Duality Transformation in the Continuum

Partition function of the $SU(2)$ Yang–Mills theory in $3d$

$$\mathcal{Z} = \int DA_i \; \exp\left(-\frac{1}{2g_3^2}\int d^3x \; \mathrm{Tr}\; F_{ij}^2\right), \qquad \left(g_3^2 = \text{mass}\right)$$

$$\to \int \prod_{\text{links}} dU_{\text{link}} \; \exp\left(\sum_{\text{plaquettes}} \beta \, \frac{\mathrm{Tr}\; U_{\text{plaq}} + \text{c.c.}}{2\;\mathrm{Tr}\;1}\right), \qquad \left(\beta = \frac{4}{a g_3^2} \to \infty, \quad a \to 0\right)$$

$$= \sum_{J_P,\, j_l} \prod_{\text{plaq's}} (2J_P + 1)\, e^{-\frac{2J_P(J_P+1)}{\beta}} \prod_{\text{links}} (2j_l + 1)$$

$$\times \prod_{\text{cubes}} \left\{ \begin{matrix} j & j & j \\ J & J & J \end{matrix} \right\} \prod_{\text{lattice sites}} \left\{ \begin{matrix} j & j & j \\ j & j & j \end{matrix} \right\}$$

$$\stackrel{\beta\to\infty}{\longrightarrow} \quad \int Dw^A(x)\cdot \mathrm{Jac}\cdot \exp\int d^3x \left(-\frac{g_3^2}{2}\, g_{ii} + \frac{i}{2} R\sqrt{g}\right), \qquad \mathrm{Jac} = \det(\partial_i\partial_j\, w^A)\; g^{-\frac{5}{4}},$$

$$g_{ij} = \partial_i w^A\, \partial_j w^A, \quad A = 1\ldots 6; \qquad R = \text{scalar curvature}\;.$$

This is an identical reformulation of the $3d$ $SU(2)$ quantum Yang–Mills theory in terms of **local gauge-invariant variables** $w^A(x)$ **or** $g_{ij}(x)$. Einstein–Hilbert, *plus* "æther".

Number of gauge-invariant d.o.f.'s:

in $A^a_\mu(x)$: $(d-1)(N^2-1)|_{N=3,\,d=3} = 6$, same as in $g_{ij}(x)$ or in six scalar functions, the "external coordinates" parameterizing the dual space $w^A(x)$, $A = 1\ldots 6$.

Perturbation theory, $g_3^2 \to 0$:

The Einstein–Hilbert term is $O(g_3^0)$, hence it has to vanish in the limit $g_3^2 \to 0$. The general parametrization of the flat $3d$ space is by by **three** functions $w^a(x), a = 1\ldots 3$. The action is that of three massless scalar fields — exactly as needed for three (at $N\!=\!2$) transverse gluons!

Derivation of the same action but directly in the continuum [Lunev (1992)]

The YM partition function can be written with the help of an additional Gaussian integration over the "dual field strength" G_i^a (a=1,2,3 are color and i=1,2,3 are Euclidean indices):

$$\mathcal{Z} = \int DA_i^a \, \exp\left(-\frac{1}{4g_3^2}\int d^3x \, F_{ij}^a(A)F_{ij}^a(A)\right) \qquad \left[F_{ij}^a(A) = \partial_i A_j^a - \partial_j A_i^a + \epsilon^{abc}A_i^b A_j^c\right]$$

$$= \int DG_i^a \, DA_i^a \, \exp\int d^3x \left[-\frac{g_3^2}{2}G_i^a G_i^a + \frac{i}{2}\epsilon^{ijk}F_{ij}^a(A)G_k^a\right].$$

Gaussian integration over A_i^a can be easily performed, the saddle-point $\bar{A}_i^a$ being defined by the equation

$$\epsilon^{ijk}(\partial_j\delta^{ab} + \epsilon^{acb}\bar{A}_j^c)G_k^b = \epsilon^{ijk}\, D_j^{ab}(\bar{A})G_k^b = 0.$$

This equation is generally satisfied if

$$D_j^{ab}(\bar{A})G_k^b = \Gamma^i_{jk} G_i^a$$

where Γ is symmetric in subscripts. This eqn. is known to be satisfied in general relativity, provided one makes the following identification:

