Instantons in Quantum Chromodynamics

Periodicity of the Yang–Mills potential energy

Let us temporarily work in the $A_0^a = 0$ gauge, called Weyl or Hamiltonian gauge, and forget about fermions (=quarks) for a while. The remaining pure YM or "pure glue" theory is nonetheless non-trivial, since gluons are self-interacting. For simplicity I start from the $SU(2)$ gauge group.

The spatial YM potentials $A_i^a(\mathbf{x}, t)$ can be considered as an infinite set of the coordinates of the system, where $i = 1, 2, 3, \quad a = 1, 2, 3$ and $\mathbf{x}$ are "labels" denoting various coordinates. The YM action is

$$S = \frac{1}{4g^2} \int d^4x \; F^a_{\mu\nu} F^a_{\mu\nu} = \int dt \left(\frac{1}{2g^2} \int d^3\mathbf{x} \; \mathbf{E}^2 - \frac{1}{2g^2} \int d^3\mathbf{x} \; \mathbf{B}^2 \right) \qquad (1)$$

where $\mathbf{E}$ is the electric field strength,

$$E_i^a(\mathbf{x},t) = \dot{A}_i^a(\mathbf{x},t) \tag{2}$$

(the dot stands for the time derivative), and $\mathbf{B}$ is the magnetic field stregth,

$$B_i^a(\mathbf{x},t) = \frac{1}{2}\epsilon_{ijk}\left(\partial_j A_k^a - \partial_k A_j^a + \epsilon^{abc} A_j^b A_k^c\right). \tag{3}$$

Apparently, the first term in eq. (1) is the kinetic energy of the system of coordinates $\{A_i^a(\mathbf{x},t)\}$ while the second term is minus the potential energy being just the magnetic energy of the field. The simple and transparent form of eq. (2) is the advantage of the Weyl gauge. Upon quantization the electric field is replaced by the variational derivative, $E_i^a(x) \to -ig^2\delta/\delta A_i^a(x)$, if one uses the 'coordinate representation' for the wave functional. The functional Schrödinger equation for the wave functional $\Psi[A_i^a(x)]$ takes the form

$$\mathcal{H}\Psi[A_i] = \int d^3x \left\{-\frac{g^2}{2}\frac{\delta^2}{(\delta A_i^a(x))^2} + \frac{1}{2g^2}(B_i^a(x))^2\right\}\Psi[A_i] = \mathcal{E}\Psi[A_i] \tag{4}$$

where $\mathcal{E}$ is the eigenenergy of the state in question. The YM vacuum is the ground state of the Hamiltonian (4), corresponding to the lowest energy $\mathcal{E}$.

Let us introduce an important quantity called the Pontryagin index or the four-dimensional topological charge of the YM fields:

$$Q_T = \frac{1}{32\pi^2}\int d^4x\ F^a_{\mu\nu}\tilde{F}^a_{\mu\nu}, \qquad \tilde{F}^a_{\mu\nu} \equiv \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}F^a_{\alpha\beta}. \tag{5}$$

The integrand in eq. (5) happens to be a full derivative of the four-vector K_μ:

$$\frac{1}{32\pi^2}F^a_{\mu\nu}\tilde{F}^a_{\mu\nu} = \partial_\mu K_\mu, \qquad K_\mu = \frac{1}{16\pi^2}\epsilon_{\mu\alpha\beta\gamma}\left(A^a_\alpha\partial_\beta A^a_\gamma + \frac{1}{3}\epsilon^{abc}A^a_\alpha A^b_\beta A^c_\gamma\right). \tag{6}$$

Therefore, assuming the fields A_μ are decreasing rapidly enough at spatial infinity, one can rewrite the 4-dimensional topological charge (5) as

$$Q_T = \int d^4x(\partial_0 K_0 - \partial_i K_i) = \int dt\frac{d}{dt}\int d^3\mathbf{x}K_0. \tag{7}$$

Introducing the *Chern–Simons number*

$$N_{CS} = \int d^3\mathbf{x}\; K_0 = \frac{1}{16\pi^2}\int d^3\mathbf{x}\,\epsilon^{ijk}\left(A_i^a\partial_j A_k^a + \frac{1}{3}\epsilon^{abc}A_i^a A_j^b A_k^c\right) \qquad (8)$$

we see from eq. (7) that Q_T can be rewritten as the difference of the Chern–Simons numbers characterizing the fields at $t = \pm\infty$:

$$Q_T = N_{CS}(+\infty) - N_{CS}(-\infty). \qquad (9)$$

The Chern–Simons number of the field has an important property that it can change by integers under large gauge transformations. Indeed, under a general time-independent gauge transformation,

$$A_i \to U^\dagger A_i U + iU^\dagger\partial_i U, \quad A_i \equiv A_i^a\frac{\tau^a}{2}, \qquad (10)$$

the Chern–Simons number transforms as follows:

$$N_{CS} \to N_{CS} + N_W + \frac{i}{8\pi^2}\int d^3x\,\epsilon^{ijk}\partial_j\,\mathrm{Tr}\,(\partial_i U U^\dagger A_k). \qquad (11)$$

The last term is a full derivative and can be omitted if, *e.g.*, A_i decreases sufficiently fast at spatial infinity. N_W is the winding number of the gauge transformation (10):

$$N_W = \frac{1}{24\pi^2} \int d^3\mathbf{x}\, \epsilon^{ijk} \left[(U^\dagger \partial_i U)(U^\dagger \partial_j U)(U^\dagger \partial_k U) \right]. \tag{12}$$

