Princeton, Nov 10, 2003

Exotic baryons as chiral solitons

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2. 'Classical' theory of chiral solitons
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4. Quark content of chiral solitons

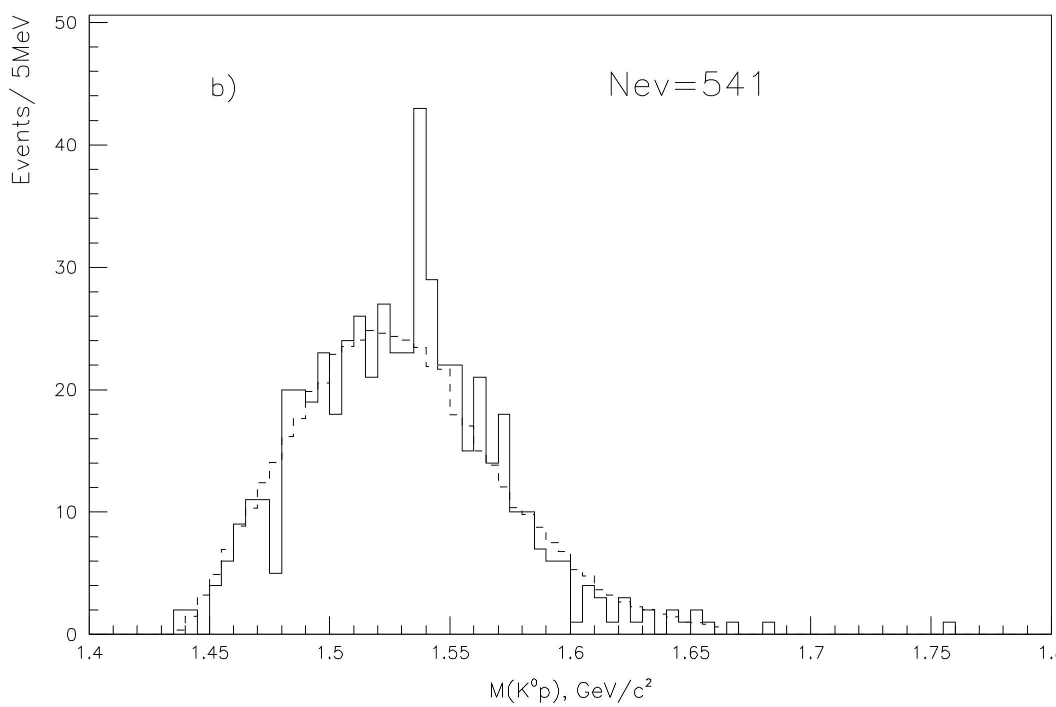
Discovery of pentaquark baryons

D.D., Victor Petrov and Maxim Polyakov,

Exotic Anti-Decuplet of Baryons: Prediction from Chiral Solitons, *Zeit. Physik* (1997)

Abstract

We predict an exotic baryon (having spin $1/2$, isospin 0 and strangeness $+1$) with a relatively low mass of about $1530\,\mathrm{MeV}$ and total width of less than $15\,\mathrm{MeV}$. It seems that this region of masses has avoided thorough searches in the past.



Θ^+ production in the ITEP xenon bubble chamber: $K^+n \to \Theta^+ \to K^0p$ [Barmin et al. (2003)]. Conclusion by the authors:

$$\mathcal{M}_{\Theta^+} = 1539 \pm 2\,\mathrm{MeV}, \quad \Gamma < 9\,\mathrm{MeV}, \quad \sigma = 4.4.$$

Narrow Θ^+ seen also in six other experiments.

'Classical' theory of chiral solitons

In the chiral limit ($m_u \simeq 4\,\mathrm{MeV} \to 0, m_d \simeq 7\,\mathrm{MeV} \to 0, m_s \simeq 150\,\mathrm{MeV} \to 0$) one can write down QCD at low momenta as a non-linear action depending on the octet of the pseudoscalar (pseudo-) Goldstone fields π, K, η:

$$S[\pi(x)] = S_{\mathrm{Re}}[\pi] + S_{\mathrm{WZ}}[\pi] = O(N_c),$$

$$S_{\mathrm{Re}}[\pi] = \frac{F_\pi^2}{4}\int d^4x\, \mathrm{Tr}\; L_\mu L_\mu \; + \; \text{higher derivatives},$$

$$L_\mu = iU^\dagger \partial_\mu U, \qquad U = \exp(i\pi^A\lambda^A),$$

$$S_{\mathrm{WZ}}[\pi] = \frac{N_c}{240\pi^2}\int d^5x\, \epsilon_{\alpha\beta\gamma\delta\epsilon}\, \mathrm{Tr}\; L_\alpha L_\beta L_\gamma L_\delta L_\epsilon \quad -$$

Wess–Zumino–Witten term, a full derivative in $5d$.

