Non-perturbative methods in Quantum Field Theory

in application to particle and condensed matter physics

Lecture notes available at http://www.nordita.dk/ diakonov and at http://www.phys.psu.edu/ guest8

Content of the course

1. What do condensed matter and particle physics have in common?

2. Path (or functional) integrals in Quantum Mechanics, Statistical Mechanics and Field Theory

3. Two-dimensional models: O(N), CP(N). Asymptotic freedom

4. Friedan's renormalization in curved space in aplication to spin systems

5. Nonperturbative 1/N expansion. Spontaneous mass generation and restoration of symmetry

6. Yang–Mills theory in 2,3,4 dimensions

7. Yang–Mills theory on a lattice. Methods and results

8. Classical solutions. Instantons and solitons

9. Quantum fluctuations about instantons

10. Self-organizing instanton ensemble

11. Tunneling of Bose–Einstein Condensate (BEC). Macroscopic ‘self-trapping’. ‘Swallow-tail catastrophe’ in BEC on optical lattices

12. Instantons in 2d models. Domain walls

13. Quantum fluctuations about solitons. The Skyrmion. The new 5-quark baryon

14. Quantum determinants: exact, approximate and numerical methods

15. Quantum anomalies. Interpretation of anomalies

16. Duality transformation in gauge theory. Yang–Mills theory as Quantum Gravity. Spontaneous mass generation

17. Instantons in Quantum Chromodynamics

18. Spontaneous chiral symmetry breaking in strong interactions

19. Monopoles and the confinement of color

20. Sponges as an object of Quantum Gravity.

What do condensed matter
and particle physics
have in common?

Short answer:

1. The two disciplines deal with fields
[spin density, EM field, electron and quark wave functions...]

2. The fields can fluctuate from point to point, i.e. they are quantum fields

3. The adequate language is Feynman's path integrals

4. Theoretical methods are essentially the same

Feynman's formulation of QM:

Let us consider a non-relativistic particle with mass m in a one-dimensional potential well $V(x)$. Its Lagrangian is

$$L = \frac{m\dot{x}^2}{2} - V(x) \tag{1}$$

and its energy is $H = \frac{m\dot{x}^2}{2} - V(x)$. In Quantum mechanics one introduces the Hamiltonian

$$\mathcal{H} = -\frac{1}{2m}\frac{d^2}{dx^2} + V(x). \tag{2}$$

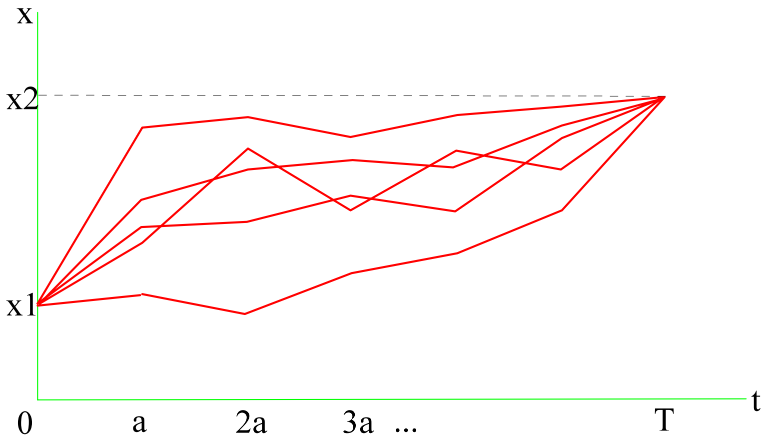
To find the (quantized) energy levels E_n and the stationary wave functions $\psi_n(x)$ one solves the Schrödinger eqn:

$$\mathcal{H}\psi_n(x) = E_n\,\psi_n(x) \tag{3}$$

The evolution operator is $e^{-i\mathcal{H}T}$ where T is time.

