Four Dimensions: Duality Transformation in the Continuum

$$
\begin{aligned}
\mathcal{Z} &= \int DA_\mu \, \exp \int d^4x \, \left(-\frac{1}{2g^2} \mathrm{Tr}\, F_{\mu\nu} F_{\mu\nu} \right) \\
&= \int DA_\mu \, DG_{\mu\nu} \, \exp \int d^4x \, \left(-\frac{g^2}{2} \, \mathrm{Tr}\, G_{\mu\nu} G_{\mu\nu} + \frac{i}{2} \, \epsilon^{\alpha\beta\mu\nu} \, \mathrm{Tr}\, G_{\alpha\beta} F_{\mu\nu} \right) \quad (1)
\end{aligned}
$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu A_\nu]$ is the standard Yang–Mills field strength and $\epsilon^{\alpha\beta\mu\nu}$ is the antisymmetric tensor. To avoid possible confusion we write down explicitly all indices. To be specific, the gauge group is $SU(N)$ with $N^2 - 1$ generators t^a, $\quad \mathrm{Tr}\, t^a t^b = \delta^{ab}/2$. Eq. (1) is called the 1st order formalism.

Both terms in eq. (1) are invariant under (N^2-1)-function gauge transformation,

$$\left\{\begin{array}{lcl} \delta A_\mu &=& [\mathcal{D}_\mu\,\alpha], \\ \delta G_{\mu\nu} &=& [G_{\mu\nu}\,\alpha] \end{array}\right. \tag{2}$$

where $\mathcal{D}_\mu = \partial_\mu - iA_\mu^a t^a$ is the Yang–Mills covariant derivative, $[\mathcal{D}_\mu \mathcal{D}_\nu] = -iF_{\mu\nu}$.

Because of the Bianchi identity, $\epsilon^{\mu\nu\rho\sigma}\,[\mathcal{D}_\nu F_{\rho\sigma}] = 0$, the second (mixed) term in eq. (1) is, in addition, invariant under the $4\cdot(N^2-1)$-function **'dual' gauge transformation**,

$$\left\{\begin{array}{lcl} \delta A_\mu &=& 0, \\ \delta G_{\mu\nu} &=& [\mathcal{D}_\mu \beta_\nu] - [\mathcal{D}_\nu \beta_\mu]. \end{array}\right. \tag{3}$$

Taking a particular combination of the functions in eqs. (2,3),

$$\alpha = v^\mu A_\mu, \qquad \beta_\mu = v^\lambda G_{\lambda\mu}, \tag{4}$$

leads to the transformation

$$\delta G_{\mu\nu} = -G_{\lambda\nu}\,\partial_\mu v^\lambda - G_{\mu\lambda}\,\partial_\nu v^\lambda - \partial_\lambda G_{\mu\nu}\,v^\lambda, \tag{5}$$

being the known transformation of a (covariant) tensor under general coordinate transformation. Therefore, the 'mixed' term is diffeomorphism-invariant, and is known as BF gravity. It defines a topological field theory of the Schwarz type. Moreover, it is invariant not under four but as much as $4 \cdot (N^2 - 1)$ local transformations; four diffeomorphisms are but their small subset. We shall see later on that the additional local transformations mix up fields with different spins.

The first term in eq. (1) is not invariant under the dual gauge transformation (3), therefore it is not invariant under diffeomorphisms. For that reason, we call it the 'æther' term: it distinguishes the Yang–Mills theory from a non-propagating topological BF gravity represented by the second (mixed) term in the action (1).

$SU(2)$, $d = 4$ BF gravity in basis-independent formulation

In the 1st order formalism, the integral (1) over the Yang–Mills connection A_μ is gaussian, and one can integrate it out.

In this section, we write down the result of the A_μ integration in a compact form which makes clear the 12-function symmetry of the BF action. From the point of view of the

Yang–Mills theory, it solves the problem of reformulating it in terms of local gauge-invariant variables. From the point of view of the BF gravity, we rewrite it in the basis-invariant formalism.