At large number of colors N_c baryons are **solitons** [Skyrme (1961), Witten (1983)] corresponding to the local minimum of this action, with baryon charge

$$B = N_{\mathrm{wind}} = \frac{1}{24\pi^2}\int d^3x\, \epsilon_{ijk}\, \mathrm{Tr}\; L_i L_j L_k$$

$$= \frac{1}{\pi}\int_0^\infty dr \frac{d}{dr}\left(P(r) - \tfrac{1}{2}\cos 2P(r)\right) = 1, \quad \text{if}$$

$$\pi^A(\mathbf{x}) = \begin{cases} n^a P(r), & A = a = 1,2,3, \\ 0, & A = 4,5,6,7,8 \end{cases},$$

$$P(r) = \text{soliton profile}; \quad P(0) = -\pi,\; P(\infty) \sim \frac{1}{r^2} \to 0.$$

Quantum fluctuations about classical minimum are, generally, suppressed as $1/N_c$:

$$\begin{aligned} \mathcal{E}[\pi_{\text{class}} + \delta\pi] &= \mathcal{M}_0 + \tfrac{1}{2}\,\delta\pi\, W\, \delta\pi + \ldots \\ \mathcal{M}_0 &= O(N_c), \text{ quadratic form} \qquad W = O(N_c) \\ \Rightarrow \delta\pi &= O\left(\frac{1}{\sqrt{N_c}}\right). \end{aligned}$$

This is incorrect for zero modes of W: fluctuations in the flat directions – translations and rotations – are not suppressed and must be treated exactly.

Skyrmion rotation

$$\pi(\mathbf{x},t) = R(t)\,\pi_{\text{class}}(\mathbf{x})\,R^\dagger(t).$$

One can introduce 'right' Ω_A and 'left' $\tilde{\Omega}_A$ angular velocities

$$\begin{aligned} \Omega_A &= -i\mathrm{Tr}\,(R^\dagger \dot{R} \lambda^A), \\ \tilde{\Omega}_A &= -i\mathrm{Tr}\,(\dot{R} R^\dagger \lambda^A), \qquad \Omega^2 = \tilde{\Omega}^2 = 2\mathrm{Tr}\,\dot{R}^\dagger \dot{R} \end{aligned}$$

and get the rotational lagrangian

$$\mathcal{L}^{\rm rot} = \frac{I_1}{2}\sum_{a=1}^{3}\Omega_a^2 + \frac{I_2}{2}\sum_{A=4}^{7}\Omega_A^2 + \frac{N_c}{2\sqrt{3}}\Omega_8.$$

The real part of the action cannot depend on the spurious rotation $R_8 = \exp(i\alpha_8(t)\lambda^8)$ but the WZW term turns out to be linear in Ω_8 [Witten (83), Guadagnini (84)].

Canonical quantization of rotations. The hamiltonian is

$$\begin{aligned}\mathcal{H}_{\rm rot} &= \Omega_A J_A - \mathcal{L}_{\rm rot} = \frac{1}{2I_1}\sum_{a=1}^{3} J_a^2 + \frac{1}{2I_2}\sum_{A=4}^{7} J_A^2,\\ J_A &= \frac{\partial \mathcal{L}}{\partial \Omega_A}, \qquad J_8 = \frac{N_c}{2\sqrt{3}}.\end{aligned}$$

There are two sets of ‘angular momenta’:

$$\begin{aligned}[J_A J_B] &= i f_{ABC} J_C, \qquad [\tilde{J}_A \tilde{J}_B] = i f_{ABC}\tilde{J}_C,\\ \left[J_A \tilde{J}_B\right] &= 0, \qquad J_A^2 = \tilde{J}_A^2.\end{aligned}$$

