Chiral symmetry breaking by instantons

Chiral symmetry breaking by definition

The QCD Lagrangian with N_f massless flavors is known to posses a large global symmetry, namely a symmetry under $U(N_f) \times U(N_f)$ independent rotations of left- and right-handed quark fields. This symmetry is called *chiral* [1]. Instead of rotating separately the 2-component Weyl spinors corresponding to left- and right-handed components of quark fields, one can make independent vector and axial $U(N_f)$ rotations of the full 4-component Dirac spinors – the QCD lagrangian is invariant under these transformations too.

Meanwhile, axial transformations mix states with different P-parities. Therefore, were that symmetry exact, one would observe parity degeneracy of all states with otherwise the same quantum numbers. In reality the splittings between states with the same quantum numbers but opposite parities are huge. For example, the splitting between the vector ρ

[1] The word was coined by Lord Kelvin in 1894 to describe molecules not superimposable on its mirror image.

and the axial a_1 meson is $(1260 - 770) \simeq 500\,\mathrm{MeV}$; the splitting between the nucleon and its parity partner is even larger: $(1535 - 940) \simeq 600\,\mathrm{MeV}$.

The splittings are too large to be explained by the small bare or current quark masses which break the chiral symmetry from the beginning. Indeed, the current masses of light quarks are: $m_u \simeq 4\,\mathrm{MeV}, \quad m_d \simeq 7\,\mathrm{MeV}, \quad m_s \simeq 150\,\mathrm{MeV}$. The conclusion one can draw from these numbers is that the chiral symmetry of the QCD Lagrangian is broken down *spontaneously*, and very strongly. Consequently, one should have light (pseudo) Goldstone pseudoscalar hadrons – their role is played by pions which indeed are by far the lightest hadrons.

The order parameter associated with chiral symmetry breaking is the so-called *chiral* or *quark condensate*:

$$\langle \bar{\psi}\psi \rangle \simeq -(250\ MeV)^3. \tag{1}$$

It should be noted that this quantity is well defined only for massless quarks, otherwise it is somewhat ambiguous. By definition, this is the quark Green function taken at one point; in momentum space it is a closed quark loop:

$$\langle\bar{\psi}\psi\rangle = -N_c \int \frac{d^4k}{(2\pi)^4 i} \, \mathrm{Tr} \, \frac{Z(k)}{M(k) - \not{k}}. \tag{2}$$

If the quark propagator is massless and has only the 'slash' term, the trace over the spinor indices in the loop gives an identical zero. Therefore, chiral symmetry breaking implies that a massless (or nearly massless) quark develops a non-zero dynamical mass $M(k)$, $i.e.$ a 'non-slash' term in the propagator. There are no reasons for this quantity to be a constant independent of the momentum; moreover, we understand that it should anyhow vanish at large momentum. Sometimes it is called the constituent quark mass, however a momentum-dependent *dynamical quark mass* $M(k)$ is a more adequate term which I shall use below.

The spontaneous generation of the dynamical quark mass (equivalent to the spontaneous chiral symmetry breaking, SCSB) is the most important feature of QCD being key to the whole hadron phenomenology. The theory's task is to get $M(k)$ in the form

$$M(k) = \Lambda f(k/\Lambda) \tag{3}$$

where Λ is the renormalization-invariant combination of the UV cutoff and the bare coupling constant, and f is some function. Instantons enable one to get $M(k)$ in the needed form and to find the function. But first let us derive some general relations.