The gaussian integration over A_μ in eq. (1) is equivalent to the saddle-point approximation. The saddle point (which we denote by $\bar{A}_\mu$) is found from varying the 'mixed' term in A_μ:

$$\epsilon^{\lambda\mu\alpha\beta}\,[\mathcal{D}_\mu(\bar{A})\,G_{\alpha\beta}]=0. \tag{6}$$

We need to solve this equation in respect to the saddle-point YM connection $\bar{A}_\mu$ and to substitute it back into the BF action

$$S_2=\frac{i}{2}\int d^4x\,\epsilon^{\alpha\beta\mu\nu}\,\mathrm{Tr}\,G_{\alpha\beta}F_{\mu\nu}(\bar{A}). \tag{7}$$

The goal is to write down the result for S_2 through gauge-invariant combinations made of the dual field strength $G_{\alpha\beta}$ and to reveal its 12-function symmetry.

Gauge-invariant variables

First of all, we need a convenient parametrization of $G^a_{\alpha\beta}$ which have $6\cdot 3$ degrees of freedom (dof's), out of which $18-3=15$ are gauge-invariant. Our main variable will be an antisymmetric tensor $T^i_{\alpha\beta}=-T^i_{\beta\alpha}$ (the Greek indices run from 1 to 4 whereas the Latin indices run from 1 to 3). Given T, one constructs the quantity

$$(\sqrt{g})^3 \quad \stackrel{d}{=} \quad \frac{1}{48}\left(\epsilon_{ijk}\,T^i_{\alpha\beta}T^j_{\gamma\delta}T^k_{\epsilon\eta}\right)\,\left(\epsilon_{lmn}\,T^l_{\kappa\lambda}T^m_{\mu\nu}T^n_{\rho\sigma}\,\right)\,\epsilon^{\alpha\beta\kappa\lambda}\epsilon^{\gamma\delta\mu\nu}\,\epsilon^{\epsilon\eta\rho\sigma}.$$

With its help we construct the contravariant antisymmetric tensor,

$$T^{i\,\mu\nu}\stackrel{d}{=}\frac{1}{2\sqrt{g}}\,\epsilon^{\mu\nu\alpha\beta}\,T^i_{\alpha\beta} \tag{8}$$

and require the ortho-normalization condition,

$$T^i_{\alpha\beta}T^{j\,\alpha\beta}=\delta^{ij}. \tag{9}$$

This condition is 'dimensionless' in T, therefore it imposes not 6 but only 5 constraints on the covariant tensor $T^i_{\alpha\beta}$ which thus carries $18-5=13$ dof's. General solution to eq. (9) is given by

$$T^i_{\alpha\beta} = \eta^i_{AB}\, e^A_\alpha\, e^B_\beta \tag{10}$$

where e^A_α can be called the tetrad; η^i_{AB} is the 't Hooft symbol

$$\eta^i_{AB} = \frac{1}{2i}\mathrm{Tr}(\tau^+_A\tau^-_B - \tau^+_B\tau^-_A)\tau^i \qquad \tau^\pm_A = (\pm i\tau, \mathbf{1})\,, \quad A = 1...4. \tag{11}$$

All algebraic statements can be verified by exploiting the η-symbol algebra. There are 16 dof's in the tetrad, however three rotations under one of the $SO(3)$ subgroups of the $SO(4)$ Euclidean group does not enter into the combination (10), therefore, the r.h.s. of eq. (10) carries, as it should, 13 dof's.

We next introduce the metric tensor,

$$g_{\mu\nu} \stackrel{d}{=} \frac{1}{6}\,\epsilon^{ijk}\, T^i_{\mu\alpha} T^{j\;\alpha\beta} T^k_{\beta\nu} = e^A_\mu\, e^A_\nu = \frac{1}{6}\epsilon^{abc}\,\frac{\epsilon^{\alpha\beta\rho\sigma}}{2\sqrt{g}}\, G^a_{\mu\alpha} G^b_{\rho\sigma} G^c_{\beta\nu}. \tag{12}$$

It explains the previous notation: eq. (8) is consistent with the determinant of this metric tensor.

Finally, we parameterize the dual field strength as

$$G^a_{\alpha\beta} = d^a_i \, T^i_{\alpha\beta} = d^a_i \, \eta^i_{AB} \, e^A_\alpha \, e^B_\beta \, , \tag{13}$$

where the new variable d^a_i (we shall call it the triad) is subject to the normalization constraint $\det \, d^a_i = 1$ and therefore contains 8 dof's. In fact, the combination (13) is invariant under simultaneous $SO(3)$ rotations of T^i and d_i, therefore the r.h.s. of eq. (13) contains $13 + 8 - 3 = 18$ dof's, as does the l.h.s. Thus, eq. (13) is a complete parametrization of $G^a_{\mu\nu}$.