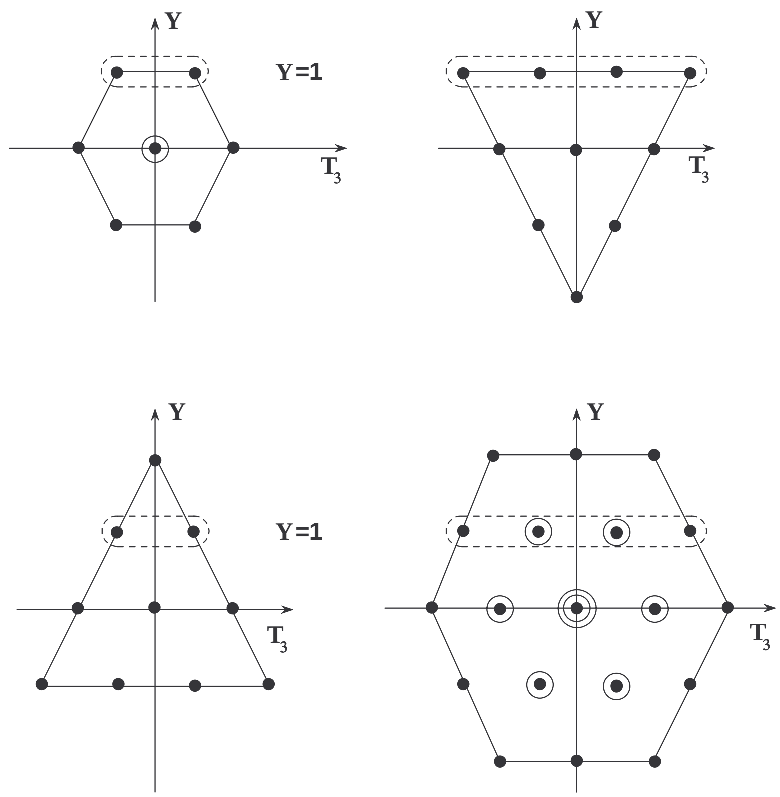
$\tilde{J}_A$ are generators of **left** shifts of R, i.e. **flavor** generators, and J_A are generators of **right** shifts, i.e. **spin** generators.

$$U = R\,(\mathbf{n}\cdot\tau)\,R^\dagger$$

Rotational eigenfunctions are ($J_8 = \frac{\sqrt{3}}{2}Y$, hypercharge)

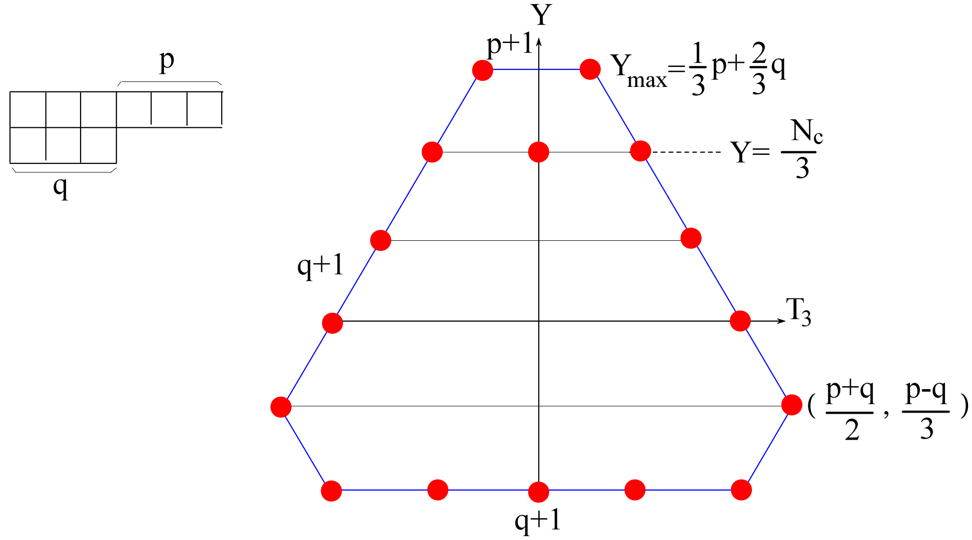
$$\Psi(R) = D^{\rm irep}_{J_3, Y'=\frac{N_c}{3};\ T_3, Y}(R).$$

Only those baryon $SU(3)$ multiplets appear as skyrmion rotational states, which contain hypercharge $Y = \frac{N_c}{3}$; the number of particles with $Y = \frac{N_c}{3}$ gives the spin multiplicity $2J+1$, i.e. the spin is also known. Examples of allowed multiplets at $N_c = 3$:



The first two are exactly the lowest multiplets in nature; the antidecuplet with spin and parity $\frac{1}{2}^+$ must be also observed! [D.D. and Petrov (84), Chemtob (84)]

What is the generalization of these irep's to arbitrary odd $N_c \neq 3$?



