -p \end{pmatrix}$$

$$\times \begin{pmatrix} j_4 & j_2 & j_6 \\ -n & l & p \end{pmatrix} \begin{pmatrix} j_4 & j_5 & j_3 \\ n & -o & m \end{pmatrix} = \begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{Bmatrix}.$$

Since the three j's of any $3jm$ symbol satisfy the triangle inequalities, e.g. $|j_1 - j_2| \leq j_3 \leq j_1 + j_2$, etc., the following four triades of the $6j$ symbols have to satisfy the triangle inequalities: $(j_1 j_2 j_3)$, $(j_1 j_5 j_6)$, $(j_2 j_4 j_6)$ and $(j_3 j_4 j_5)$; otherwise, the $6j$ symbol is zero.

The $6j$ symbols are symmetric under permutation of any of two columns and under interchange of the upper and lower arguments simultaneously in any two columns.

A full contraction of **six** $3jm$ symbols yields the $9j$ symbol:

$$\sum \begin{pmatrix} j_1 & j_2 & j_3 \\ k & l & m \end{pmatrix} \begin{pmatrix} j_4 & j_5 & j_6 \\ n & o & p \end{pmatrix} \begin{pmatrix} j_7 & j_8 & j_9 \\ q & r & s \end{pmatrix}$$

$$\cdot \quad \begin{pmatrix} j_1 & j_4 & j_7 \\ k & n & q \end{pmatrix} \begin{pmatrix} j_2 & j_5 & j_8 \\ l & o & r \end{pmatrix} \begin{pmatrix} j_3 & j_6 & j_9 \\ m & p & s \end{pmatrix}$$

$$= \left\{ \begin{array}{ccc} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \\ j_7 & j_8 & j_9 \end{array} \right\}.$$

A convenient reference book on D-functions, $3jm$, $6j$ and $9j$ symbols is D. Varshalovich, A. Moskalev and V. Khersonskii, *Quantum Theory of Angular Momenta*, World Scientific (1988)

The trace of the D^J-function is called the character of the representation J,

$$\chi^J(U) \stackrel{d}{=} \sum_{m=-J}^{J} D^J_{mm}(U) = \frac{\sin(J+\frac{1}{2})\omega}{\sin\frac{1}{2}\omega},$$

$$\mathrm{Tr}U = 2\cos\tfrac{1}{2}\omega = \cos\frac{\beta}{2}\cos\frac{\alpha+\gamma}{2}.$$

For example, Integration over plaquette variables factorizes into:

$$\int dU \exp\left(\beta \frac{\mathrm{Tr}\ U + \mathrm{Tr}\ U^\dagger}{4}\right) D^J_{mn}(U)$$

$$= \quad \delta_{mn} \frac{1}{\pi} \int_0^{2\pi} d\omega \ \sin^2 \frac{\omega}{2} \ \frac{\sin(J+\frac{1}{2})\omega}{\sin \frac{1}{2}\omega} \ e^{\beta \cos \frac{1}{2}\omega}$$

$$= \quad \delta_{mn} \frac{2}{\beta} I_{2J+1}(\beta)$$

where $I_{2J+1}(\beta)$ is the modified Bessel function with the index $2J+1$. Its asymptotics at large β is:

$$\frac{2}{\beta} I_{2J+1}(\beta) \quad = \quad \frac{2}{\beta} I_1(\beta)\ T_J(\beta),$$

$$T_J(\beta) \quad \xrightarrow{\beta\to\infty} \quad \exp\left[-\frac{2J(J+1)}{\beta}\right]$$

