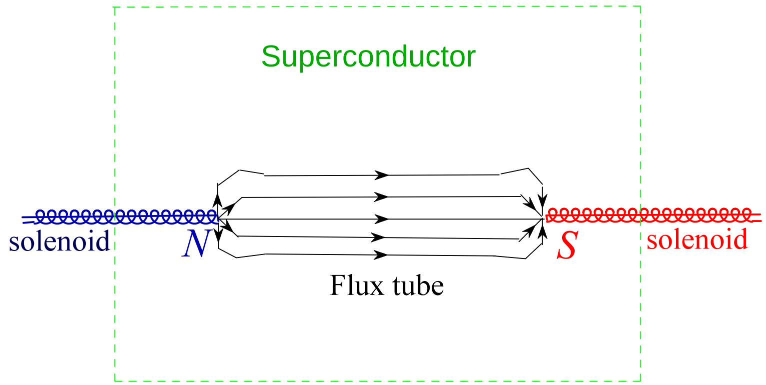
Mandelstam – Polyakov – ’t Hooft mechanism of confinement *or* dual Meissner effect

The usual Meissner effect: magnetic field cannot penetrate into the superconductor (except by burning out a narrow tube where superconductivity is destroyed = Abrikosov vortex).

Two infinitely thin and long solenoids are, at their endpoints, sources of the Coulomb-like magnetic field.



Energy of the magnetic ‘monopole-antimonopole’ pair $= E_{\perp} \cdot L \Longrightarrow$ linear potential energy between monopoles.

Dual Meissner effect:

- condensation of magnetic monopoles
- quarks are sources of (colored) electric field
- Along the tube connecting quarks the magnetic condensate is destroyed
- electric field squeezed inside the tube
  = Abrikosov–Nielsen–Olesen vortex

Estimate of the string tension

Landau–Ginsburg effective theory of supersonductivity:

$$\mathcal{E} = \int d^3r \left[\frac{\mathbf{B}^2}{2} + |(\partial_i - ieA_i)\phi|^2 + \lambda^2(\phi^2 - v^2)^2\right],$$

$$m_W = ev, \qquad m_H = \lambda v, \qquad \mathbf{B} = \mathrm{curl}\,\mathbf{A}.$$

Dimensionless quantities: $\phi' = \frac{\phi}{v}$, $A_i' = \frac{A_i}{m_W}$,

$x = r m_W, \;\; \kappa = \frac{\lambda}{e}.$

$$\mathcal{E} \;\; = \;\; L v^2 \int d^2 x \left[\frac{\mathbf{B}'^2}{2} + \left| (\partial_i - ieA'_i)\phi' \right|^2 + \kappa^2 (\phi'^2 - 1)^2 \right],$$

$$\text{vortex transverse size } \rho_0 \sim \frac{1}{\sqrt{\kappa}}.$$

String tension $= \sigma =$ energy / length of the tube $= v^2 \left[O(1) + O\left(\frac{m_H}{m_W} \right) \right].$

Londons' limit: $m_H \to \infty \quad \Longrightarrow$ infinite-energy vortex
Bogomolny–Prasad–Sommerfeld limit: $m_H \to 0.$
type-I superconductor: $(m_H > m_W)$: no vortices
type-II superconductor: $(m_H < m_W)$: yes

One needs an analog of type II superconductor with magnetic monopoles condensed.

Polyakov's realization of confinement

$d = 2+1$ Georgi–Glashow model: Yang–Mills SU(2) fields interacting with the Higgs field in the triplet representation:

$$S = \int d^3x \left\{ \frac{\left(F_{ij}^a\right)^2}{4g^2} + \tfrac{1}{2}\left(D_i^{ab}\phi^b\right)^2 + \lambda^2\left[(\phi^a)^2 - v^2\right]^2 \right\}.$$

't Hooft–Polyakov monopole is a local minimum of this action:

$$\phi^a = \mp n^a v \Phi(r), \qquad n^a = \frac{x^a}{r}, \quad \Phi(r) \to \begin{cases} 0, & r \to 0 \\ 1, & r \to \infty \end{cases}$$

$$A_i^a = \epsilon_{aij}\, n_j \frac{1 - R(r)}{r}, \qquad R(r) \to \begin{cases} 1, & r \to 0 \\ 0, & r \to \infty \end{cases}$$

Magnetic field strength, $B_i \overset{r\to\infty}{\sim} \frac{n_i}{r^2}$, is that of the magnetic monopole !

$$\text{Action} = \frac{v}{g}\chi\left(\frac{m_H}{m_W}\right) \gg 1, \qquad \chi(0) = 4\pi \ (\text{in the BPS limit})$$

To add up hedgehogs, one has first to comb their hair!