The $SU(2)$ unitary matrix U of the gauge transformation (10) can be viewed as a mapping from the 3-dimensional space onto the 3-dimensional sphere of parameters S^3. If at spatial infinity we wish to have the same matrix U independently of the way we approach the infinity (and this is what is usually assumed), then the spatial infinity is in fact one point, so the mapping is topologically equivalent to that from S^3 to S^3. This mapping is known to be non-trivial, meaning that mappings with different winding numbers are irreducible by smooth transformations to one another. The winding number of the gauge transformation is, analytically, given by eq. (12). As it is common for topological characteristics, the integrand in (12) is in fact a full derivative. For example, if we take the matrix $U(\mathbf{x})$ in a "hedgehog" form, $U = \exp[i(r \cdot \tau)/r\, P(r)]$, eq. (12) can be rewritten as

$$N_W = \frac{2}{\pi} \int dr \frac{dP}{dr} \sin^2 P = \frac{1}{\pi} \left[P - \frac{\sin 2P}{2} \right]_0^\infty = \text{integer} \tag{13}$$

since $P(r)$ both at zero and at infinity needs to be multiples of π if we wish $U(\mathbf{r})$ to be unambigiously defined at the origin and at the infinity.

Let us return now to the potential energy of the YM fields,

$$\mathcal{V} = \frac{1}{2g^2} \int d^3\mathbf{x} \left(B_i^a\right)^2 . \tag{14}$$

One can imagine plotting the potential energy surfaces over the Hilbert space of the coordinates $A_i^a(\mathbf{x})$. It will be some complicated mountain country. If the field happens to be a pure gauge, $A_i = iU^\dagger \partial_i U$, the potential energy at such points of the Hilbert space is naturally zero. Imagine that we move along the "generalized coordinate" being the Chern–Simons number (8), fixing all other coordinates whatever they are. Let us take some point $A_i^a(\mathbf{x})$ with the potential energy $\mathcal{V}$. If we move to another point which is a gauge transformation of $A_i^a(\mathbf{x})$ with a winding number N_W, its potential energy will be exactly the same as it is strictly gauge invariant. However the Chern–Simons "coordinate" of the new point will be shifted by an integer N_W from the original one. We arrive to the conclusion first pointed out by Faddeev and Jackiw and Rebbi in 1976, that the potential energy of the YM fields is *periodic* in the particular coordinate called the Chern–Simons number.

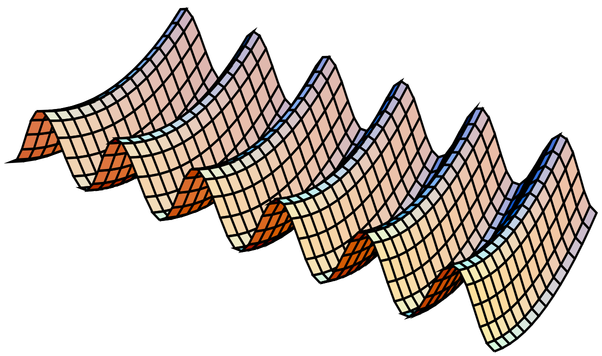

Figure 1: Potential energy of the gluon field is periodic in one direction and oscillator-like in all other directions in functional space.

Instantons in simple words

In perturbation theory one deals with zero-point quantum-mechanical fluctuations of the YM fields near one of the minima, say, at $N_{CS} = 0$. The non-linearity of the YM theory is taken into account as a perturbation, and results in series in g^2 where g is the gauge coupling. In that approach one is apparently missing a possibility for the system to tunnel to another minimum, say, at $N_{CS} = 1$. The tunneling is a typical non-perturbative effect in the coupling constant.

Instanton is a large fluctuation of the gluon field in imaginary (or Euclidean) time corresponding to quantum tunneling from one minimum of the potential energy to the neighbor one. Mathematically, it was discovered by Belavin, Polyakov, Schwarz and Tyupkin (1975); the tunneling interpretation was given by V. Gribov (1976). The name 'instanton' has been introduced by 't Hooft (1976) who studied many of the key properties of those fluctuations. Anti-instantons are similar fluctuations but tunneling in the opposite direction in Fig. 1. Physically, one can think of instantons in two ways: on the one hand it is a tunneling *process* occuring in time, on the other hand it is a localized *pseudoparticle* in the Euclidean space.

Following the WKB approximation, the tunneling amplitude can be estimated as $\exp(-S)$, where S is the action along the classical trajectory in imaginary time, leading from the minimum at $N_{CS} = 0$ at $t = -\infty$ to that at $N_{CS} = 1$ at $t = +\infty$.

According to eq. (9) the 4-dimensional topological charge of such trajectory is $Q_T = 1$. To find the best tunneling trajectory having the largest amplitude one has thus to minimize the YM action (1) provided the topological charge (5) is fixed to be unity. This can be done using the following trick. Consider the inequality

$$0 \leq \int d^4x \left(F^a_{\mu\nu} - \tilde{F}^a_{\mu\nu}\right)^2$$

$$= \int d^4x \left(2F^2 - 2F\tilde{F}\right) = 8g^2 S - 64\pi^2 Q_T, \tag{15}$$

hence the action is restricted from below:

$$S \geq \frac{8\pi^2}{g^2} Q_T = \frac{8\pi^2}{g^2}. \tag{16}$$

Therefore, the minimal action for a trajectory with a unity topological charge is equal to $8\pi^2/g^2$, which is achieved if the trajectory satisfies the *self-duality* equation:

$$F^a_{\mu\nu} = \tilde{F}^a_{\mu\nu}. \tag{17}$$

Notice that any solution of eq. (17) is simultaneously a solution of the general YM equation of motion $D_\mu^{ab} F_{\mu\nu}^b = 0$: that is because the "second pair" of the Maxwell equations, $D_\mu^{ab} \tilde{F}_{\mu\nu}^b = 0$, is satisfied identically.