We start by writing down the QCD partition function. Functional integrals are well defined in Euclidean space which is obtained by the following formal substitutions of Minkowski space quantities:

$$\begin{aligned} ix_{M0} &= x_{E4}, \quad x_{Mi} = x_{Ei}, \quad A_{M0} = iA_{E4}, \quad A_{Mi} = A_{Ei}, \\ i\bar{\psi}_M &= \psi_E^\dagger, \quad \gamma_{M0} = \gamma_{E4}, \quad \gamma_{Mi} = i\gamma_{Ei}, \quad \gamma_{M5} = \gamma_{E5}. \end{aligned} \tag{4}$$

Neglecting for brevity the gauge fixing and Faddeev–Popov ghost terms, the QCD partition function with quarks can be written as

$$\mathcal{Z} = \int DA_\mu D\psi D\psi^\dagger \exp\left[-\frac{1}{4g^2}\int F_{\mu\nu}^2 + \sum_f^{N_f}\int \psi_f^\dagger (i\nabla\!\!\!/ + im_f)\psi_f\right]$$

$$= \int DA_\mu \exp\left[-\frac{1}{4g^2}\int F_{\mu\nu}^2\right]\prod_f^{N_f} \mathsf{det}(i\nabla\!\!\!/ + im_f). \tag{5}$$

The chiral condensate of a given flavor f is, by definition,

$$\langle\bar{\psi}_f\psi_f\rangle_M = -i\langle\psi_f^\dagger\psi_f\rangle_E = -\frac{1}{V}\frac{\partial}{\partial m_f}\left(\ln\mathcal{Z}\right)_{m_f\to 0}. \tag{6}$$

The Dirac operator has the form

$$i\nabla\!\!\!/ = \gamma_\mu(i\partial_\mu + A_\mu^{\mathrm{I\bar{I}}} + a_\mu) \tag{7}$$

where $A_\mu^{\mathrm{I\bar{I}}}$ denotes the classical field of the $I\bar{I}$ ensemble and a_μ is a presumably small field of quantum fluctuations about that ensemble, which I shall neglect as it has little impact on chiral symmetry breaking. Integrating over DA_μ in eq. (5) means averaging over the $I\bar{I}$ ensemble with the partition function, therefore one can write

$$\mathcal{Z} = \overline{\mathsf{det}(i\nabla\!\!\!/ + im)} \tag{8}$$

where I temporarily restrict the discussion to the case of only one flavor for simplicity. Because of the im term the Dirac operator in (8) is formally not Hermitian; however the determinant is real due to the following observation. Suppose we have found the eigenvalues and eigenfunctions of the Dirac operator,

$$i\not\nabla\Phi_n = \lambda_n\Phi_n, \tag{9}$$

then for any $\lambda_n \neq 0$ there is an eigenfunction $\Phi_{n'} = \gamma_5\Phi_n$ whose eigenvalue is $\lambda_{n'} = -\lambda_n$. This is because γ_5 anticommutes with $i\nabla$. Owing to this the fermion determinant can be written as

$$\begin{aligned}\det(i\not\nabla+im) &= \prod_n(\lambda_n+im)=\sqrt{\prod(\lambda_n^2+m^2)} = \exp\left[\frac{1}{2}\sum_n \ln(\lambda_n^2+m^2)\right] \\ &= \exp\left[\frac{1}{2}\int_{-\infty}^{\infty} d\lambda\,\nu(\lambda)\ln(\lambda^2+m^2)\right], \quad \nu(\lambda)\equiv\sum_n\delta(\lambda-\lambda_n),\end{aligned}\tag{10}$$

where I have introduced the *spectral density* $\nu(\lambda)$ of the Dirac operator $i\nabla$. Note that the last expression is real and even in m, which is a manifestation of the QCD chiral invariance. Differentiating eq. (10) in m and putting it to zero one gets according to the general eq. (6) a formula for the chiral condensate:

$$\begin{aligned}\langle\bar{\psi}\psi\rangle &= -\frac{1}{V}\frac{\partial}{\partial m}\left[\frac{1}{2}\int d\lambda\,\overline{\nu(\lambda)}\ln(\lambda^2+m^2)\right]_{m\to 0} \\ &= -\frac{1}{V}\int_{-\infty}^{\infty} d\lambda\,\overline{\nu(\lambda)}\frac{m}{\lambda^2+m^2}\bigg|_{m\to 0} \end{aligned}\qquad(11)$$