The probability amplitude that the particle goes from point x_1 to point x_2 during time T is

$$
\begin{aligned}
A_{12} \quad &= \quad \langle x_2| e^{-\frac{iHT}{\hbar}} |x_1\rangle = \sum_n \psi_n^*(x_2)\, e^{-iE_nT}\, \psi_n(x_1) \\
&= \quad \int_{x(0)=x_1}^{x(T)=x_2} Dx(t)\ \exp \frac{i}{\hbar} \int_0^T dt \left[\frac{m\dot{x}^2(t)}{2} - V\left(x(t)\right) \right]
\end{aligned}
$$



Path integral over trajectories
with x(0)=x1, x(T)=x2

Discretized action

$$S = \sum_n a \left[\frac{m}{2} \left(\frac{x(t_n) - x(t_{n-1})}{a} \right)^2 - V\left(x(t_n)\right) \right] .$$

Path integral can be understood as the limit of an infinite number of ordinary integrations

– over the intermediate points $x_1 \dots x_N$:

$$A_{12} = \lim_{N \to \infty} \mathcal{N} \prod_{n=1}^{N} \int dx_n \, e^{\frac{i}{\hbar} S(x_1, \dots, x_N)}.$$

If $S \gg \hbar$ quantum mechanics becomes classical: small fluctuations of trajectories lead to a large variation of the action, and phase factors for close trajectories $e^{iS/\hbar}$ annihilate each other. Only those trajectories contribute to the path integral, whose small variation doesn't change the action, $\delta S = 0$. But this condition is the Euler–Lagrange equation of motion:

$$-\frac{d}{dt}\frac{dL}{d\dot{x}} + \frac{dL}{dx} = 0\,!$$

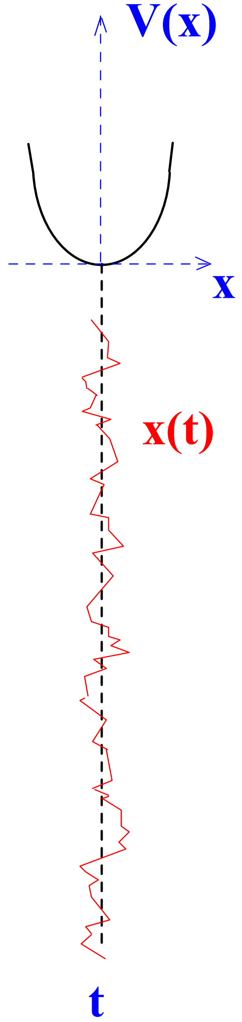
In classical theory one differentiates
In quantum theory one integrates

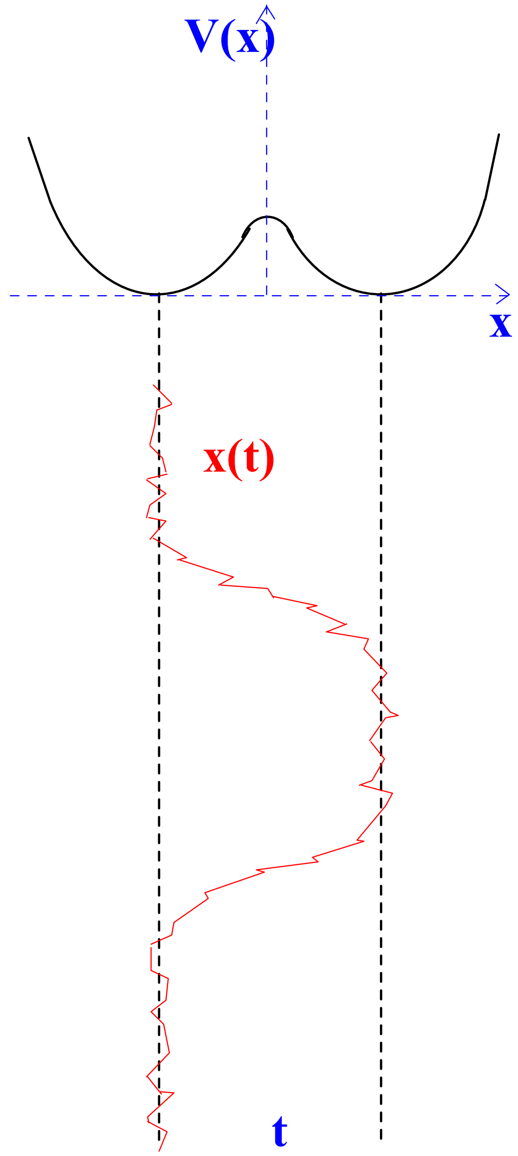
To cut out the ground state: take $t \longrightarrow -i\tau,\ T \longrightarrow -iT,\ (T \longrightarrow \infty)$. Put $x_1 = x_2$ and integrate over it. This is called the partition function: (henceforth $\hbar \longrightarrow 1$)

$$\begin{aligned} \mathcal{Z} &= \sum_n e^{-E_n T} \\ &= \int Dx(\tau)\ \exp - \int_0^T d\tau \left[\frac{m}{2} \left(\frac{dx}{d\tau} \right)^2 + V\left(x(\tau)\right) \right] . \end{aligned}$$

In statistical mechanics one implies that $T = \frac{1}{kT^{\mathrm{o}}} = \beta$ where T^{o} is temperature. In particle physics T is the observation (Euclidean) time. If one is interested only in the ground (or vacuum) state E_0 one cuts it out by taking temperature $T^{\mathrm{o}} \longrightarrow 0$ or observation time $T \longrightarrow \infty$.