Thus, the tunneling amplitude can be estimated as

$$\mathcal{A} \sim \exp(-\mathrm{Action}) = \exp\left(-\frac{1}{4g^2}\int d^4x\, F_{\mu\nu}^2\right) = \exp\left(-\frac{8\pi^2}{g^2}\right) = \exp\left(-\frac{2\pi}{\alpha_s}\right). \tag{18}$$

It is non-analytic in the gauge coupling constant and hence instantons are missed in all orders of the perturbation theory. However, it is not a reason to ignore tunneling. For example, tunneling of electrons from one atom to another in a metal is also a non-perturbative effect but we would get nowhere in understanding metals had we ignored it.

Instanton configurations

To solve eq. (17) let us recall a few facts about the Lorentz group $SO(3,1)$. Since we are talking about the tunneling fields which can only develop in imaginary time, it means that we have to consider the fields in Euclidean space-time, so that the Lorentz group is just $SO(4)$ isomorphic to $SU(2) \times SU(2)$. The gauge potentials A_μ belong to the $(\frac{1}{2}, \frac{1}{2})$ representation of the $SU(2) \times SU(2)$ group, while the field strength $F_{\mu\nu}$ belongs to the reducible $(1,0) + (0,1)$ representation. In other words it means that one linear combination of $F_{\mu\nu}$ transforms as a vector of the left $SU(2)$, and another combination transforms as a vector of the right $SU(2)$. These combinations are

$$F_L^A = \eta^A_{\mu\nu}(F_{\mu\nu} + \tilde{F}_{\mu\nu}), \qquad F_R^A = \bar{\eta}^A_{\mu\nu}(F_{\mu\nu} - \tilde{F}_{\mu\nu}), \tag{19}$$

where $\eta, \bar{\eta}$ are the so-called 't Hooft symbols, see below. We see therefore that a self-dual field strength is a vector of the left $SU(2)$ while its right part is zero. Keeping that experience in mind we look for the solution of the self-dual equation in the form

$$A^a_\mu = \bar{\eta}^a_{\mu\nu}\, x_\nu\, \frac{1 + \Phi(x^2)}{x^2}. \tag{20}$$

$$\eta^a_{ij} = \epsilon_{aij}, \quad \eta^a_{4j} = -\eta^a_{j4} = -\delta a\, j,$$

$$\bar{\eta}^a_{ij} = \epsilon_{aij}, \quad \bar{\eta}^a_{4j} = -\bar{\eta}^a_{j4} = +\delta a\, j.$$

Using the formulae for the 't Hooft's η symbols one can easily check that the YM action can be rewritten as

$$S = \frac{8\pi^2}{g^2}\frac{3}{2}\int d\tau \left[\frac{1}{2}\left(\frac{d\Phi}{d\tau}\right)^2 + \frac{1}{8}(\Phi^2 - 1)^2\right], \qquad \tau = \ln\left(\frac{x^2}{\rho^2}\right). \tag{21}$$

This can be recognized as the action of the double-well potential whose minima lie at $\Phi = \pm 1$, and τ plays the role of "time"; ρ is an arbitrary scale. The trajectory which tunnels from 1 at $\tau = -\infty$ to -1 at $\tau = +\infty$ is

$$\Phi = -\tanh\left(\frac{\tau}{2}\right), \tag{22}$$

and its action (21) is $S = 8\pi^2/g^2$, as needed. Substituting the solution (22) into (20) we get

$$A^a_\mu(x) = \frac{2\bar{\eta}^\mu_{\nu a}\rho^2}{x^2(x^2+\rho^2)}. \tag{23}$$

The correspondent field strength is

$$F^a_{\mu\nu} = -\frac{4\rho^2}{(x^2+\rho^2)^2}\left(\bar\eta^\mu_{\nu a} - 2\bar\eta^\mu_{\alpha a}\frac{x_\alpha x_\nu}{x^2} - 2\bar\eta^\beta_{\nu a}\frac{x_\mu x_\beta}{x^2}\right), \qquad F^a_{\mu\nu}F^a_{\mu\nu} = \frac{192\rho^4}{(x^2+\rho^2)^4}, \tag{24}$$

and satisfies the self-duality condition (17).

The *anti-instanton* corresponding to tunneling in the opposite direction, from $N_{CS} = 1$ to $N_{CS} = 0$, satisfies the *anti*-self-dual equation, with $\tilde F \to -\tilde F$; its concrete form is given by eqs.(23, 24) with the replacement $\bar\eta \to \eta$.

Eqs.(23, 24) describe the field of the instanton in the singular Lorenz gauge; the singularity of A_μ at $x^2 = 0$ is a gauge artifact: the gauge-invariant field strength squared is smooth at the origin. Formulae for instantons are more compact in the Lorenz gauge, and I shall use it further on.

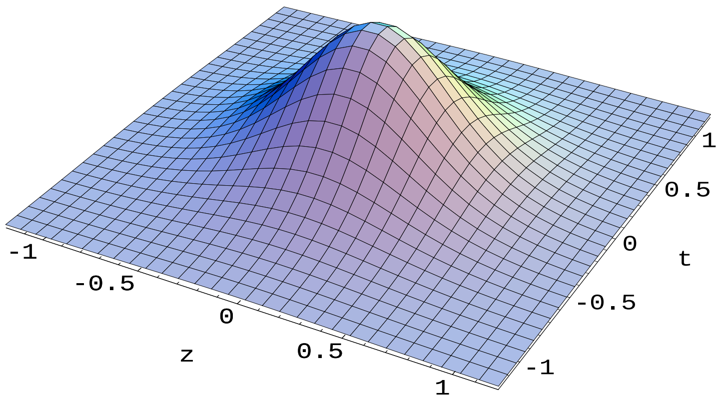

Figure 2: Action density of the YM instanton as function of $z,\ t$ at fixed $x = y = 0$.