- $G_i^a \equiv e_i^a$ where e_i^a is a **dreibein** such that the **metric tensor** is the standard

$$g_{ij} = e_i^a e_j^a, \qquad g^{ij} = e^{ai} e^{aj}, \qquad e_i^a e^{bi} = \delta^{ab}, \qquad \det e_i^a = \sqrt{g};$$

- Γ is the standard **Christoffel symbol**,

$$\Gamma_{i,jk} = \frac{1}{2}\left(\partial_k g_{ik} + \partial_j g_{ik} - \partial_i g_{jk}\right), \qquad \Gamma^i_{jk} = g^{il}\Gamma_{l,jk};$$

- $\bar{A}_i^c \equiv -\frac{1}{2}\epsilon^{abc}\omega_i^{ab}$ where ω is the **spin connection**,

$$\omega_i^{ab} = -\omega_i^{ba} = \frac{1}{2}e^{ak}(\partial_i e_k^b - \partial_k e_i^b) - \frac{1}{2}e^{bk}(\partial_i e_k^a - \partial_k e_i^a) - \frac{1}{2}e^{ak}e^{bj}e_i^c(\partial_k e_j^c - \partial_j e_k^c).$$

It is easy to check that with these definitions the above eqns. are fulfilled. One finds an explicit solution for the saddle-point potential $\bar{A}_i^a$ through the dual field strength $G_i^a = e_i^a$.

Pursuing further the analogy with Riemannian geometry one can define the **covariant derivative** in curved space,

$$(\nabla_i)_l^k = \partial_i \delta_l^k + \Gamma_{il}^k,$$

whose commutator is the **Riemann tensor**,

$$\begin{aligned}[\nabla_i \nabla_j]_l^k = R^k_{\ lij} &= \partial_i \Gamma^k_{jl} - \partial_j \Gamma^k_{il} + \Gamma^k_{im}\Gamma^m_{jl} - \Gamma^k_{jm}\Gamma^m_{il} \\ &= \epsilon^{abc} F^a_{ij}(\bar{A}) e^{bk} e^c_l, \qquad e^c_l \equiv G^c_l,\end{aligned}$$

related, as we see, to the YM field strength at the saddle point $\bar{A}$. More directly, we get from the YM field strength at the saddle point expressed in terms of gravity quantities,

$$F^a_{ij}(\bar{A}) = \frac{1}{2} R^m_{\ lij}\, \epsilon^{abc}\, e^b_m e^{cl}.$$

Therefore, the second term in the action is, at the saddle point, nothing but the Einstein–Hilbert action:

$$\epsilon^{ijk} F^a_{ij}(\bar{A}) G^a_k = \sqrt{g} R, \qquad R \equiv R^{m}_{\;lmn} g^{ln},$$

where we have used the identity

$$\epsilon^{abc} e^a_k e^b_m = \sqrt{g}\, \epsilon_{kmn}\, e^{cn}.$$

The first term in the action is

$$G^a_i G^a_i = e^a_i e^a_i = g_{ii}.$$

Integrating over A^a_i gives a pre-exponential factor $(\det G^a_i)^{-\frac{3}{2}} = g^{-\frac{3}{4}}$. The resulting integral in is over the 9 components of the dreibein G^a_i which can be rotated by the 3-parameter gauge transformation O^{ab}. However, both the action and the measure depend on $G^a_i(x)$ only through the metric tensor g_{ij} which is **gauge-invariant**. Writing

$$d^{(9)} e^a_i = d^{(3)} O^{ab}\, d^{(6)} g_{ij}\, g^{-\frac{1}{2}}$$

one can pass to integrating over the metric. One gets finally the YM partition function identically rewritten in terms of 6 gauge-invariant variables:

$$\mathcal{Z} = \int Dg_{ij}\, g^{-\frac{5}{4}} \exp \int d^3x \left[-\frac{g_3^2}{2}\, g_{ii} + \frac{i}{2} R(g)\sqrt{g} \right], \qquad g_{ij} = \partial_i w^A \partial_j w^A.$$