It is now clear how to organize the 15 gauge-invariant variables made of $G^a_{\mu\nu}$. These are the 5 dof's contained in a symmetric 3×3 tensor

$$h_{ij} \stackrel{d}{=} d^a_i \, d^a_j , \qquad \det \, h = 1, \tag{14}$$

and 13 dof's of $T^i_{\alpha\beta}$. However, h_{ij} and $T^i_{\alpha\beta}$ will always enter contracted in i, j (as it follows from eq. (13)), so that the dof's associated with the simultaneous $SO(3)$ rotation will drop out. In other words, one can choose h_{ij} to be diagonal and containing only 2 dof's.

Christoffel symbols, covariant derivative, Riemann tensor

We are now prepared to solve the saddle-point eq. (6) and to express the BF action (7) in a nice geometric way. We substitute (13) into eq. (6) and rewrite it as

$$0=\frac{1}{2}\epsilon^{\lambda\mu\alpha\beta}\,D^{ab}_{\mu}(\bar{A})\,\left(d^b_i\,T^i_{\alpha\beta}\right)=\frac{1}{\sqrt{g}}D^{ab}_{\mu}\left(d^b_i\,\sqrt{g}\,T^{i\,\lambda\mu}\right)$$

$$=\frac{1}{\sqrt{g}}\partial_{\mu}\left(\sqrt{g}\,T^{i\,\lambda\mu}\right)d^a_i+T^{i\,\lambda\mu}\,D^{ab}_{\mu}d^b_i. \tag{15}$$

The action of the YM covariant derivative on the triad can be decomposed in the triad again:

$$D^{ab}_{\mu}\,d^b_i=\gamma^j_{\mu i}\,d^a_j \tag{16}$$

which serves the definition of the 'minor' Christoffel symbol $\gamma^j_{\mu i}$ (not to be confused with the ordinary Christoffel symbol in $d=4$). With its help we define the 'minor' covariant derivative,

$$(\nabla_{\mu})^j_i\stackrel{d}{=}\partial_{\mu}\,\delta^j_i+\gamma^j_{\mu i}, \tag{17}$$

and the 'minor' Riemann tensor,

$$R^j_{\ i\,\mu\nu} = [\nabla_\mu\,\nabla_\nu]^j_i = \partial_\mu\gamma^j_{\nu i} - \partial_\nu\gamma^j_{\mu i} + \gamma^j_{\mu k}\gamma^k_{\nu i} - \gamma^j_{\nu k}\gamma^k_{\mu i}. \tag{18}$$

The saddle point eq. (15) can be compactly written as

$$\begin{aligned} (\nabla_\mu)^j_i\left(\sqrt{g}\,T^{i\,\lambda\mu}\right) &= 0 \quad \text{or} \\ T_{\kappa\lambda;\,\mu} + T_{\lambda\mu;\,\kappa} + T_{\mu\kappa;\,\lambda} &= 0, \end{aligned} \tag{19}$$

meaning that the antisymmetric tensor $T^{\alpha\beta}$ is 'covariantly constant'. Another consequence of eqs. (15,16) is that the symmetric tensor h_{ij} is covariantly constant, too:

$$h_{ik;\,\mu} \stackrel{d}{=} \partial_\mu\,h_{ik} - \gamma^j_{\mu i}\,h_{kj} - \gamma^j_{\mu k}h_{ij} = 0. \tag{20}$$

The 'minor' Christoffel symbol can be found explicitly; it consists of a symmetric and antisymmetric parts:

$$\gamma^j_{\mu i} = \tfrac{1}{2}h^{jn}(\partial_\mu h_{ni} + \epsilon_{nik}\,S^k_\mu),$$

$$
\begin{aligned}
S^k_\mu &= T^k_{\nu\beta}T^l_{\mu\alpha}g^{\alpha\beta}\left[h_{lm}\partial_\lambda T^{m\,\lambda\nu}\right.\\
&\qquad \left.+\frac{1}{2g}T^{m\,\lambda\nu}\partial_\lambda(g\,h_{lm})\right],
\end{aligned}
\tag{21}
$$

where we have used contravariant upper indices to denote inverse matrices h^{jn}, $g^{\alpha\beta}$.