Instanton collective coordinates

The instanton field, eq. (23), depends on an arbitrary scale parameter ρ which we shall call the instanton size, while the action, being scale invariant, is independent of ρ. One can obviously shift the position of the instanton to an arbitrary 4-point z_μ – the action will not change either. Finally, one can rotate the instanton field in colour space by constant unitary matrices U. For the $SU(2)$ gauge group this rotation is characterized by 3 parameters, *e.g.* by Euler angles. For a general $SU(N_c)$ group the number of parameters is $N_c^2 - 1$ (the total number of the $SU(N_c)$ generators) *minus* $(N_c - 2)^2$ (the number of generators which do not affect the left upper 2×2 corner where the standard $SU(2)$ instanton (23) is residing), that is $4N_c - 5$. These degrees of freedom are called instanton orientation in colour space. All in all there are

$$4\ (\mathrm{centre}) \quad + \quad 1\ (\mathrm{size}) \quad + \quad (4N_c - 5)\ (\mathrm{orientations}) \quad = \quad 4N_c \tag{25}$$

so-called collective coordinates describing the field of the instanton, of which the action is independent.

It is convenient to introduce 2×2 matrices

$$\sigma_\mu^\pm = (\pm i\overrightarrow{\sigma}, 1), \qquad x^\pm = x_\mu \sigma_\mu^\pm, \tag{26}$$

such that

$$2i\tau^a \eta_{\nu a}^\mu = \sigma_\mu^+ \sigma_\nu^- - \sigma_\nu^+ \sigma_\mu^-, \qquad 2i\tau^a \bar{\eta}_{\nu a}^\mu = \sigma_\mu^- \sigma_\nu^+ - \sigma_\nu^- \sigma_\mu^+, \tag{27}$$

then the instanton field with arbitrary center z_μ, size ρ and color orientation U in the $SU(N_c)$ gauge group can be written as

$$A_\mu = A_\mu^a t^a = \frac{-i\rho^2 U[\sigma_\mu^-(x-z)^+ - (x-z)_\mu]U^\dagger}{(x-z)^2[\rho^2+(x-z)^2]}, \qquad \mathrm{Tr}\,(t^a t^b) = \frac{1}{2}\delta^{ab}, \tag{28}$$

or as

$$A_\mu^a = \frac{2\rho^2 O^{ab}\bar\eta^\mu_{\nu b}(x-z)_\mu}{(x-z)^2[\rho^2+(x-z)^2]}, \qquad O^{ab} = \mathrm{Tr}\,(U^\dagger t^a U \sigma^b), \qquad O^{ab}O^{ac} = \delta^{bc}. \tag{29}$$

This is the explicit expression for the $4N_c$-parameter instanton field in the $SU(N_c)$ gauge theory, written down in the singular Lorenz gauge.

Gluon condensate

The QCD perturbation theory implies that the fields $A_i^a(\mathbf{x})$ are performing quantum zero-point oscillations; in the lowest order these are just plane waves with arbitrary frequencies. The aggregate energy of these zero-point oscillations, $(\mathbf{B}^2+\mathbf{E}^2)/2$, is divergent as the fourth power of the cutoff frequency, however for any state one has $\langle F_{\mu\nu}^2\rangle = 2\langle \mathbf{B}^2 - \mathbf{E}^2\rangle = 0$, which is just a manifestation of the virial theorem for harmonic oscillators: the average potential energy is equal the kinetic one (I am temporarily in the Minkowski space). One can prove that this is also true in any order of the perturbation theory in the coupling constant, provided one does not violate the Lorentz symmetry and the renormalization properties of the theory. Meanwhile, we know from the QCD sum rules phenomenology that the QCD vacuum possesses what is called *gluon condensate* (introduced by Shifman, Vainshtein and Zakharov in 1977):

$$\frac{1}{32\pi^2}\langle F_{\mu\nu}^a F_{\mu\nu}^a\rangle = \frac{1}{16\pi^2}\langle \mathbf{B}^2 - \mathbf{E}^2\rangle \simeq (200\ MeV)^4 \quad > \ 0. \tag{30}$$

Instantons suggest an immediate explanation of this basic property of QCD. Indeed, instanton is a tunneling process, it occurs in imaginary time; therefore in Minkowski space one has $E_i^a = \pm i B_i^a$ (this is actually the duality eq. (17)). Therefore, during the tunneling $\mathbf{B}^2 - \mathbf{E}^2$ is positive, and one gets a chance to explain the gluon condensate. In Euclidean space the electric field is real as well as the magnetic one, and the gluon condensate is just the average action density. Let us make a quick estimate of its value. Let the total number of instantons and anti-instantons (henceforth I's and $\bar{I}$'s for short) in the 4-dimensional volume V be N. Assuming that the average separations of instantons are larger than their average sizes (to be justified below), we can estimate the total action of the ensemble as the sum of individual actions (see eq. (16)):

$$\langle F^2_{\mu\nu} \rangle V = \int d^4x F^2_{\mu\nu} \simeq N \cdot 32\pi^2, \tag{31}$$

hence the gluon condensate is directly related to the instanton density in the 4-dimensional Euclidean space-time:

$$\frac{1}{32\pi^2} \langle F^a_{\mu\nu} F^a_{\mu\nu} \rangle \simeq \frac{N}{V} \equiv \frac{1}{\bar{R}^4}. \tag{32}$$

In order to get the phenomenological value of the condensate one needs thus to have the average separation between pseudoparticles

$$\bar{R} \simeq \frac{1}{200\,\mathrm{MeV}} = 1\,\mathrm{fm} \quad \text{[SVZ, Shuryak (1978)]} \tag{33}$$