How do typical trajectories $x(\tau)$ look like?


V(x)
x
x(t)
t


V(x)
x
x(t)
t

Text-book example: harmonic oscillator,

$$V(x) = \tfrac{m}{2}\omega_0^2\, x^2.$$

Since $x(t+T) = x(t)$ we can decompose arbitrary trajectory (over which we have to integrate!) in periodic functions,

$$x(t) = \sum_{n=-\infty}^{\infty} c_n \frac{1}{\sqrt{T}} \exp\left(2\pi i\, n\frac{t}{T}\right), \qquad c_{-n} = c_n^*$$

and the path integral will be understood as a product of integrals over all Fourier coefficients,

$$\int Dx(t) = \prod_n \int \frac{dc_n}{\sqrt{2\pi}}.$$

We shall need the ortho-normalization condition,

$$\int_0^T dt \, \frac{1}{\sqrt{T}} \, \exp\left(2\pi i \, m \frac{t}{T}\right) \, \frac{1}{\sqrt{T}} \, \exp\left(2\pi i \, n \frac{t}{T}\right) = \delta_{m+n,0} .$$

We get in the exponent of the path integral

$$\frac{m}{2} \int dt \, (\dot{x}^2 + \omega_0^2 \, x^2) = \frac{m}{2} \sum_{n=-\infty}^{\infty} c_{-n} \left[\left(\frac{2\pi n}{T}\right)^2 + \omega_0^2 \right] c_n$$

so that

$$\prod_n \int \frac{dc_n}{\sqrt{2\pi}} \, \exp\left(- \sum_n c_{-n} \, \lambda_n \, c_n \right) \sim \prod_n \frac{1}{\sqrt{\lambda_n}} .$$

Harmonic oscillator's partition function is

$$\mathcal{Z} = \prod_n \sqrt{\frac{1}{\left(\frac{2\pi n}{T}\right)^2 + \omega_0^2}} = \frac{1}{\omega_0} \prod_{n=1}^{\infty} \frac{1}{\left(\frac{2\pi n}{T}\right)^2 + \omega_0^2}$$

$$= \frac{1}{\omega_0} \, \frac{\frac{\omega_0 T}{2}}{\sinh \frac{\omega_0 T}{2}} = \frac{T}{e^{\frac{\omega_0 T}{2}} - e^{-\frac{\omega_0 T}{2}}}$$

$$= T \, e^{-\frac{\omega_0 T}{2}} \left(1 + e^{-\omega_o T} + e^{-2\omega_0 T} + e^{-3\omega_0 T} + \ldots\right)$$

$$= (T) \sum_n e^{-E_n T} \overset{T\to\infty}{=} e^{-\frac{\omega_0 T}{2}} , \qquad E_n = \omega_0 \left(\frac{1}{2} + n\right)$$

[Actually, an infinite ω_0-independent product $\prod_{n=1}^{\infty} \left(\frac{2\pi n}{T}\right)^2$ as due to the integration measure has been inserted to make the ratio finite].

[don't forget that in statistical mechanics $\omega_0 \longrightarrow \hbar\omega_0$, $T \longrightarrow \frac{1}{kT^o}$]

Important lesson:

Path Integral $=$ Product of integrals over Fourier coefficients

Quantum mechanics can be called $0+1$-dimensional field theory. In true field theory the 'coordinates' are fields depending on real space coordinates $\overrightarrow{x}$.

Typical path integral in QFT:

$$\mathcal{Z} = \int D\phi(\overrightarrow{x}, t)\ \exp\left(-S[\phi, \partial\phi, \ldots]\right)\ .$$

For example,

$$S[\phi, \partial\phi, ...] = \int_0^T dt \int d^d x \left(\frac{1}{2}\partial_i\phi\partial_i\phi + \frac{\lambda}{4}\phi^4 \right) \; .$$

It is a field theory in $d+1$ dimensions.

We shall study many such examples ...