where $\overline{\nu(\lambda)}$ means averaging over the instanton ensemble together with the weight given by the fermion determinant itself. The latter, however, may be cancelled in the so-called quenched approximation where the back influence of quarks on the dynamics is neglected. Theoretically, this is justified at large N_c. Naively, one would think that the r.h.s. of eq. (11) is zero at $m \to 0$. That would be correct for a finite-volume system with a discrete spectral density. However, if the volume goes to infinity faster than m goes to zero (which is what one should assume in the thermodynamic limit) the second factor in the integrand becomes a representation of a δ-function,

$$\frac{m}{\lambda^2+m^2}\stackrel{m\to 0}{\longrightarrow}\mathrm{sign}(m)\,\pi\,\delta(\lambda), \qquad(12)$$

so that one obtains:

$$\langle\bar{\psi}\psi\rangle = -\frac{1}{V}\mathrm{sign}(m)\pi\overline{\nu(0)}. \tag{13}$$

It is known as the Banks–Casher (1981) relation. The chiral condensate is thus proportional to the averaged spectral density of the Dirac operator at zero eigenvalues.

The appearance of the sign function is not accidental: it means that at small m QCD partition function depends on m non-analytically:

$$\ln \mathcal{Z} = V\left(c_0 + \pi\overline{\nu(0)}|m| + c_2 m^2 \ln(|m|) + \ldots\right). \tag{14}$$

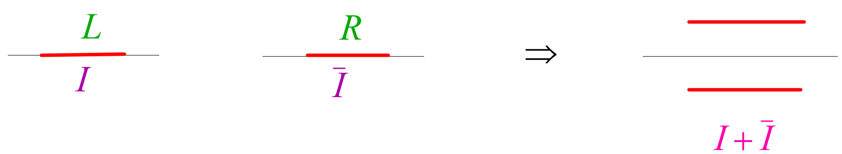
The fact that the partition function is even in m is the reflection of the original invariance of the QCD under γ_5 rotations; the fact that it is non-analytic in the symmetry-breaking parameter m is typical for spontaneous symmetry breaking.

Physics: quarks hopping from one instanton to another

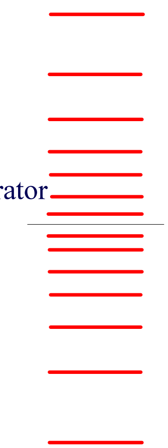
The key observation is that the Dirac operator in the background field of one (anti) instanton has an exact zero mode with $\lambda = 0$ ['t Hooft (1976)]. It is a consequence of a general Atiah–Singer index theorem; in our case it is guaranteed by the unit Pontryagin index or the topological charge of the instanton field. These zero modes are 2-component Weyl spinors: *right*-handed for instantons and *left*-handed for antiinstantons. Explicitly, the zero modes are ($\alpha = 1 ... N_c$ is the color and $i, j, k = 1, 2$ are the spinor indices):

$$
\begin{aligned}
[\Phi_R(x - z_1)]_i^\alpha &= \phi(x - z_1, \rho_1)(x - z_1)^+_{ij} U^\alpha_{1k} \epsilon^{jk}, \\
[\Phi_L(x - z_2)]_i^\alpha &= \phi(x - z_2, \rho_2)(x - z_2)^-_{ij} U^\alpha_{2k} \epsilon^{jk}, \\
\phi(x, \rho) &= \frac{\rho}{\pi (2x^2)^{1/2} (x^2 + \rho^2)^{3/2}}.
\end{aligned}
\qquad (15)
$$

Here $z_{1\mu}, \rho_1, U_1$ are the center, size and orientation of an instanton and $z_{2\mu}, \rho_2, U_2$ are those of an antiinstanton, respectively, ϵ^{jk} is the 2×2 antisymmetric matrix.