<u>'Stringy' or singular gauge</u>

The theory is invariant under gauge transformations:

$$\phi = \phi^a \frac{\tau_a}{2} \quad \rightarrow \quad S(x)\phi S^\dagger(x),$$

$$A_i = A_i^a \frac{\tau_a}{2} \quad \rightarrow \quad S(x)A_i S^\dagger(x) + iS\partial_i S^\dagger.$$

Choose the unitary gauge-transformation matrix $S(x)$ such that

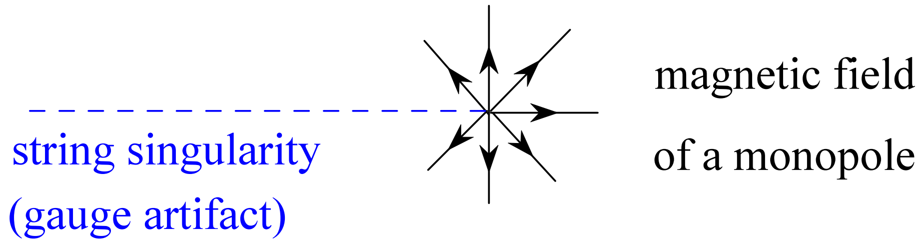
$$S(\mathbf{n}\cdot\tau)S^\dagger = \tau_3 \quad \Longrightarrow \quad S(\theta,\phi) = e^{-i\frac{\phi}{2}\tau_3}\, e^{i\frac{\theta}{2}\tau_2}\, e^{i\frac{\phi}{2}\tau_3}.$$

$$\phi' \quad = \quad S\,\phi\,S^\dagger = \frac{\tau_3}{2}\,v\,\Phi(r) \to \frac{\tau_3}{2}\,v,$$

$$A'_i \quad = \quad \begin{cases} A'_r & = & 0 \\ A'_\theta & = & \text{only inside core} \\ A'_\phi & = & \text{inside core } \mp \frac{\tau_3}{2r}\,\frac{1-\cos\theta}{\sin\theta}. \end{cases}$$

Magnetic field strength outside the monopole core at $r \gg 1/m_W$:

$$B'_r = \left(\text{curl}\,A'\right)_r = \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\,A'_\phi\right) = \pm\frac{\tau_3}{2r^2}.$$



Monopole interactions

One can add up (anti)monopoles only in the singular ('stringy') gauge where the Higgs field $\phi^a \stackrel{r\to\infty}{\longrightarrow} \delta^{a3}\, v$.
The interaction of two (anti)monopoles is Coulomb-like at large separations:

$$U_{\rm int} = \frac{4\pi}{g^2}\frac{1}{r_{12}}\left(\pm 1 - e^{-r_{12}m_H}\right).$$

Two gluons (out of three) $W^{\pm} = \frac{A^1 \pm i A^2}{\sqrt{2}}$ have large masses $m_W = gv$ and decouple. The third gluon (the 'photon') is massless, but there are monopoles around.

Monopole ensemble

Monopole 'weight' or 'fugacity'

$$\zeta = (\text{pre–exponent}) \cdot \exp(-\text{Action}) = \text{controllably small}.$$

At small momenta, the theory becomes the theory of plasma of magnetic charges of opposite sign. The grand canonical partition function is

$$\mathcal{Z} = \sum_{N_+,N_-} \frac{\zeta^{N_+}}{N_+!}\frac{\zeta^{N_-}}{N_-!} \prod_m \int d^3 z_m \, \exp\left(-\frac{4\pi}{g^2} \sum_{m<n} \frac{q_m q_n}{|z_m - z_n|} \right).$$

<u>"Bosonization trick"</u>

Monopole charge density

$$\rho(x) = \sum q_m \, \delta(x - z_m)$$

Consider a functional integral

$$\begin{aligned} & \int D\chi(x) \, \exp\left(- \int d^3 x \, \left[\tfrac{1}{2} (\partial_i \chi)^2 - i \frac{4\pi}{g} \chi \rho \right] \right) \\ = \quad & \exp \frac{1}{2} \int d^3 x \, \left(\frac{4\pi}{g} \right)^2 \rho \, \frac{1}{\triangle} \, \rho \end{aligned}$$

$$
\begin{aligned}
&= \exp -\frac{1}{2}\frac{4\pi}{g^2}\iint d^3x d^3y\, \rho(x)\,\frac{1}{|x-y|}\,\rho(y) \\
&= \exp -\frac{4\pi}{g^2}\sum_{m<n}\frac{q_m q_n}{|z_m - z_n|}.
\end{aligned}
$$

Therefore, the plasma partition function can be written down as a functional integral:

$$
\mathcal{Z} = \sum_{N_\pm}\frac{\zeta^{N_+}}{N_+!}\frac{\zeta^{N_-}}{N_-!}\int D\chi\, e^{-\frac{1}{2}\int d^3x\,(\partial_i\chi)^2}\prod_m \int d^3 z_m e^{i\frac{4\pi}{g}\sum q_m \chi(z_m)}
$$

Inside, is a series for the exponent:

$$
\sum_N \frac{\zeta^N}{N!}\left(\int d^3z\, e^{\pm\, i\frac{4\pi}{g}\,\chi(z)}\right)^N = \exp\left(\int d^3z\, e^{\pm\, i\frac{4\pi}{g}\,\chi(z)}\right).
$$

