Duality Transformation on the Lattice, cont'd

Standard lattice partition function (in $3d$ $SU(2)$)

$$\mathcal{Z} = \int DA_i \; \exp\left(-\frac{1}{2g_3^2}\int d^3x \; \mathrm{Tr}\; F_{ij}^2\right)$$

$$\to \int \prod_{\mathrm{links}} dU_{\mathrm{link}} \; \exp\left(\sum_{\mathrm{plaquettes}} \beta \, \frac{\mathrm{Tr}\; U_{\mathrm{plaq}} + \mathrm{c.c.}}{2 \, \mathrm{Tr}\; 1}\right)$$

$$\beta = \frac{4}{a g_3^2} \to \infty, \quad a \to 0, \quad g_3^2 \;\; \text{fixed} \;\; (= \text{mass})$$

Observables: gauge-invariant closed Wilson loops

$$W_J[C] \quad = \quad \frac{1}{2J+1}\mathrm{Tr}\,\mathrm{P}\;\exp\; i\oint_C A_i\,dx^i$$

$$\to \quad \frac{1}{2J+1}\mathrm{Tr}_J \prod_{\mathrm{links}} U_{\mathrm{link}}\ldots$$

Expect (but not proven so far!)

- $<$ large Wilson loop $> \sim \; \exp(-g_3^4 \cdot \text{Area})$
- $< W(x)W(y) > \sim \; \exp(-g_3^2 \cdot \text{Separation})$

Duality transformation. Insert a unity for every plaquette:

$$1 = \prod_{\text{plaquettes}} \int dU_{\text{plaq}}\, \delta(U_{\text{plaq}}\,,\; U_1 U_2 U_3 U_4)$$

$$\delta(U,V) = \sum_{J=0,\frac{1}{2},1,\frac{3}{2},\ldots} (2J+1)\; D^J_{m_1 m_2}(U^\dagger) D^J_{m_2 m_1}(V),$$

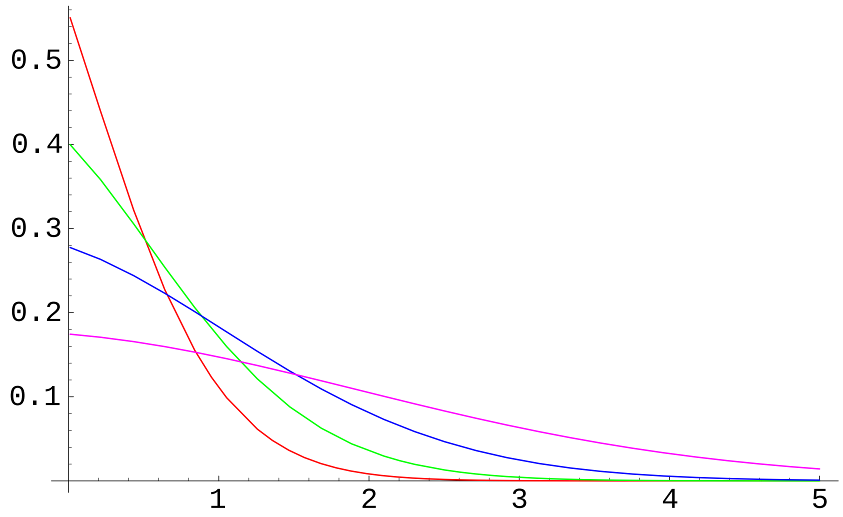
$$V = U_1 U_2 U_3 U_4.$$

Integration over plaquette variables is factorized:

$$\int dU \exp\left(\beta \frac{\text{Tr } U + \text{Tr } U^\dagger}{2 \text{ Tr } 1}\right) D^J_{mn}(U^\dagger) = \delta_{mn} \frac{2}{\beta} I_{2J+1}(\beta)$$

$$\frac{2}{\beta} I_{2J+1}(\beta) = \frac{2}{\beta} I_1(\beta)\; T_J(\beta), \quad T_J(\beta) \to \exp\left[-\frac{2J(J+1)}{\beta}\right]$$

Angular momenta J are canonical conjugate to Euler angles in the U's. It is similar to passing from x-space to p-space. However, since U's are compact, J's are discrete $(= 0, \frac{1}{2}, 1 \ldots)$. In the continuum limit $J \sim \sqrt{\beta} \gg 1$, become continuous. J's have the meaning of dual field strength $(E \to B)$.