There is another point of view on the gluon condensate which I describe briefly. In principle, all information about field theory is contained in the partition function being the functional integral over the fields. In the Euclidean formulation it is

$$\mathcal{Z} = \int DA_\mu \exp\left(-\frac{1}{4g^2}\int d^4x F_{\mu\nu}^2\right) \xrightarrow{T\to\infty} e^{-\mathcal{E}T}, \tag{34}$$

where I have used that at large (Euclidean) time T the partition function picks up the ground state or vacuum energy $\mathcal{E}$. For the sake of brevity I do not write the gauge fixing and Faddeev–Popov ghost terms. If the state is homogeneous, the energy can be written as $\mathcal{E} = \theta_{44}V^{(3)}$ where $\theta_{\mu\nu}$ is the stress-energy tensor and $V^{(3)}$ is the 3-volume of the system. Hence, at large 4-volumes $V = V^{(3)}T$ the partition function is $\mathcal{Z} = \exp(-\theta_{44}V)$. This θ_{44} includes zero-point oscillations and diverges badly. A more reasonable quantity is the partition function, normalized to the partition function understood as a perturbative expansion about the zero-field vacuum[1],

$$\frac{\mathcal{Z}}{\mathcal{Z}_{\mathrm{P.T.}}} = \exp\left[-(\theta_{44} - \theta_{44}^{\mathrm{P.T.}})V\right]. \tag{35}$$

We expect that the non-perturbative vacuum energy density $\theta_{44} - \theta_{44}^{\mathrm{P.T.}}$ is a negative

[1]The latter can be distinguished from the former by imposing a condition that it does not contain integration over singular Yang–Mills potentials; recall that the instanton potentials are singular at the origins.

quantity since we have allowed for tunneling: as usual in quantum mechanics, it lowers the ground state energy. If the vacuum is isotropic, one has $\theta_{44} = \theta_{\mu\mu}/4$. Using the trace anomaly,

$$\theta_{\mu\mu} = \frac{\beta(\alpha_s)}{16\pi\alpha_s^2}\left(F^a_{\mu\nu}\right)^2 \simeq -b\frac{F^2_{\mu\nu}}{32\pi^2}, \tag{36}$$

where $\beta(\alpha_s)$ is the Gell-Mann–Low function,

$$\beta(\alpha_s) \equiv \frac{d\alpha_s(\mu)}{d\ln\mu} = -b_1\frac{\alpha_s^2(\mu)}{2\pi} - \frac{b_2}{2}\frac{\alpha_s^3(\mu)}{(2\pi)^2} - ..., \tag{37}$$

with $b_{1,2}$ given below, one gets:

$$\frac{\mathcal{Z}}{\mathcal{Z}_{\mathrm{P.T.}}} = \exp\left(\frac{b_1}{4}V\langle F^2_{\mu\nu}/32\pi^2\rangle_{\mathrm{NP}}\right) \tag{38}$$

where $\langle F^2_{\mu\nu}\rangle_{\mathrm{NP}}$ is the gluon field vacuum expectation value which is due to non-perturbative fluctuations, i.e. the gluon condensate. The aim of any QCD-vacuum builder is to minimize the vacuum energy or, equivalently, to maximize the gluon condensate. It is important that it is a renormalization-invariant quantity, meaning that its dependence on the ultraviolet cutoff μ and the bare charge $\alpha_s(\mu)$ given at this cutoff is such that it is actually cutoff-independent. Such a combination is called Λ, see eq. (46). The gluon condensate has to be proportional to Λ^4 by dimensions.

The fact that the vacuum energy or, equivalently, the gluon condensate is a renormalization-invariant quantity leads to an infinite number of low-energy theorems [Novikov, Shifman, Vainshtein and Zakharov (1980)]. Translated into the instanton-vacuum language, the renormalizability of the QCD implies that the probability that there are N I's and $\bar{I}$'s in the vacuum is [Diakonov and Petrov (1984)]

$$P(N) \sim \exp\left[-\frac{b}{4}\left(\ln\frac{N}{\langle N\rangle} - 1\right)\right], \tag{39}$$

where $\langle N\rangle \simeq V\langle F^a_{\mu\nu}F^a_{\mu\nu}\rangle/(32\pi^2)$ is the *average* number of I's and $\bar{I}$'s .

One-instanton weight

The notion "instanton vacuum" implies that one assumes that the QCD partition function (34) is mainly saturated by an ensemble of interacting I's and $\bar{I}$'s , together with quantum fluctuations about them. Instantons are necessarily present in the QCD vacuum if only because they lower the vacuum energy with respect to the purely perturbative (divergent) one. The question is whether they give the dominant contribution to the gluon condensate, and to other basic quantities. To answer this question one has to compute the partition function (34) assuming that it is mainly saturated by instantons, and to compare the obtained gluon condensate with the phenomenological one.

The starting point of this calculation is the contribution of one isolated instanton to the partition function (34), or the one-instanton weight. We have already estimated the tunneling amplitude in eq. (18) but it is not sufficient: the prefactor is very important. To the 1-loop accuracy, it has been first computed by 't Hooft (1976) for the $SU(2)$ colour group, and generalized to arbitrary $SU(N)$ by C. Bernard (1978).