Dimension of a (p, q) representation

$$\mathrm{Dim}(p,q) = (p+1)(q+1)\left(1+\frac{p+q}{2}\right)$$

The eigenvalue of the quadratic Casimir operator

$$C_2(p,q) = \frac{1}{3}\left[p^2 + q^2 + pq + 3(p+q)\right].$$

The spin of the allowed multiplet is

$$J = \frac{1}{6}(4p + 2q - N_c).$$

Particles at the upper side of the hexagon have hypercharge

$$\begin{aligned} Y_{\max} &= \frac{1}{3}p + \frac{2}{3}q \\ &\equiv \frac{N_c}{3} + E, \qquad E \geq 0. \end{aligned}$$

E gives the minimal number of additional quark-antiquark pairs one needs to add on top of the usual N_c quarks to compose a multiplet, hence can be named "exoticness".

Expressing the rotational energy of a skyrmion through (E, J) instead of (p, q):

$$\mathcal{E}_{\rm rot}(J,E) = \frac{E^2 + E(\frac{N_c}{2} + 1 - J) + \frac{N_c}{2}}{2I_2} + \frac{J(J+1)}{2I_1}.$$

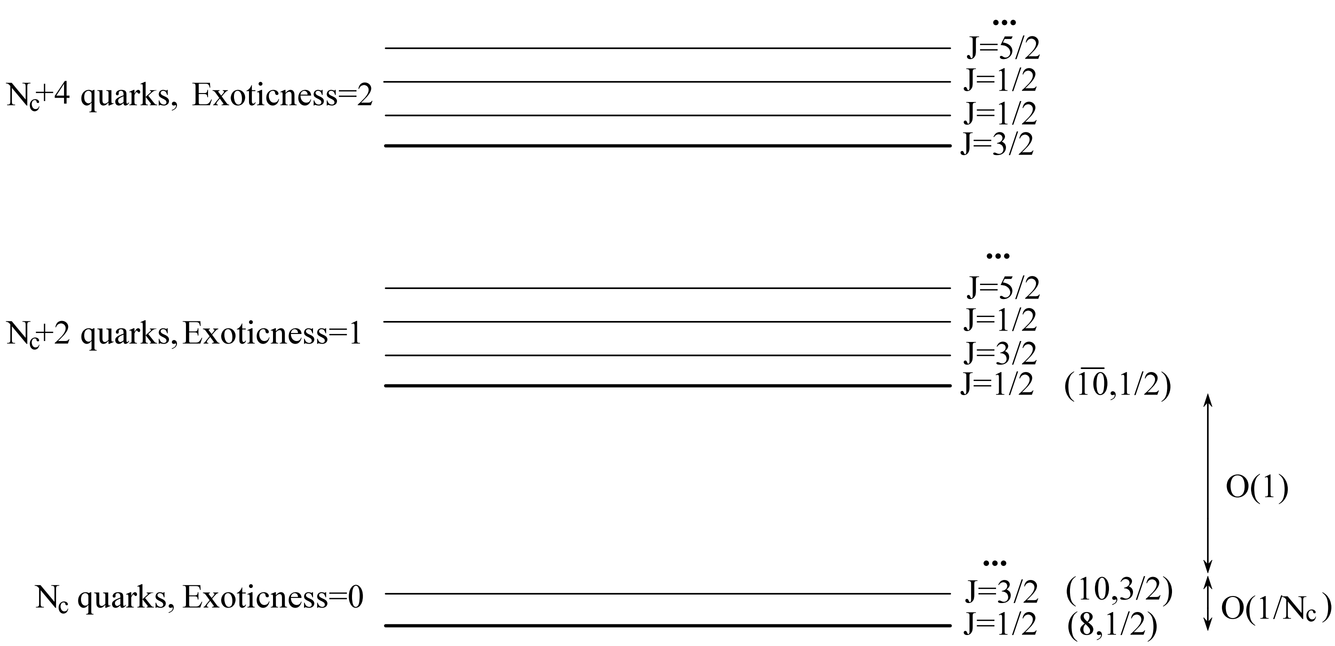
At large N_c the spectrum is equidistant in exoticness:

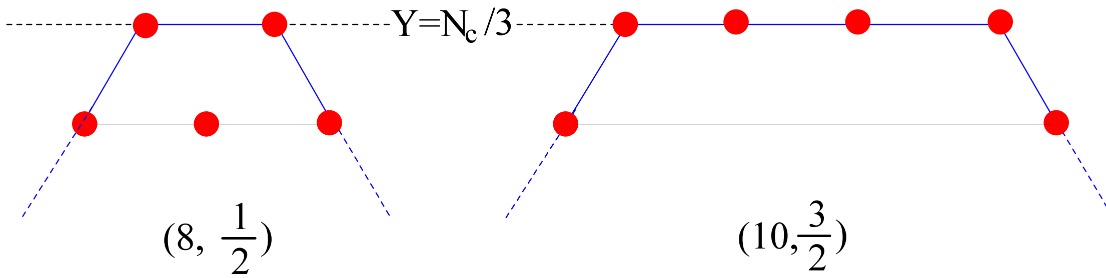
$$\mathcal{E}_{\rm rot}(E) = \frac{N_c(E+1)}{4I_2}$$

with the spacing $\frac{N_c}{4I_2} = O(N_c^0)$. It means that each time we add a quark-antiquark pair it costs at large N_c the same energy

$$\text{energy of a pair} = \frac{N_c}{4I_2} = O(1).$$

In physical terms, the energy cost of adding a pair can be small if the pair is added in the form of a Goldstone boson.


...
J=5/2
J=1/2
J=1/2
J=3/2
Nc+4 quarks, Exoticness=2
...
J=5/2
J=1/2
J=3/2
J=1/2 (10,1/2)
Nc+2 quarks, Exoticness=1
O(1)
...
J=3/2 (10,3/2)
J=1/2 (8,1/2)
Nc quarks, Exoticness=0
O(1/Nc)



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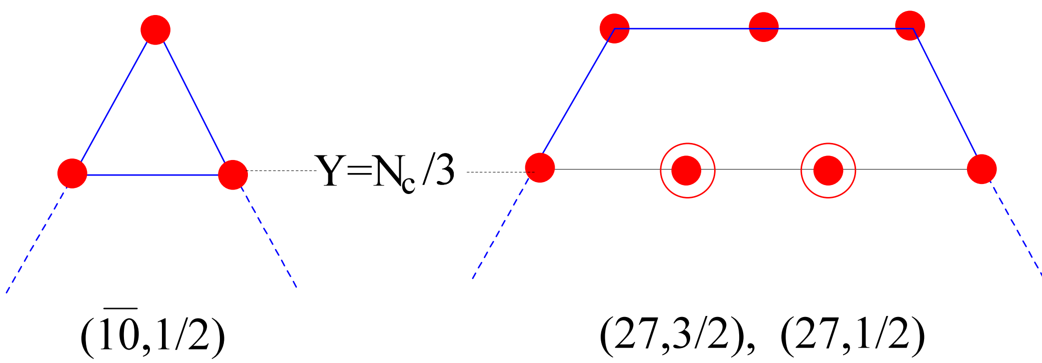
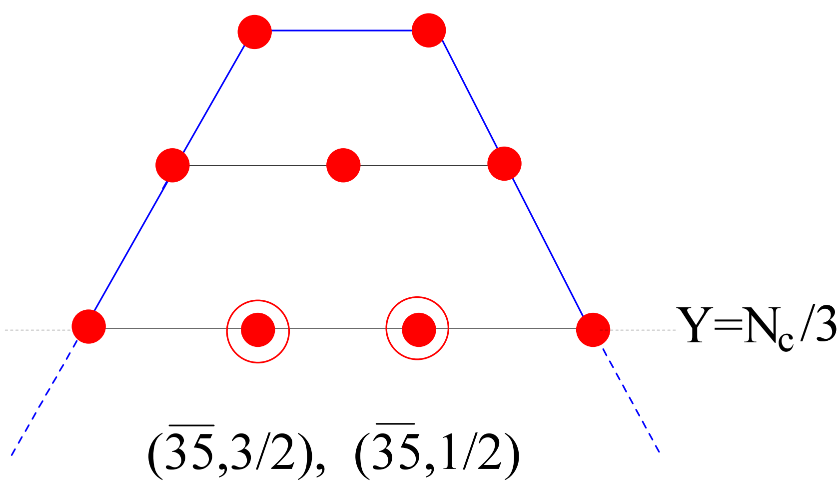
Non-exotic ($E = 0$) multiplets that can be composed of N_c quarks.