Probabilities of having different J's on plaquettes, for given lattice $\beta = 2.6, 5, 10, 24$.

Probabilities of having various J's in a typical lattice simulation at $\beta = 2.6$:

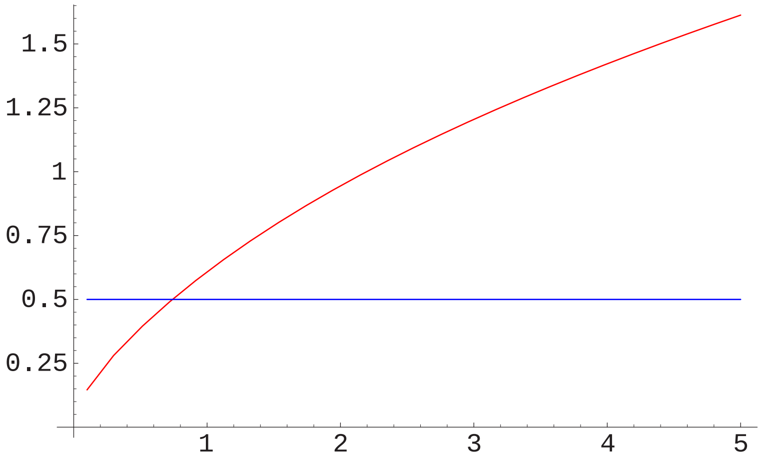
J	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2	≥ 3
probability	56%	29%	11%	3%	1%	$<$0.5%

$$\text{probability} = \frac{I_{2J+1}(\beta)}{\sum_J I_{2J+1}(\beta)}$$

at $\beta = 10$:

J	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2	≥ 3
probability	28%	24%	18%	13%	8%	5%

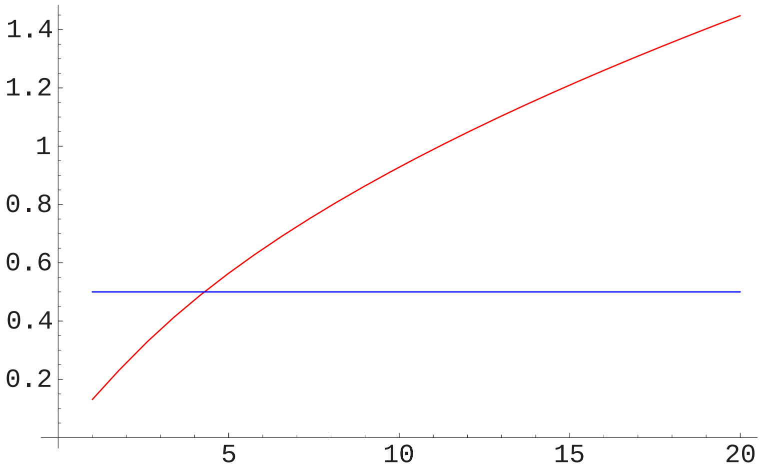
$U(1)$ lattice [="photodynamics"]: average $\bar{m}$ on plaquette, as function of β



Measuring the Wilson loop, one **has** to span a surface with half-integer m's. When $\bar{m} < \frac{1}{2}$ one always **looses** when one spans the loop by $m = \frac{1}{2}$ plaquettes $\Longrightarrow$ area law.

Confinement in $U(1)$ lattice theory is lost at $\beta \simeq 1$. In the continuum limit ($\beta \to \infty$) in Quantum Electrodynamics there is only the Coulomb law $\sim 1/r$, and no linear potential. The linear potential in QED arises as a lattice artifact at $\beta \leq 1$.

$SU(2)$ lattice: average J on plaquette, as function of β

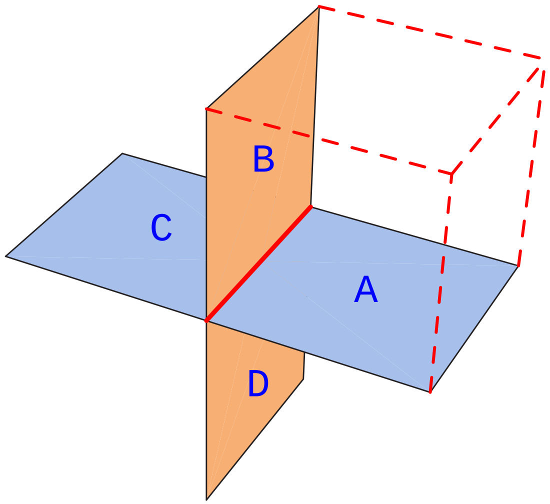


At $\beta < 4.5$ we do not give any chance to the lattice to produce anything but the area law.