The general field can be decomposed as a sum of a classical field of an instanton $A_\mu^I(x,\xi)$ where ξ is a set of $4N_c$ collective coordinates characterizing a given instanton (see eq. (28)), and of a presumably small quantum field $a_\mu(x)$:

$$A_\mu(x) = A_\mu^I(x,\xi) + a_\mu(x). \tag{40}$$

There is a subtlety in this decomposition due to the gauge freedom. The action is

$$\mathrm{Action} = \frac{1}{4g^2}\int d^4x\ F_{\mu\nu}^2 = \frac{8\pi^2}{g^2} + \frac{1}{g^2}\int d^4x\ D_\mu F_{\mu\nu} a_\nu + \frac{1}{2g^2}\int d^4x\ a_\mu W_{\mu\nu} a_\nu + O(a^3). \tag{41}$$

Here the term linear in a_μ drops out because the instanton field satisfies the equation of motion. The quadratic form $W_{\mu\nu}$ has $4N_c$ zero modes related to the fact that the action does not depend on $4N_c$ collective coordinates. This brings in a divergence in the functional integral over the quantum field a_μ which, however, can and should be qualified as integrals over the collective coordinates: centre, size and orientations. Formally the functional integral over a_μ gives

$$\frac{1}{\sqrt{\det\ W_{\mu\nu}(A^I)}}, \tag{42}$$

which must be i) normalized (to the determinant of the free quadratic form, i.e. with no background field), ii) regularized (for example by using the Pauli–Villars method), and iii)

accounted for the zero modes. Actually one has to compute a "quadrupole" combination,

$$\left[\frac{\text{det}'W\ \text{det}(W_0+\mu^2)}{\text{det}W_0\ \text{det}(W+\mu^2)}\right]^{-\frac{1}{2}}, \tag{43}$$

where W_0 is the quadratic form with no background field and μ^2 is the Pauli–Villars mass playing the role of the ultraviolet cutoff; the prime reminds that the zero modes should be removed and treated separately. The resulting one-instanton contribution to the partition function (normalized to the free one) is [G. 't Hooft, C. Bernard]:

$$\frac{\mathcal{Z}_{1-\text{inst}}}{\mathcal{Z}_{\text{P.T.}}} = \int d^4 z_\mu \int d\rho \int d^{4N_c-5}U\, d_0(\rho), \tag{44}$$

$$d_0(\rho) = \frac{C(N_c)}{\rho^5}\left[\frac{2\pi}{\alpha_s(\mu)}\right]^{2N_c}(\mu\rho)^{\frac{11}{3}N_c}\exp\left(-\frac{2\pi}{\alpha_s(\mu)}\right). \tag{45}$$

The fact that there are all in all $4N_c$ integrations over the collective coordinates z_μ, ρ, U reflects $4N_c$ zero modes in the instanton background. The combination of the UV cutoff

μ and the coupling constant given at that cutoff $\alpha_s(\mu)$ is

$$\Lambda = \mu \, \exp\left(-\frac{3}{11\,N_c}\,\frac{2\pi}{\alpha_s(\mu)}\right) \tag{46}$$

The numerical coefficient $C(N_c)$ depends implicitly on the regularization scheme used. In the Pauli–Villars scheme exploited above

$$C(N_c) = \frac{4.60\,\exp(-1.68 N_c)}{\pi^2 (N_c-1)!(N_c-2)!}. \tag{47}$$

If the scheme is changed, one has to change the coefficient $C(N_c) \to C'(N_c) = C(N_c) \cdot (\Lambda/\Lambda')^b$. One has [A. and P. Hasenfratz (1984)]: $\Lambda_{\mathrm{P.V.}} = e^{\frac{1}{22}} \Lambda_{\overline{\mathrm{MS}}} = 40.66\, e^{-\frac{3\pi^2}{11 N_c^2}} \Lambda_{\mathrm{lat}} = \ldots$ ($\Lambda_{\mathrm{lat}} = \frac{1}{a}$).

Eq. (45) cannot yet be expressed through the 2-loop renormalization-invariant combination Λ (46) as it is written to the 1-loop accuracy only. In the 2-loop approximation the instanton weight is given by [D.D. and Petrov (1984)]

$$
\begin{aligned}
d_0(\rho) &= \frac{C(N_c)}{\rho^5}\beta(\rho)^{2N_c}\exp\left[-\beta^{\mathrm{II}}(\rho)+\left(2N_c-\frac{b_2}{2b_1}\right)\frac{b_2}{2b_1}\frac{\ln\beta(\rho)}{\beta(\rho)}+O\left(\frac{1}{\beta(\rho)}\right)\right] \\
&\sim \frac{1}{\rho^5}(\Lambda\rho)^{\frac{11}{3}N_c},
\end{aligned}
\tag{48}
$$

where $\beta(\rho) \equiv 2\pi/\alpha_s(\rho)$ and $\beta^{\mathrm{II}}(\rho)$ are the inverse charges to the 1-loop and 2-loop accuracy, respectively (not to be confused with the Gell-Mann–Low function!):

$$
\beta^{\mathrm{II}}(\rho) = \beta(\rho) + \frac{b_2}{2b_1}\ln\frac{2\beta(\rho)}{b_1}, \tag{49}
$$

$$
\beta(\rho) = b_1\ln\frac{1}{\Lambda\rho}, \qquad b_1 = \frac{11}{3}N_c, \qquad b_2 = \frac{34}{3}N_c^2. \tag{50}
$$

These equations express the one-instanton weight $d_0(\rho)$ through the cutoff-independent combination Λ (46), and the instanton size ρ. This is how the ‘transmutation of dimensions' occurs in the instanton calculus and how Λ enters into the game. Henceforth all dimensional quantities will be expressed through Λ, which is very much welcome.

Notice that the integral over the instanton sizes in eq. (44) diverges as a high power of ρ at large ρ: this is of course the consequence of asymptotic freedom. It means that individual instantons tend to swell. This circumstance plagued the instanton calculus for many years. If one attempts to cut the ρ integrals "by hand", one violates the renormalization properties of the YM theory, as mentioned in the previous section. Actually the size integrals appear to be cut from above due to instanton interactions.