Diagonalization of the would-be zero modes



Spectral density of Dirac operator in the instanton ensemble

For infinitely separated I and $\bar{I}$ one has thus two degenerate states with exactly zero eigenvalues. As usual in quantum mechanics, this degeneracy is lifted through the diagonalization of the Hamiltonian, in this case the 'Hamiltonian' is the full Dirac operator. The two "wave functions" which diagonalize the "Hamiltonian" are the sum and the difference of the would-be zero modes, one of which is a 2-component left-handed spinor, and the other is a 2-component right-handed spinor. The resulting wave functions are 4-component Dirac spinors; one can be obtained from another by multiplying by the γ_5 matrix. As the result the two would-be zero eigenstates are split symmetrically into two 4-component Dirac states with *non-zero* eigenvalues equal to the overlap integral between the original states:

$$\lambda \;=\; \pm|T_{\mathrm{I\bar{I}}}|, \tag{16}$$

$$T_{\mathrm{I\bar{I}}} = \int d^4x\,\Phi_1(x-z_1,U_1)^\dagger(-i\nabla\!\!\!/)\Phi_2(x-z_2,U_2) \stackrel{z_{12}\to\infty}{\longrightarrow} -\frac{2\rho_1\rho_2}{z_{12}^4}\mathrm{Tr}\,(U_1^\dagger U_2 z_{12\,\mu}\sigma^+_\mu).$$

We see that the splitting between the would-be zero modes falls off as the third power of the distance between I and $\bar{I}$; it also depends on their relative orientation. The fact that two levels have eigenvalues $\pm\lambda$ is in perfect agreement with the γ_5 invariance mentioned in the previous section.

When one adds more I's and $\bar{I}$'s each of them brings in a would-be zero mode. After the diagonalization they get split symmetrically with respect to the $\lambda = 0$ axis. Eventually, for an $I\bar{I}$ ensemble one gets a continuous band spectrum with a spectral density $\nu(\lambda)$ which is even in λ and finite at $\lambda = 0$.

Let the total number of I's and $\bar{I}$'s in the 4-dimensional volume V be N. The spread κ of the band spectrum of the would-be zero modes is given by their average overlap (16):

$$\kappa^2 = \frac{N}{V}\int d^4 z_{12} dU_{12} \, |T_{12}|^2 = \frac{N\rho^4}{VN_c}\int \frac{d^4k}{(2\pi)^4}\frac{F^4(k\rho)}{k^2} = 6.62107\,\frac{N\rho^2}{VN_c} \qquad (17)$$

where

$$F(k\rho) = 2t\left[I_0(t)K_1(t) - I_1(t)K_0(t) - \frac{1}{t}I_1(t)K_1(t)\right]_{t=\frac{k\rho}{2}} \stackrel{k\to\infty}{\longrightarrow} \frac{6}{(k\rho)^3}, \qquad F(0) = 1, \qquad (18)$$

is the Fourier transform of the instanton zero mode (15); the modified Bessel functions are involved here. Numerically, if one takes the instanton density $N/V = (1\,\mathrm{fm})^{-4}$ and the average instanton size $\rho = \frac{1}{3}\,\mathrm{fm}$ one obtains $\kappa = 100\,\mathrm{MeV}$.

In the random instanton ensemble, one gets the following spectral density of the Dirac operator [D.D. and Petrov (1986)]:

$$\nu(\lambda) = \frac{N}{\pi\kappa}\sqrt{1 - \frac{\lambda^2}{4\kappa^2}} \qquad (19)$$

From eq. (13) one immediately finds the value of the chiral condensate:

$$\langle\bar{\psi}\psi\rangle = -\frac{1}{R^2\rho}\sqrt{\frac{N_c}{6.62}} = -(253\,\mathrm{MeV})^3 \qquad (20)$$

which is quite close to the phenomenological value [2].

[2]Contrary to the gluon condensate, the chiral or quark condensate is somewhat dependent on the scale where one estimates it. The above number refers to the scale given by the average instanton size, that is 600 MeV.