Integration over link variables involves four Wigner D-functions, as every link is shared by four plaquettes:

$$\int dU\, D^{J_A}_{m_1 m_2}(U)\, D^{J_B}_{m_3 m_4}(U)\, D^{J_C}_{m_5 m_6}(U)\, D^{J_D}_{m_7 m_8}(U),$$

$$J_A \otimes J_B = \oplus\, j \qquad (3jm \text{ symbols}) \qquad \begin{pmatrix} J_A & J_B & j \\ m_1 & m_3 & m \end{pmatrix} \begin{pmatrix} J_A & J_B & j \\ m_2 & m_4 & m' \end{pmatrix}$$



Summation over 'magnetic' numbers m closes in a 3-cube, and one gets an infinite network of $6j$ symbols involving J's living on the plaquettes and j's living on the links.

The lattice partition function can be identically rewritten through a product of 6j symbols:

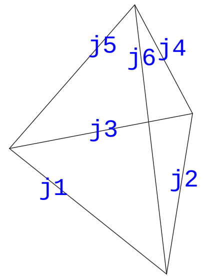
$$\mathcal{Z} = \sum_{J_P,\, j_l} \prod_{\rm plaq's} (2J_P+1)\, T_{J_P}(\beta)\, (-1)^{2J_P} \prod_{\rm links} (2j_l+1)$$

$$\times \prod_{\text{even cubes corners}} \left\{ \begin{array}{ccc} j & j & j \\ J & J & J \end{array} \right\} \prod_{\text{lattice sites}} \left\{ \begin{array}{ccc} j & j & j \\ j & j & j \end{array} \right\}.$$

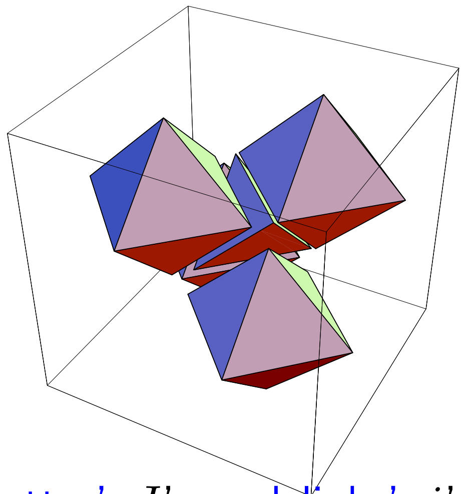
[Anishety, Cheluvaraja, Sharatchandra and Matur (1993); Diakonov and Petrov (1999)]

6j symbol is a table of 6 numbers with four triades satisfying triangle inequalities; it can be graphically represented by a tetrahedron whose edges' lengths are equal to $j_1, j_2, \ldots, j_6$:

$$\left\{ \begin{matrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{matrix} \right\} =$$



The tetrahedra of the partition function completely triangulate the dual space WITHOUT HOLES:



A generic configuration of plaquettes' J's and links' j's does not triangulate a flat but rather a curved $3d$ dual space. However, a curved Riemannian manifold can be embedded into a 6-dimensional flat space, at least locally.

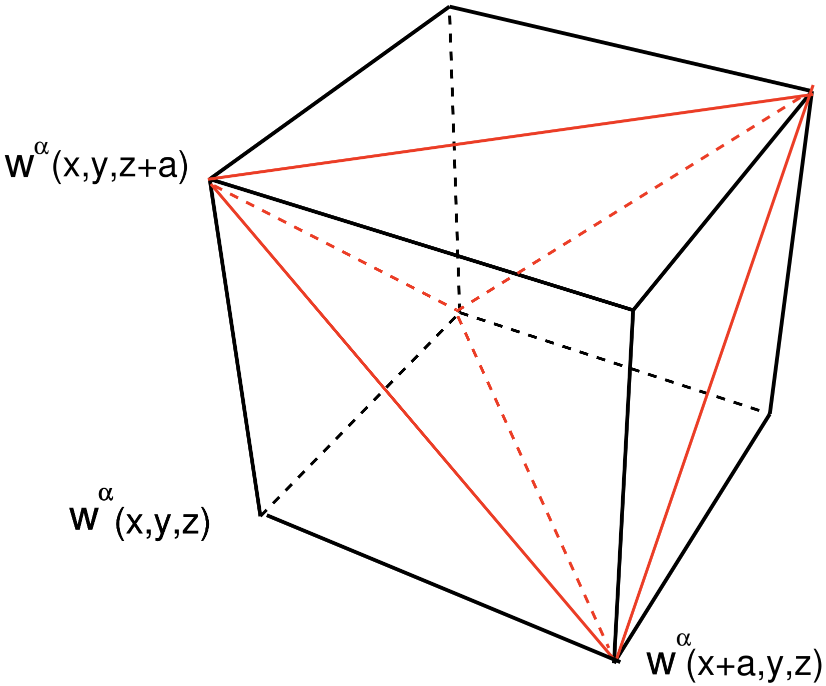
A d-dimensional curved space can be embedded into $D = \frac{1}{2}d(d+1)$ flat dimensions:

d	$D = \frac{1}{2}d(d+1)$
2	3
3	6
4	10

Introduce $6d$ coordinates of the dual lattice $w^A(x)$ and $6d$ angular momenta:

$$\begin{aligned} J_x^A\left(x+\frac{a}{2}, y, z\right) &= w^A(x+a, y, z) - w^A(x, y, z), \\ &\to a\,\partial_x w^A + O(a^2). \end{aligned}$$

All triangle inequalities of the $6j$ symbols will be satisfied automatically!



The momenta $J^A, A = 1, 2 \ldots 6$ apparently satisfy the identity

$$J_z^A\left(x,y,z+\frac{a}{2}\right) - J_x^A\left(x+\frac{a}{2},y,z\right) = w^A(x,y,z+a) - w^A(x+a,y,z)$$

$$= J_z^A\left(x+a,y,z+\frac{a}{2}\right) - J_x^A\left(x+\frac{a}{2},y,z+a\right),$$

and similarly for other components. This is nothing but a discretized version of the Bianchi identity,

$$\epsilon_{ijk}\partial_i J_k^A = 0, \qquad A = 1, ..., 6.$$

Therefore, in 6 dimensions one recovers the simple (flat) form of the Bianchi identity for the dual field strength. One can say that the complicated (nonlinear) form of the usual non-Abelian Bianchi identity is a result of the projection of the flat Bianchi identity onto the curved colour space.

In the continuum limit the 6-dimensional coordinates of the sites of the dual lattice become continuous functions of space; we denote them as $w^A(x)$. Since 6 functions depend only on 3 coordinates there are three relations between $w^A(x)$ at any point; these relations define the curved 3-dimensional manifold whose triangulation is given by the set of J's and j's. The induced metric tensor of the manifold is determined by

$$g_{ij}(x) = \partial_i w^A \partial_j w^A.$$

Continuum limit ($a \to 0,\ J \sim \sqrt{\beta} \to \infty$):

$$\sum_{J=0,1/2,1,\ldots} (2J+1)\ldots \longrightarrow 2\int_0^\infty dJ^2 \ldots\ ,$$

Asymptotics of $6j$ symbols when all J's are large (Ponzano and Regge (68) + Schulten and Gordon (75)):

$$\left\{ \begin{array}{ccc} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{array} \right\} = \frac{1}{\sqrt{12\pi V(j)}} \cos\left(\sum_n j_n\, \theta_n \right),$$

where $V(j)$ is the tetrahedron volume and θ_n are dihedral angles (between faces).

For many 6j symbols, take a product of cosines (written as exponents) and assemble together all dihedral angles related to a given J or j. The angle defects,

$$\Theta(J) = \sum_{n=1}^{4} \theta_n^{(J)} - 2\pi, \qquad \text{borrowed from } (-1)^{2J}\ (!),$$

$$\Theta(j) = \sum_{n=1}^{6} \theta_n^{(j)} - 4\pi,$$

are zero in flat space. The product of $6j$ symbols measures the curvature of the dual $3d$ space.