Exotic ($E = 1$) multiplets that can be composed of N_c quarks and one extra $\bar{Q}Q$ pair.



Exotic ($E = 2$) multiplets needing two extra pairs.

Since the rotational splitting for exotic multiplets is $O(1)$, a question arises, can we trust it? Why don't rotation mix up with normal fluctuations (e.g. vibrations) of a skyrmion?

Rotation is slow

$$\sum_{A=1}^{3} J_A^2 = J(J+1), \quad \sum_{A=4}^{7} J_A^2 \quad \xrightarrow{N_c\to\infty} \quad \frac{N_c}{2}(E+1),$$

As operators, $\hat{\Omega}_{1\text{–}3} = \hat{J}_{1\text{–}3}/I_1$ and $\hat{\Omega}_{4\text{–}7} = \hat{J}_{4\text{–}7}/I_2$. Hence

$$\sqrt{<\Omega^2_{1\text{–}3}>} \quad \leq \quad \frac{\sqrt{J_1^2+J_2^2+J_3^2}}{I_1} = O\left(\frac{J}{N_c}\right),$$

$$\sqrt{<\Omega^2_{4\text{–}7}>} \quad \leq \quad \frac{\sqrt{J_4^2+J_5^2+J_6^2+J_7^2}}{I_2} = O\left(\frac{\sqrt{E+1}}{\sqrt{N_c}}\right).$$

We see that $\Omega_{1\text{–}3} \ll \Omega_{4\text{–}7} \ll 1$ both for non-exotic and exotic multiplets. For any exoticness $E \ll N_c$ the soliton rotation is slow.

Interplay between vibrational and rotational excitations leads to a $1/N_c$ correction to the 'exotic' rotational levels.

Quantum fluctuations about $SU(3)$ skyrmions

We wish to evaluate the evolution operator in the saddle-point approximation:

$$\mathcal{Z} = \int D\pi(x,t)\, \exp\left[iS_{\rm Re} + iS_{\rm WZ}\right].$$

General parametrization

$$\pi(x,t) = R(t)\ \left[\pi_{\rm class}(x) + \delta\pi(x,t)\right]\ R^\dagger(t)$$

is not unique since part of $\delta\pi$ contains an infinitesimal rotation. Insert a unity:

$$\begin{aligned} 1 &= ({\rm Jac})^{-1} \int DR(t)\, D\delta\pi(x,t)\, \delta\left(\pi - R\left[\pi_{\rm class} + \delta\pi\right] R^\dagger\right) \\ &\times\ \delta\left(\Omega_A \int \delta\pi\, k_A\right). \end{aligned}$$

Decompose the general fluctuation $\delta\pi(x,t)$ in (ortho-normalized) eigenfunctions of the quadratic form:

$$\begin{aligned} W\,\psi_n &= \kappa_n\,\psi_n, \qquad \kappa_n \geq 0, \\ \delta\pi(x,t) &= d_A(t)\psi_0^A(x) + \sum c_n(t)\psi_n(x) \\ \psi_0^A(x) &= [\lambda^A\,\hat{n}]\,\sin P(r) \quad \text{rotational zero mode} \end{aligned}$$

δ-function is used to eliminate integration over the Fourier coefficients of the zero mode $d_A(t)$, expressing them through $c_n(t)$. Then

$$\mathcal{Z} = \int DR(t)\, Dc_n(t)\, \text{Jac}\, \exp\left(iN_c\left[\Omega^2 + O(c_n^2, \dot{c}_n^2) + O(\Omega^2 c_n)\right]\right).$$

Would be standard stuff, but there are two beautiful peculiarities:

- U(1) gauge invariance (!)
- the WZW term

Decompose $SU(3)$ adjoint variables in terms of $SU(2)$ triplets, doublets and singlets:

$$\begin{aligned} i\,\text{Tr}\, R^\dagger \dot{R} = \Omega &= \Omega_a \tau^a + \Omega^\alpha \lambda^\dagger_\alpha + \Omega^\dagger_\alpha \lambda^\alpha + \Omega_8 \lambda^8, \\ \delta\pi &= \pi_a \tau^a + K^\alpha \lambda^\dagger_\alpha + K^\dagger_\alpha \lambda^\alpha + \eta \lambda^8, \\ K^\alpha &= \text{Tr}\, \delta\pi\, \lambda^\alpha, \qquad \lambda^\alpha = \begin{pmatrix} \lambda^4 + i\lambda^5 \\ \lambda^6 + i\lambda^7 \end{pmatrix}. \end{aligned}$$