The second term, the Einstein–Hilbert action with a purely imaginary Newton constant is invariant under **local** 3-dimensional diffeomorphisms $w^A(x) \to w^A(x'(x))$, and depends actually only on three variables. The first term is not diffeomorphism-invariant, and we call it the "æther" term as it corresponds to an isotropic and homogeneously distributed "matter" which, however, spoils the general covariance. It distinguishes the YM theory from the pure gravity theory (which is a topological field theory [Witten (1989)], and is responsible for the propagation of transverse gluons in the perturbative regime. We note that the integration measure also differs from the diffeomorphism-invariant measure $\prod_{i\le j} dg_{ij} g^{-2}$. Finally one can pass on to external coordinates $w^A(x),\ A = 1\ldots 6$:

$$\prod_{i\le j} dg_{ij} = \prod_{A=1}^{6} dw^A \det_{6\times 6}(\partial_i \partial_j w^A)\,.$$

Could the diffeomorphism-invariance be anticipated? Yes:

The Bianchi identity, $\epsilon^{ijk}D_i^{ab}F_{jk}^b = 0$, guarantees that the second (mixed) term in the action is invariant under local (N^2-1)-function **dual gauge transformation**

$$\begin{aligned} \delta A_i^a &= 0, \\ \delta e_i^a &= D_i^{ab}(A)\,\beta^b, \end{aligned}$$

as well as the ordinary gauge transformation

$$\begin{aligned} \delta A_i^a &= D_i^{ab}(A)\,\alpha^b, \\ \delta e_i^a &= f^{abc}\,e_i^b\,\alpha^c. \end{aligned}$$

A particular choice of the transformation parameters,

$$\alpha^a = v^i A_i^a, \qquad \beta^a = v^i G_i^a$$

leads to the transformation

$$\delta g_{ij} = \partial_j v^k \, g_{ik} + \partial_i v^k \, g_{kj} + v^k \, \partial_k g_{ij}$$

which is the known transformation law for the metric tensor under an infinitesimal diffeomorphism

$$x^i \to x^i + v^i(x)\,.$$

Hence the 2^{nd} term in the action must be invariant under the diffeomorphism. From dimensions, it can be only $R\sqrt{g}$. The numerical coefficient must be computed.

Higher gauge groups $SU(N)$

$SU(2),\ 2^2 - 1 = 3$

A_i^a has 9 dof's, out of which 6 are gauge-invariant; these are the components of g_{ij}, i.e. spin 0 (1 dof) and spin 2 (5 dof's) fields, 6=1+5.

$SU(3),\ 3^2 - 1 = 8$

A_i^a has 24 dof's, out of which 16 are gauge-invariant; these are g_{ij} (6 dof's) and symmetric h_{ijk} i.e. spin 1 (3 dof's) and spin 3 (7 dof's) fields, 16=6+10.

$SU(4),\ 4^2 - 1 = 15$

A_i^a has 45 dof's, out of which 30 are gauge-invariant; these are g_{ij} (6 dof's), h_{ijk} (10 dof's) and spin 2 (5 dof's) and spin 4 (9 dof's) fields, 30=6+10+14.

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$SU(N),\ N^2 - 1$

A_i^a has $3\cdot(N^2-1)$ dof's, out of which $2\cdot(N^2-1)$ are gauge-invariant; these are spins from 0 to N.

In fact, all spins are encountered twice, except the 'edge' spins $0, 1, N-1, N$ which are encountered once.

The YM partition function can be identically rewritten as [D.D. and Victor Petrov, hep-th/0108097]

$$\mathcal{Z} = \int D(\text{spin } 0 \ldots \text{spin } N \text{ fields}) \exp(S_1+S_2),$$

$$S_1 = \frac{g_3^2}{2}\int d^3x\, g_{ii}, \quad (\text{`aether}'),$$

$$S_2 = \frac{i}{N}\int d^3x \text{ ``}R\sqrt{g}'' \quad (\text{generalized Einstein gravity})$$

S_2 is invariant under (N^2-1)-function transformations, out which 3 are the usual diffeomorphisms of the 'dual space'. The rest local transformations mix up fields with different spins. At $N \longrightarrow \infty$ we get an infinite tower of spins which are mixed up by local transformations – like in string theory.

Regge trajectory slope

$$\alpha' \stackrel{?}{=} \frac{1}{\left(g_3^2\, N\right)^2}$$