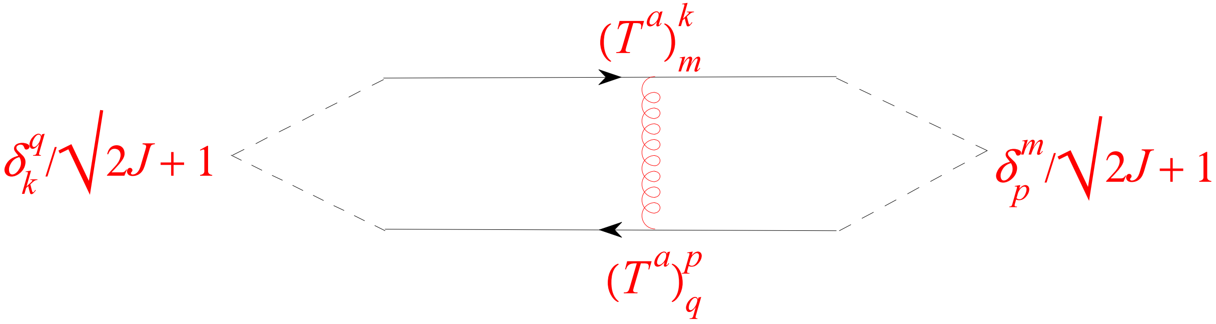
NB: continuum limit: $J \sim \beta \gg 1$!

In $2d$ because of the orthogonality of D-functions all plaquettes have the same J outside the loop and the same J' inside the loop, with $|J - j_s| \leq J' \leq J + j_s$ where j_s is the 'color spin' of the source along the loop. The average Wilson loop is, therefore, exactly computable in $2d$:

$$\langle W_{j_s}(S)\rangle = \frac{\sum_J [T_J(\beta)]^{\frac{V}{a^2}} \sum_{J'=|J-j_s|}^{J+j_s} [T_{J'}(\beta) \,/\, T_J(\beta)]^{\frac{S}{a^2}}}{\sum_J [T_J(\beta)]^{\frac{V}{a^2}}}$$

$$\to [T_{j_s}(\beta)]^{\frac{S}{a^2}} \to \exp\left[-\frac{g_2^2}{2} j_s(j_s+1)\, S\right],$$

– the needed area behavior (with the 'Casimir' string tension) $\Rightarrow$ confinement in $d = 1{+}1$!



$$\mathrm{Tr}(T^a\,T^a) = J(J+1)(2J+1)$$

$$-\frac{\partial^2}{\partial\,x^2}\phi = g^2\delta(x) \Rightarrow \phi = \frac{g^2}{2}\,|x|$$

$$V = \frac{g^2}{2}\,J(J+1)\,|x-y|$$

is the Coulomb energy of a quark and an antiquark with 'isospin' J in $2d$.

3jm symbols (≈ Clebsch–Gordan coeff's):

$$\int dU\ D^{J_1}_{a_1 b_1}(U)\ D^{J_2}_{a_2 b_2}(U)\ D^{J_3}_{a_3 b_3}(U)$$

$$= \begin{pmatrix} J_1 & J_2 & J_3 \\ a_1 & a_2 & a_3 \end{pmatrix} \begin{pmatrix} J_1 & J_2 & J_3 \\ b_1 & b_2 & b_3 \end{pmatrix}$$

6j symbols = a contraction of four 3jm's:

$$\sum_{klmnop} (-1)^{j_4+n+j_5+o+j_6+p} \begin{pmatrix} j_1 & j_2 & j_3 \\ k & l & m \end{pmatrix} \begin{pmatrix} j_1 & j_5 & j_6 \\ k & o & -p \end{pmatrix}$$

$$\times \begin{pmatrix} j_4 & j_2 & j_6 \\ -n & l & p \end{pmatrix} \begin{pmatrix} j_4 & j_5 & j_3 \\ n & -o & m \end{pmatrix} = \begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{Bmatrix}$$

9j symbols = a contraction of six 3jm's:

$$\sum \begin{pmatrix} j_1 & j_2 & j_3 \\ k & l & m \end{pmatrix} \begin{pmatrix} j_4 & j_5 & j_6 \\ n & o & p \end{pmatrix} \begin{pmatrix} j_7 & j_8 & j_9 \\ q & r & s \end{pmatrix} \begin{pmatrix} j_1 & j_4 & j_7 \\ k & n & q \end{pmatrix}$$

$$\times \begin{pmatrix} j_2 & j_5 & j_8 \\ l & o & r \end{pmatrix} \begin{pmatrix} j_3 & j_6 & j_9 \\ m & p & s \end{pmatrix} = \begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \\ j_7 & j_8 & j_9 \end{Bmatrix}$$