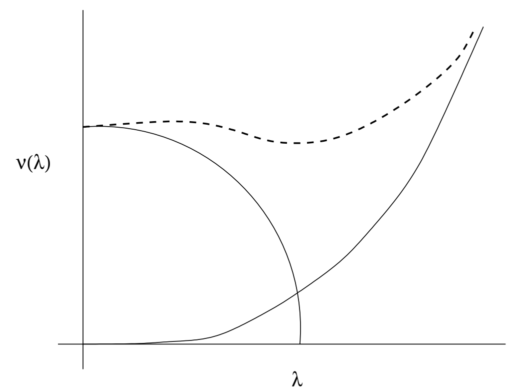


Figure 1: Schematic eigenvalue distribution of the Dirac operator. The solid lines are the zero mode and free contributions, the dashed line an estimate of the full spectrum.

I would like to stress that the chiral condensate is not linear in the instanton density N/V what one would naively expect but rather proportional to its square root (the gluon condensate is, naturally, linear). If the instanton density goes to zero the spectral density of the Dirac operator tends to a δ-function at zero eigenvalues. This is what one expects from the zero modes in the infinitely-dilute limit.

Eq. (19) is known as the Wigner semicircle spectrum. For high eigenvalues $\lambda \gg \kappa$

the spectral density is asymptotically given by that of free massless quarks:

$$\nu(\lambda) \approx \frac{N_c}{4\pi^2}\lambda^3 \,. \tag{21}$$

Schematically, the combination of the low- and high-energy spectra are shown in Fig. 1 where the intereference with the intermediate modes with $\lambda \geq 2\kappa$ has been ignored.

We see thus that the spontaneous chiral symmetry breaking by instantons can be interpreted as a delocalization of the "would-be" zero modes, induced by the background instantons, resulting from quarks hopping between them [D.D. and Petrov (1984, 1986)]. Imagine random impurities (atoms) spread over a sample with final density, such that each atom has a localized bound state for an electron. Due to the overlap of those localized electron states belonging to individual atoms, the levels are split into a band, and the electrons become delocalized. That means conductivity of the sample, the so-called Mott–Anderson conductivity. In our case the localized zero quark modes of individual instantons randomly spread over the volume get delocalized due to their overlap, which means chiral symmetry breaking.

This analogy between chiral symmetry breaking in QCD and the problem of electrons in condensed matter systems with random impurities goes even further. The acquisition of a dynamical mass by a quark is fully analogous to the appearance in the Green function of an electron in a metal with impurities of a finite relaxation time (but in our case this time depends on the momentum). The appearance of the massless pole in the pseudoscalar channel corresponding to the Goldstone pion is analogous to the formation of a diffusion mode in the density-density correlation function. For the recent development of these and related ideas, see work by Verbaarschot, Nowak, Zahed.

Recently, the instanton mechanism of the SCSB has been scrutinized by direct lattice methods which generally support the mechanism described above [DeGrand et al. (1998-)].

Quark propagator and dynamical quark mass

The spectral density of the Dirac operator, averaged over the instanton vacuum, carries very limited information, although one can already see that chiral symmetry is spontaneously broken. In fact, one can compute analytically much more complicated correlation functions in the instanton vacuum, such as the quark propagator and correlators of mesonic currents. What is difficult to calculate analytically can be done by numeric simulations of the instanton ensemble [Shuryak (1982-1995)].

Each time a quark 'hops' from one random instanton to an anti-instanton (and *vice versa*) it has to change its helicity, because instanton's zero mode is right-handed while the anti-instanton's one is left-handed, see the schematic drawing in Fig. 3. Delocalization implies quarks make an infinite number of such jumps. An infinite number of helicity-flip transitions generates a non-slash term in the quark propagator, i.e. the dynamically-generated mass $M(p)$, see Fig. 4. It implies the spontaneous chiral symmetry breaking.