A direct calculation of 96 dihedral angles yields:

$$\exp\ i\left[\sum J\,\Theta(J) + \sum j\,\Theta(j)\right] = \exp\ i\sum_x a^3\,\frac{1}{2}\sqrt{g}\,R$$

$$= \exp\ \frac{i}{2}\int d^3x\ \sqrt{g}\ R,$$

i.e. the Einstein–Hilbert action of general relativity, but for the curved dual space! [General formula: Regge (1961)]

The metric tensor is built from $6d$ coordinates $w^A(x)$, and one can calculate all differential geometry's quantities:

$$
\begin{aligned}
g_{ij} &= \partial_i w^A \partial_j w^A, \qquad A = 1, \ldots, 6 \\
\Gamma_{ij,\,k} &= \partial_i \partial_j w^A \partial_k w^A \qquad \to R_{ijkl} \to R_{ij} \to R \\
R &= \frac{1}{72\, g^2}\, \epsilon_{ABCDEF}\, \epsilon_{A'B'C'D'EF}\, \epsilon^{ijk}\, \epsilon^{i'j'k'}\, \epsilon^{pll'} \epsilon^{qmm'} \\
&\cdot \quad w_i^A w_{i'}^{A'} w_j^B w_{j'}^{B'} w_k^C w_{k'}^{C'} w_{lm}^D w_{l'm'}^{D'} w_p^G w_q^G \,.
\end{aligned}
$$

The necessary and sufficient condition for the space to be flat is that it is described by only three (out of the general six) external coordinates $w^{1,2,3}(x,y,z)$.

The Yang–Mills partition function can be exactly (!) rewritten as [Lunev (92), Anishetty et al. (99), DP (99)]

$$\begin{aligned} \mathcal{Z} &= \int Dw^A \,\mathrm{Jac}\, \exp \int d^3x \left(-\frac{g_3^2}{2}\, \partial_i w^A\, \partial_i w^A + \frac{i}{2}\sqrt{g} R \right) \\ \mathrm{Jac} &= \det(\partial_i \partial_j\, w^A)\; g^{-\frac{5}{4}}. \end{aligned}$$

In this form it was derived from lattice-regularized YM theory [DP, hep-th/9912268] but can be derived directly in the continuum theory as well.

As usual in differential geometry one can define the Christoffel symbol,

$$\Gamma_{i,jk}(x) = \frac{1}{2}(\partial_j g_{ik} + \partial_k g_{ij} - \partial_i g_{jk}) = \partial_i w^A \partial_j \partial_k w^A,$$

and the Riemann tensor,

$$R_{ijkl}(x) = \frac{1}{2}(\partial_j\partial_k g_{il} + \partial_i\partial_l g_{jk} - \partial_j\partial_l g_{ik} - \partial_i\partial_k g_{jl}) + \Gamma_{m,jk}\Gamma^m_{il} - \Gamma_{m,jl}\Gamma^m_{ik}.$$

The contravariant tensor is inverse to the covariant one,

$$g^{ij} g_{jk} = \delta^i_k,$$

and can be used to rise indices, and for contractions. The determinant of the metric tensor is

$$g = \det g_{ij} = \frac{1}{3!}\epsilon^{ijk}\,\epsilon^{lmn} g_{il} g_{jm} g_{kn},$$

and the contravariant metric tensor is

$$g^{ij} = \frac{1}{2g}\epsilon^{ikl}\epsilon^{jmn}\partial_k w^A \partial_m w^A \partial_l w^B \partial_n w^B.$$

The Ricci tensor is

$$R_{ik} = g^{lm}\left[\left(\partial_i\partial_k w^A \partial_l\partial_m w^A - \partial_i\partial_l w^A \partial_k\partial_m w^A\right)\right.$$

$$+g^{np}\left(\partial_n w^A \partial_i\partial_l w^A \partial_p w^B \partial_k\partial_m w^B + \partial_n w^A \partial_i\partial_l w^A \partial_m w^B \partial_k\partial_p w^B\right.$$

$$\left.\left.-\partial_n w^A \partial_i\partial_k w^A \partial_p w^B \partial_l\partial_m w^B - \partial_p w^A \partial_i\partial_n w^A \partial_m w^B \partial_k\partial_l w^B\right)\right].$$

Finally, the scalar curvature is obtained as a full contraction,

$$R = g^{ik} g^{jl} R_{ijkl} = g^{ik} R_{ik}$$

$$= \frac{1}{72\, g^2}\, \epsilon_{ABCDEF}\, \epsilon_{A'B'C'D'EF}\, \epsilon^{ijk}\, \epsilon^{i'j'k'}\, \epsilon^{pll'} \epsilon^{qmm'}$$

$$\cdot w_i^A w_{i'}^{A'} w_j^B w_{j'}^{B'} w_k^C w_{k'}^{C'} w_{lm}^D w_{l'm'}^{D'} w_p^G w_q^G \,.$$