<u>$U(1)$ gauge invariance.</u> Simultaneous transformation

$$R(t) \to R(t)\, e^{i\gamma(t)\lambda^8}, \qquad \pi(x,t) \to e^{-i\gamma(t)\lambda^8}\, \pi(x,t)\, e^{i\gamma(t)\lambda^8}$$

does not change anything. Under this tramformation,

$$\begin{aligned} \pi_a &\to \pi_a, \quad \Omega_a \to \Omega_a, \quad \eta \to \eta, \\ K^\alpha &\to e^{-i\gamma(t)\sqrt{3}}\, K^\alpha, \quad \Omega^\alpha \to e^{-i\gamma\sqrt{3}}\, \Omega^\alpha, \quad \Omega_8 \to \Omega_8 + 2\dot{\gamma}. \end{aligned}$$

Hence all time derivatives of kaon fluctuations enter only as **covariant derivatives**, $D_t K^\alpha = \left(\partial_t + i\frac{\sqrt{3}}{2}\Omega_8\right) K^\alpha$.

$$S_{\rm WZ} = \frac{N_c B}{2\sqrt{3}}\Omega_8 + i\frac{N_c}{4}\int d^3x\, B(r)\left[K^\dagger D_t K - (D_t K)^\dagger K\right]$$
$$= i\frac{N_c}{8}\int \left[\Omega^\dagger K - K^\dagger \Omega\right]\frac{(1-\cos P)\sin^2 P\, P'}{r^2},$$

$$S_{\rm Re} = \frac{I_1}{2}\sum_{a=1}^{3}\Omega_a^2 + \frac{I_2}{2}\sum_{A=4}^{7}\Omega_A^2 + N_c\int (D_t K)^\dagger D_t K\, H(x)$$
$$+ \; N_c\int \left[\Omega^\dagger D_t K + (D_t K)^\dagger \Omega\right] G(x).$$

Ω_8 without the time derivative appears only in the 'classical' WZ term, but there the gauge transformation shifts it by a full derivative $d\gamma/dt$. *This is an alternative way to get the main quantization condition* $\tilde{Y} = \frac{N_c}{3}$. In all other terms Ω_8 can be gauged out.

The δ-function eliminating extra degrees of freedom in $\delta\pi(x,t)$ has to be imposed on terms **linear** in Ω's. Then the mixing of rotational and 'vibrational' d.o.f.'s happens only in the cubic order. It leads to small $1/N_c$ corrections to the energies of a rigid rotator, even for 'exotic' multiplets [D.D. and Petrov (03)].

The larger N_c the more accurate would be the description of Θ^+ as a rotational state of a chiral soliton.

What is smaller: $1/N_c$ or m_s/Λ?

Theoretically, any relation can be considered. In nature:

- Splittings in baryon octet and decuplet (owing to m_s) is somewhat less than the splitting between octet and decuplet centers (which is $1/N_c$). In addition, Gell-Mann–Okubo relations are satisfied to 0.5% accuracy
- In mesons: splitting between octet η and singlet η' pseudoscalar mesons is $1/N_c$; their mixing angle is $O(m_s/N_c)$, numerically $\simeq 1/6$.

Conclusion: reasonable limit $m_s = O(\Lambda/N_c)$ or less. In baryons, m_s can be treated as a perturbation in most cases.

Summary

1. The 'classical' theory of rotating chiral solitons predicts not only standard baryon multiplets $(\mathbf{8}, \frac{1}{2})$, $(\mathbf{10}, \frac{3}{2})$ but also the exotic $(\overline{\mathbf{10}}, \frac{1}{2})$ multiplet which cannot be 'made of' three quarks but whose properties are related by symmetry with those than can be. This is the base for the successful prediction of the Θ^+.

2. The 'exotic' rotational excitations have $O(1)$ spacing. Nevertheless, corrections to the rigid rotator scheme from non-rotational excitations are small as $1/N_c$, after the correct separation of vibrations from rotations.

3. In the Chiral Quark Soliton Model it is possible to decode the quark content of the chiral soliton, and to build relativistic-invariant wave functions of both non-exotic and exotic baryons, in terms of quarks and antiquarks.