Given the Christoffel symbol, one may return to eq. (15) and find the saddle-point YM field $\bar{A}_\mu$. However, we do not need the explicit form of $\bar{A}_\mu$ to find the action (7) at the saddle point.

Action in terms of gauge-invariant variables

In order to find the Yang–Mills field strength $F_{\mu\nu}$ at the saddle point we consider the double commutator of YM covariant derivatives,

$$
[\mathcal{D}_\mu[\mathcal{D}_\nu d_i]] = [\mathcal{D}_\mu,\ \gamma^j_{\nu i}d_j] = d_j(\partial_\mu\gamma^j_{\nu i}+\gamma^j_{\mu k}\gamma^k_{\nu i}),
\tag{22}
$$

and subtract the same commutator with $(\mu\nu)$ interchanged:

$$[\mathcal{D}_\mu[\mathcal{D}_\nu d_i]] - [\mathcal{D}_\nu[\mathcal{D}_\mu d_i]] = -[d_i[\mathcal{D}_\mu \mathcal{D}_\nu]]$$

$$= i[d_i\, F_{\mu\nu}(\bar{A})] = d_j\, R^j_{\ i\,\mu\nu}. \tag{23}$$

Hence we see that YM curvature at the saddle point is expressed via the 'minor' Riemann tensor, eq. (18). Explicitly,

$$F^a_{\mu\nu}(\bar{A}) = \tfrac{1}{2}\epsilon^{abc}\, d^{bi}\, d^c_j\, R^j_{\ i\,\mu\nu} \tag{24}$$

where the inverse triad, $d^{bi} d^b_l = \delta^i_l$, has been used; $d^{bi} = h^{im} d^b_m$. We put eq. (24) into the action (7) and get finally

$$S_2 = \frac{i}{4}\int d^4x\, \sqrt{g}\, R^j_{\ i\,\mu\nu}\, T^{l\,\mu\nu}\, \epsilon_{jlk}\, h^{ki}. \tag{25}$$

This is the $SU(2)$, $d = 4$ BF gravity action in the gauge-invariant or basis-independent formulation, as it is expressed through gauge-invariant variables T and h. We notice that it is covariant both in respect to Greek and Latin indices. It is a "more general relativity".

To get the full YM action in the gauge-invariant form one has to add the first ('æther') term of eq. (1),

$$S_1 = -\frac{g^2}{4}\int d^4x\, T^i_{\mu\nu}\, h_{ij}\, T^j_{\mu\nu}, \tag{26}$$

which is gauge- but not diffeomorphism-invariant.

In the particular case when $h_{ij} = \delta_{ij}$ the action (25) can be rewritten in terms of the 4-dimensional metric tensor $g_{\mu\nu}$ being a particular combination of $T^i_{\mu\nu}$, see eq. (12). In this case the BF action (25) comes to the usual Einstein–Hilbert action,

$$S_2|_{h_{ij}=\delta_{ij}} = \frac{i}{2}\int d^4x\, \sqrt{g}R, \tag{27}$$

where R is the standard scalar curvature made of $g_{\mu\nu}$.

Another particular case is arbitrary (but constant) field h_{ij} and a conformal flat metric, $g_{\mu\nu} = \Phi\,\delta_{\mu\nu}$. In this case (being of relevance to instantons) the BF action (25) is

$$S_2|_{\rm conf.flat} \quad = \quad \frac{i}{2}\int d^4x\, \left(-\partial^2\Phi + \frac{1}{2\Phi}\partial_\lambda\Phi\partial_\lambda\Phi\right)\,(h_{ii}h_{jj} - 2h_{ij}h_{ij}). \tag{28}$$

The above choice of the gauge-invariant variables T, h is not imperative. For example, one can use the 15 variables

$$W_{\alpha\beta\gamma\delta} \stackrel{d}{=} G^a_{\alpha\beta} G^a_{\gamma\delta} = T^i_{\alpha\beta}\, h_{ij}\, T^j_{\gamma\delta} \tag{29}$$

or some other set of 15 variables, depending on what properties of the theory one wishes to fix upon.

Gauge-invariant perturbation theory

A seeming paradox is that the Yang–Mills theory has gluon degrees of freedom at short distances, whereas in a gauge-invariant formulation there is no place for explicitly color degrees of freedom. We shall show now that eqs. (25,26) possess two transversely polarized gluons (times 3 colors). This is the correct gauge-invariant content of the perturbation theory at zeroth order.