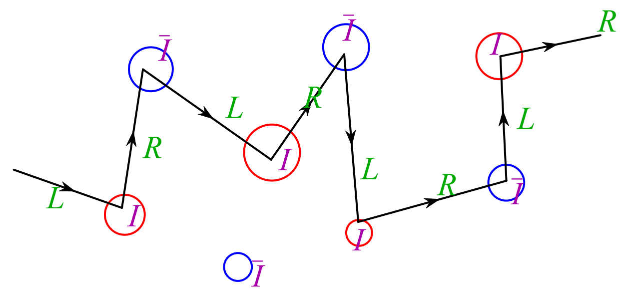


Figure 2: Quarks hopping from instantons to anti-instantons and *vice versa* flip helicity. An infinite number of such jumps generates the dynamical mass $M(p)$. Actually, one has to take into account that quarks can 'return' to the same pseudoparticle infinitely many times.

Mathematically, one has to consider the quark propagator in the gluon background being the superposition of an infinite number of I's and $\bar{I}$'s , and then average the propagator over the positions, sizes and orientations of instantons according to their partition function. This is a hopeless task, unless one exploits the fact that the packing fraction of instantons is small. The actual expansion parameter is $\alpha = \pi^2\rho^4 N/(VN_c) \sim 1/20$ which is not so bad. In the leading order in that parameter one can derive a closed equation for the quark propagator averaged over the ensemble. Its solution has the form [D.D. and Petrov (1986), Pobylitsa (1989)]

$$G(p) = Z(p)\,\frac{\not{p} + iM(p^2)}{p^2 + M^2(p^2)}. \tag{22}$$

The 'wave function renormalization' factor $Z(p)$ differs from unity by a function proportional to the above small parameter α, and this difference will be neglected. The dynamical quark mass $M(p)$ is, on the contrary, proportional to the *square root* of the packing fraction:

$$M(p^2) = \mathrm{const.}\sqrt{\frac{\pi^2 N\rho^2}{VN_c}}\,F^2(p\rho), \tag{23}$$

with the function $F(t)$ given by eq. (18); it is related to the Fourier transform of the zero mode [3].

The overall numerical constant is found from the self-consistency or gap equation [D.D. and Petrov(1986)]:

$$4N_c \int \frac{d^4p}{(2\pi)^4} \frac{M^2(p)}{M^2(p)+p^2} = \frac{N}{V}. \tag{24}$$

For the 'standard' values of the instanton ensemble, $N/V = (1\,\mathrm{fm})^{-1},\ \rho = \frac{1}{3}\,\mathrm{fm}$, one gets at zero momentum $M(0) = 345\,\mathrm{MeV}$. The dynamical mass (23) is plotted in Fig. 3 on top of the recent lattice data for this quantity obtained by an extrapolation to the chiral limit.

[3] It has been known from the perturbative analysis of the 1970's that asymptotically $M(p) \sim (\alpha_s/4\pi) < \bar\psi\psi > /p^2$ whereas eq. (23) gives at large virtuality $M(p) \sim 1/p^6$. This is because perturbative gluons are ignored in the instanton derivation. At very large p the perturbative regime $\sim 1/p^2$ has to take over.

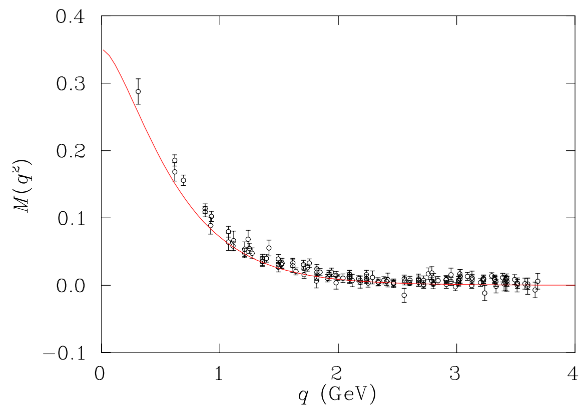


Figure 3: Dynamical quark mass $M(p)$ as function of quark virtuality. The scattered points are lattice data obtained by exprapolation to the continuum and chiral limits. Courtesy P. Bowman.