Instanton ensemble

To get a volume effect from instantons one needs to consider an $I\bar{I}$ ensemble, with their total number N proportional to the 4-dimensional volume V. Immediately a mathematical difficulty arises: any superposition of I's and $\bar{I}$'s is not, strictly speaking, a solution of the equation of motion, therefore, one cannot directly use the semiclassical approach of the previous section. One way to overcome this difficulty is to use a variational principle [Diakonov and Petrov (1984)]. Its idea is to use a modified YM action for which a chosen $I\bar{I}$ ansatz *is* a saddle point. Exploiting the convexity of the exponent one can prove that the true vacuum energy is *less* than that obtained from the modified action. One can therefore use variational parameters (or even functions) to get a best upper bound for the vacuum energy. It is not the Rayleigh-Ritz but rather the Feynman variational principle since the method has been suggested by Feynman in his famous study of the polaron problem. The gauge theory is more difficult, though: one has not to loose either gauge invariance or the renormalization properties of the YM theory. These difficulties were overcome. It should be kept in mind that we are dealing with "strong interactions", meaning that all dimenionless quantities are generally speaking of the order of unity – there are no small parameters in the theory. Therefore, one has to use certain approximate methods, and the variational principle is among the best. Today's direct lattice investigation of the $I\bar{I}$ ensemble seem to indicate that the variational principle gives rather accurate numbers in this difficult problem.

In the variational approach, the normalized (to perturbative) and regularized YM partition function takes the form of a partition function for a grand canonical ensemble of interacting pseudoparticles of two kind, I's and $\bar{I}$'s :

$$\frac{\mathcal{Z}}{\mathcal{Z}_{\rm P.T.}} \geq \sum_{N_+,N_-} \frac{1}{N_+!}\frac{1}{N_-!} \prod_n^{N_++N_-} \int d^4 z_n d\rho_n dU_I \; d_0(\rho_n) \; \exp(-U_{\rm int}), \qquad (51)$$

where $d_0(\rho)$ is the 1-instanton weight (48). The integrals are over the collective coordinates of (anti)instantons: their coordinates z, sizes ρ and orientations given by $SU(N_c)$ unitary matrices U; dU means the Haar measure normalized to unity. The instanton interaction potential $U_{\rm int}$ (to be discussed below) depends on the separation between pseudoparticles, $z_m - z_n$, their sizes $\rho_{m,n}$ and their relative orientations $U_m U_n^\dagger$. In the variational approach the interaction between instantons arise from *i*) the defect of the classical action, *ii*) the non-factorization of quantum determinants and *iii*) the non-factorization of Jacobians when one passes to integration over the collective coordinates. All three factors are ansatz-dependent, but there is a tendency towards a cancellation of the ansatz-dependent pieces. Qualitatively, in any ansatz the interactions between I's and $\bar{I}$'s resemble those of molecules: at large separations there is an attraction, at smaller separations there is a repulsion. It is very important that the interactions depend on the relative orientations of instantons: if one averages over orientations (which is the natural thing to do if the $I\bar{I}$ medium is in a disordered phase; if not, one would expect a spontaneous breaking of both Lorentz and colour symmetries, the interactions seem to be repulsive at any separations.

In general, the mere notion of the instanton interactions is notorious for being ill-defined since instanton + antiinstanton is not a solution of the equation of motion. Such a configuration belongs to a sector with topological charge zero, thus it seems to be impossible to distinguish it from what is encountered in perturbation theory. The variational approach uses brute force in dealing with the problem, and the results appear to be somewhat dependent on the ansatz used. Thanks to the inequality for the vacuum energy mentioned above, we still get quite a useful information. However, recently a mathematically unequivocal definition of the instanton interaction has been suggested, based on the one hand on analyticity and unitarity and on the other hand on certain singular solutions of the YM equations of motion. Both definitions cut off automatically contributions of the perturbation theory. The first three leading terms for the interaction potential at large separations has been computed by the two very different methods with coinciding results. At smaller separations one observes a strong repulsion.

At this point I should mention certain experience one gains from a simpler 2-

dimensional so-called CP^N model, also possessing instantons as classical Euclidean solutions. Contrary to the $4d$ YM theory, the instanton measure in that model is known exactly. In the dilute limit the instanton measure reduces to the product of integrals over instanton sizes, positions and orientations, as in eq. (51). The exact measure, however, is written in terms of the so-called 'instanton quarks' which does not suppose that instantons are dilute. The statistical mechanics of I's and $\bar{I}$'s in this model has been studied by D.D. and M. Maul (1999) both by analytical methods and by numerical simulations. Although the 'instanton quark' parametrization allows for complete 'melting' of instantons and is quite opposite in spirit to the dilute-gas ansatz, it has been observed that, owing to a combination of purely geometric and dynamic reasons, the vast majority of 'instanton quarks' form neutral clusters which can be identified with well-separated instantons. Of course, there is always a fraction of overlapping instantons in the vacuum, however, it is small even in the $2d$ case; in the $4d$ YM case both reasons mentioned above are expected to be even stronger.

Summing up the discussion, I would say that today there exists no evidence that a variational calculation with the simplest sum Ansatz is qualitatively or even quantitatively incorrect, therefore I will cite the numerics from those calculations in what follows. The main finding is that the $I\bar{I}$ ensemble (51) stabilizes at a certain density related to the Λ parameter (there is no other dimensional quantity in the theory!)