Since S_1 is proportional to the coupling constant and S_2 is not, the zeroth order corresponds to $S_2 = 0$, i.e. to the 'minor' Riemann tensor ${R^i}_{j\,\mu\nu} = 0$, that is to the flat dual space. It implies that the 'minor' Christoffel symbol $\gamma^i_{\mu j}$ is a "pure gauge",

$$\gamma^i_{\mu j} = \left(O^{-1}\right)^i_k \partial_\mu O^k_j, \qquad \det\, O \neq 0. \tag{30}$$

Indeed, in this case the Riemann tensor is zero:

$$(\nabla_\mu)^i_j\, c^j = \partial_\mu c^i + \gamma^i_{\mu j} c^j = \left(O^{-1}\right)^i_k \partial_\mu \left(O^k_j c^j\right);$$

$$R^i_{\ j\,\mu\nu} c^j = \left[(\nabla_\mu)^i_k\,(\nabla_\nu)^k_j - (\mu \leftrightarrow \nu)\right] c^j = \left(O^{-1}\right)^i_l \partial_\mu \partial_\nu \left(O^l_j c^j\right) - (\mu \leftrightarrow \nu) = 0$$

for any vector c^j, therefore, $R^i_{\ j\,\mu\nu} = 0$.

We substitute eq. (30) into eq. (20) and get

$$0 = h_{ik;\,\mu} = \partial_\mu \left[\left(O^{-1}\right)^p_l h_{pq} \left(O^{-1}\right)^q_m\right]\; O^m_i O^l_k \tag{31}$$

meaning that $h_{pq} = O^i_p O^j_q\, D_{ij}$ where D_{ij} is a constant matrix. Next, we substitute eq. (30) into eq. (19) and obtain

$$\partial_\kappa \left(O^k_j T^j_{\lambda\mu}\right) + \partial_\lambda \left(O^k_j T^j_{\mu\kappa}\right) + \partial_\mu \left(O^k_j T^j_{\kappa\lambda}\right) = 0, \tag{32}$$

whose general solution is $O^k_j T^j_{\kappa\lambda} = \partial_\kappa B^k_\lambda - \partial_\lambda B^k_\kappa$. The first term in the action (S_1) is then

$$\begin{aligned} G^a_{\mu\nu} G^a_{\mu\nu} &= h_{pq} T^p_{\mu\nu} T^q_{\mu\nu} = D_{ij} (O^i_p T^p_{\mu\nu})(O^j_q T^q_{\mu\nu}) \\ &= D_{ij} \Big(\partial_\mu B^i_\nu - \partial_\nu B^i_\mu\Big)\Big(\partial_\mu B^j_\nu - \partial_\nu B^j_\mu\Big). \end{aligned} \tag{33}$$

D_{ij} is a constant matrix and can be set to be δ_{ij} by a linear transformation of the three vector fields B^i_μ; therefore, we obtain the lagrangian of three massless gauge fields. It is the expected result.

For higher groups the BF action will involve higher spin fields, and the invariance under dual gauge transformation will translate, after excluding the YM connection, into the invariance under mixing higher spins.

For more details, see: D.D. and Victor Petrov, `hep-th/0108097`.

How do unify the Standard Model with the renormalizable (!) Quantum Gravity?

1. Start from the 'more general relativity' theory given by the action S_2. It is not only renormalizable but probably exactly solvable. If, for some reason, the field h_{ij} develops spontaneously a v.e.v. $< h_{ij} >\sim \delta_{ij}$ one gets, at 'low' energies the Einstein's quantum gravity, but renormalizable in the ultra-violet.

2. Start from the 'more general relativity' based on some large Lie group. Imagine, that part of the large group is spontaneously broken by the appearance of the "æther" term. This part will be the Yang–Mills theory of the Standard Model. The other part of the large group must be without the "æther". This part will be Quantum Gravity.

Problems

1. Learn to understand things in terms of the dual space. Explain confinement and spontaneous mass generation – e.g. from the compactification of the dual space.

2. Rewrite the $SU(N)$ YM theory in $4d$ in terms of local gauge invariant variables.

3. Derive the string theory from the YM theory rewritten in terms of local gauge-invariant variables, in the limit $N \to \infty$.

4. Unify the Standard Model with renormalizable Quantum Gravity.