Knowing the quark propagator one is able to compute the chiral condensate directly without referring to the Banks–Casher relation. By definition, the chiral condensate is the quark propagator taken at one point; in momentum space it is a closed quark loop:

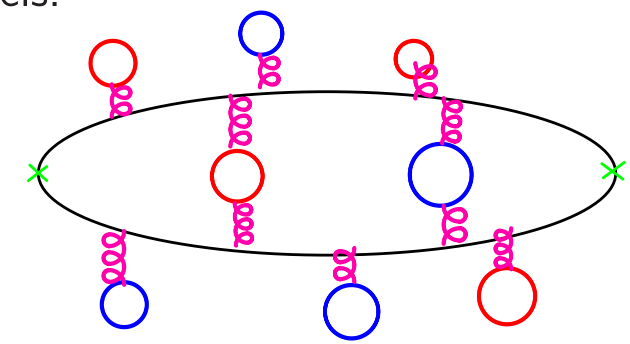
$$-\langle\bar{\psi}\psi\rangle_{\rm Mink} = i\langle\psi^\dagger\psi\rangle_{\rm Eucl} = 4N_c\int\frac{d^4p}{(2\pi)^4}\,\frac{M(p)}{p^2+M^2(p)} = {\rm const.}\sqrt{\frac{N\,N_c}{\pi^2\,V\,\rho^2}}. \quad (25)$$

Putting in the 'standard' instanton ensemble parameters one gets the same value of the condensate as before: $\langle\bar{\psi}\psi\rangle = -(253\,{\rm MeV})^3$.

Furthermore, using the small packing fraction as an expansion parameter one can also compute more complicated quantities like 2- or 3-point mesonic correlation functions of the type

$$\langle J_A(x)J_B(y)\rangle, \qquad \langle J_A(x)J_B(y)J_c(z)\rangle, \qquad J_A = \bar{\psi}\Gamma_A\psi \quad (26)$$

where Γ_A is a unit matrix in color but an arbitrary matrix in flavor and spin. Instantons influence the correlation functions in two ways: i) the quark and antiquark propagators get dressed and obtain the dynamical mass, as in eq. (22), ii) quark and antiquark may scatter simultaneously on the same pseudoparticle; that leads to certain correlations between quarks or, in other words, to effective quark interactions. These interactions are strongly dependent on the quark-antiquark quantum numbers: they are strong and attractive in the scalar and especially in the pseudoscalar and the axial channels, and rather weak in the vector and tensor channels.



We can discuss the pseudoscalar and the axial isovector channels. These are the channels where the pion shows up as an intermediate state. Since we have already obtained chiral symmetry breaking by studying a single quark propagator in the instanton vacuum, we are doomed to have a massless Goldstone pion in the appropriate correlation functions. However, it is instructive to follow how does the Goldstone theorem manifest itself in the instanton vacuum. It appears that technologically it follows from a kind of detailed balance in the pseudoscalar channel (such kind of equations are encountered in perturbative QCD where there is a delicate cancellation between real and virtual gluon emission). Since we have a concrete dynamical realization of chiral symmetry breaking we can not only check the general Ward identities of the partially conserved axial currents (which work of course) but we are in a position to find quantities whose values do not follow from general relations. One of the most important quantities is the F_π constant: it can be calculated as the residue of the pion pole. One obtains [D.D. and Petrov (1986)]:

$$
\begin{aligned}
F_\pi^2 &\approx 4N_c \int \frac{d^4p}{(2\pi)^4} \frac{M^2(p)}{[M^2(p)+p^2]^2} \\
&= \text{const.} \cdot \frac{N}{V}\, \bar{\rho}^2 \, \ln \frac{\bar{R}}{\bar{\rho}} \approx (100\,\text{MeV})^2 \qquad vs. \;\; (93 \text{ MeV})^2 \quad (\text{exper.})
\end{aligned}
\tag{27}
$$