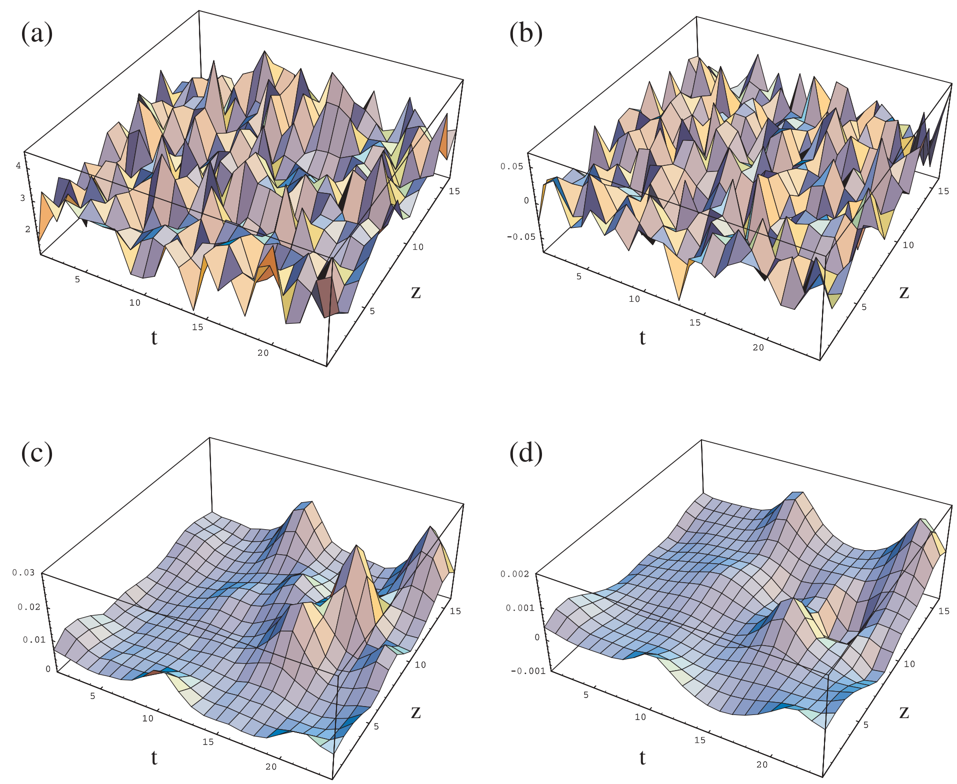

Figure 3: "Cooling" the normal zero-point oscillations reveals large fluctuations of the gluon field, which were identified with instantons and anti-instantons with random positions and sizes [Negele et al. (1995)]. The left column shows the action density and the right column shows the topological charge density for the same snapshot.

$$\frac{N}{V} \simeq \langle F_{\mu\nu}^2/32\pi^2 \rangle \simeq \frac{1}{V}\langle Q_T^2 \rangle \geq (0.75\Lambda_{\overline{\rm MS}})^4. \tag{52}$$

The average instanton size and the average separation between instantons are, respectively,

$$\bar{\rho} \quad \simeq \quad 0.48/\Lambda_{\overline{\rm MS}} \simeq 0.35\,\mathrm{fm}, \tag{53}$$

$$\bar{R} \quad = \quad \left(\frac{N}{V}\right)^{-\frac{1}{4}} \simeq 1.35/\Lambda_{\overline{\rm MS}} \simeq 0.95\,\mathrm{fm}, \tag{54}$$

if one uses $\Lambda_{\overline{\rm MS}} = 280\,\mathrm{MeV}$ as it follows from the DIS data. Earlier, very similar charactersitics, $\bar{\rho} = \frac{1}{3}\,\mathrm{fm}$, $\bar{R} = 1\,\mathrm{fm}$, have been suggested by Shuryak (1982) from studying the phenomenological applications of instantons.

Instanton interactions lead to the modification of the (divergent) size distribution function $d_0(\rho)$ (48) by a distribution decreasing at large ρ. The use of the variational principle yields a Gaussian cutoff for large sizes:

$$d_0(\rho) \to d(\rho) = d_0(\rho)\ \exp\left(-\,\mathrm{const.}\sqrt{\frac{N}{V}}\,\rho^2\right). \tag{55}$$

In fact, it is a rather narrow distribution peaked around $\bar{\rho}$ (53); therefore for practical estimates in what follows I shall just replace all instantons by the average-size one.

It should be said that, strictly speaking, nothing can prevent some instantons to be anomalously large and overlapping with other. For overlapping instantons the notion of size distribution becomes senseless. The question is quantitative: how often and how strong do instantons overlap. Given the estimate (53,54), it seems that the majority of instantons in the vacuum ensemble are well-isolated.

In the recent years instantons have been intensively studied by direct numerical simulations of gluon fields on the lattice, using various configuration-smoothing methods. A typical snapshot of gluon fluctuations in the vacuum is shown in Fig. 2. Naturally, it is heavily dominated by normal perturbative UV-divergent zero-point oscillations of the field. However, after "cooling" down these oscillations one reveals a smooth background field which was shown to be nothing but an ensemble of instantons and anti-instantons with random positions and sizes The lower part of Fig. 2 is what is left of the upper part after "cooling" that particular configuration. The average sizes and separations of instantons found vary somewhat depending on the concrete smearing method used. J. Negele et al.

give the following values

$$\bar{\rho} \simeq 0.36\,\mathrm{fm}, \qquad \bar{R} = (N/V)^{-\frac{1}{4}} \simeq 0.89\,\mathrm{fm}, \tag{56}$$

which are not far from the estimate from the variational principle. The ratio,

$$\frac{\bar{\rho}}{\bar{R}} \simeq \frac{1}{3}, \tag{57}$$

seems to be more stable: it follows from phenomenological [Shuryak], variational [Diakonov and Petrov] and lattice studies. It means that the *packing fraction*, i.e. the fraction of the 4-dimensional volume occupied by instantons appears to be rather small, $\pi^2\bar{\rho}^4/\bar{R}^4 \simeq 1/8$. This small packing fraction of the instantons gives an *a posteriori* justification for the use of the semi-classical methods. As I shall show later, it also enables one to identify adequate degrees of freedom to describe the low-energy QCD.