This is an instructive formula. The point is, F_π is anomalously small in the strong interactions scale which, in the instanton vacuum, is given by the average size of pseudoparticles, $1/\bar{\rho} \simeq 600\ MeV$. The above formula says that F_π is down by the packing fraction factor $(\bar{\rho}/\bar{R})^2 \simeq 1/9$. It can be said that F_π measures the diluteness of the instanton vacuum. However it would be wrong to say that instantons are in a dilute gas phase – the interactions are crucial to stabilize the medium and to support the known renormalization properties of the theory, therefore they are rather in a liquid phase, however dilute it may turn to be. By calculating three-point correlation functions in the instanton vacuum it is possible to determine $e.g.$ the charge radius of the pion as the Goldstone excitation [D.D. and Petrov (1986)]:

$$\sqrt{r_\pi^2} \simeq \frac{\sqrt{N_c}}{2\pi F_\pi} \simeq (340 \text{ MeV})^{-1} \qquad vs.\ (310 \text{ MeV})^{-1} \ (\text{exper.}). \tag{28}$$

In flavor-singlet pseudoscalar channel the instanton-induced interactions are not strong attraction as in the non-singlet channel. Therefore, the η' meson is not a Goldstone boson: the famous $U_A(1)$ problem is solved by instantons, as anticipated at the very beginning of the instanton era by 't Hooft. Moreover, in the limit $N_f/N_c \rightarrow 0$ instantons reproduce the theoretical Witten–Veneziano formula for the singlet η' mass, as given by

$$m_{\eta'}^2 = \frac{2N_f < Q_T^2 > /V}{F_\pi^2} \tag{29}$$

where $< Q_T^2 > /V = < (N_+ - N_-)^2 > /V$ is the topological susceptibility. In the instanton vacuum it is related to the difference between the number of I's and $\bar{I}$'s . It should be stressed that eq. (29) is correct in the chiral limit, and that the topological susceptibility is that of the pure-glue world, without quarks. As emphasized in subsection 2.7, the instanton vacuum is described by the *grand* canonical ensemble of I's and $\bar{I}$'s , with the fluctuating number of pseudoparticles $N_\pm$. In the CP-conserving vacuum, $i.e.$ for the 'instanton angle' $\theta = 0$, one finds the equal averages $< N_+ > = < N_- > \sim V$ from a saddle-point equation. At $\theta \neq 0$ the saddle-point values for $< N_\pm >$ are complex conjugate to each other [D.D., Petrov, Polyakov and Weiss (1986,1993)]. The square of the difference between the numbers of I's and $\bar{I}$'s is behaving in the normal thermodynamic way, $< (N_+ - N_-)^2 > \sim V$, and gives rise to the topological susceptibility $< Q_T^2 >$.

When the back influence of quarks on the instanton ensemble is taken into account, which is a $O(N_f/N_c)$ effect, the average $< (N_+ - N_-)^2 >$ gets dynamically suppressed since at $N_+ \neq N_-$ the number of left- and right-handed zero modes are not equal, and the fermion determinant goes to zero in the chiral limit. This is in accordance with the general anomalous Ward identites in the $U_1(A)$ channel [Veneziano (1980)].

The Witten–Veneziano formula (29) is an idealization at $N_c \to \infty$, $m \to 0$. For non-zero quark masses there is a singlet-octet mixing [Veneziano (1980) D.D. and Eides (1981)] resulting in physical η, η' mesons. Actually the mixing angle appears to be rather small – about 10°.

All quantities exhibit the natural behaviour in the number of colors N_c:

$$\langle F_{\mu\nu}^2 \rangle \sim \frac{N}{V} = O(N_c), \qquad \langle \bar{\psi}\psi \rangle = O(N_c), \qquad F_\pi^2 = O(N_c),$$
$$\bar{\rho} = O(1), \qquad M(0) = O(1), \qquad \sqrt{r_\pi^2} = O(1), \quad \text{etc.} \tag{30}$$

A systematic numerical study of various correlation functions in the instanton vacuum has been performed by Shuryak, Verbaarschot and Schäfer. In all cases considered the results agree well or very well with experiments and phenomenology.

Recent review on instantons: D. Diakonov, *Instantons at work*, hep-ph/0212026.