Hence

$$\mathcal{Z} = \int D\chi \, \exp\left\{ -\int d^3x \, \left[\tfrac{1}{2}(\partial_i \chi)^2 - 2\zeta \, \cos\left(\frac{4\pi}{g} \chi \right) \right] \right\}.$$

$$\begin{aligned} -2\zeta \, \cos\left(\frac{4\pi}{g} \chi \right) &\simeq -2\zeta + \tfrac{1}{2}\mu^2 \chi^2, \\ \mu^2 &= 2\zeta \left(\frac{4\pi}{g} \right)^2 \qquad \text{Debye mass!} \end{aligned}$$

The plasma partition function is mathematically equivalent to the 'Sine-Gordon' field theory. The field χ has the meaning of the dual potential: it gets a nonzero mass owing to the Debye screening in the monopole plasma.

$$\mathcal{Z} = \exp\left[2\zeta V + \frac{1}{12\pi} \left(2\zeta \, \frac{4\pi}{g} \right)^{\frac{3}{2}} + \ldots \right]$$

Average monopole density:

$$\frac{\bar{N}}{V} = \frac{1}{V}\frac{\partial \ln \mathcal{Z}}{\partial \ln \zeta} = 2\zeta + \text{Coulomb corrections}$$

In the weak-coupling regime $g \ll v$ monopoles are heavy, their density is small, everything is under control, this theory becomes exact!

Other ways to present plasma

$$\mathcal{Z} = \sum_{N_\pm} \frac{\zeta^{N_+ + N_-}}{N_+! N_-!} \prod_m \int d^3 z_m \, e^{-\frac{4\pi}{g^2}\sum \frac{q_m q_n}{|z_m - z_n|}}$$

$$\times \quad \int D\rho(x)\, \delta\left(\rho - \sum q_m \delta(x - z_m)\right)$$

$$\delta(\dots) \quad = \quad \int D\mu(x)\, \exp\left[i \int d^3 x\, \mu\rho - i \sum q_m \mu(z_m)\right]$$

$$= \int D\rho D\mu D\chi \ \exp - \int d^3x \left[\tfrac{1}{2}(\partial_i \chi)^2 - 2\zeta \cos\left(\frac{4\pi}{g}\chi - \mu \right) - i\mu\rho \right] .$$

New variables: $\frac{4\pi}{g}\chi - \mu = \psi; \quad \frac{4\pi}{g}\chi + \mu = \phi \longleftarrow$ integrate over

$$\mathcal{Z} = \int D\rho D\psi \ \exp \int d^3x \left[\frac{1}{2} \left(\frac{4\pi}{g} \right)^2 \rho \frac{1}{\triangle} \rho - i\psi\rho + 2\zeta \cos\psi \right]$$

$$= \int D\rho \ \exp \int d^3x \left[\frac{1}{2} \left(\frac{4\pi}{g} \right)^2 \rho \frac{1}{\triangle} \rho - V(\rho) \right]$$

$$V(\rho) = \rho \ \ln \left(\frac{\rho}{2\zeta} + \sqrt{1 + \frac{\rho^2}{4\zeta^2}} \right) + 2\zeta \sqrt{1 + \frac{\rho^2}{4\zeta^2}}$$

$$\xrightarrow{\rho \ll \zeta} -2\zeta + \frac{\rho^2}{4\zeta}$$

Unusual: $\mathrm{div}\, B^3 = -4\pi\rho,\ \mathrm{curl}\, B^3 = 0$ (!)

The 'kinetic energy' of monopole density is nothing but the magnetic energy:

$$-\int d^3x\,\frac{1}{2}\left(\frac{4\pi}{g}\right)^2\rho\,\frac{1}{\triangle}\,\rho = \frac{1}{2g^2}\int d^3x\,B_i\partial_i\,\frac{1}{\triangle}\,\partial_j B_j = \int d^3x\,\frac{B_iB_i}{2g^2}.$$

Hence

$$\mathcal{Z} = \int DB_i\,\delta(\mathrm{curl}B)\,D\psi\,\exp\int d^3x\left[-\frac{B^2}{2g^2} + i\frac{\mathrm{div}B}{4\pi}\psi + 2\zeta\cos\psi\right].$$

Confinement (= Area law for large Wilson loops

Wilson loop

$$W = \mathrm{Tr}\ \mathrm{P}\ \exp\ i\oint dx^i\,A_i^a\frac{\tau_a}{2}.$$

At large distances from monopoles only colour A_i^3 component survives $\Longrightarrow$ can use the Stokes theorem:

$$W = \mathrm{Tr}\ \exp\, i \int d^2 S^i\, B_i^3 \,\frac{\tau_3}{2} = e^{\frac{i}{2}\Phi} + e^{-\frac{i}{2}\Phi},$$

where $\Phi = \int d^2 S^i\, B_i^3$ is the flux of the magnetic field created by monopoles and antimonopoles in the plasma, through the surface spanned over the Wilson loop.

Wilson loop averaged over the ensemble of monopoles: have to plunge the source term $\exp\left(\frac{i}{2}\int d^2 \overrightarrow{S}\cdot\overrightarrow{B}\right)$ into the partition function, and integrate over all possible magnetic fields.

For large loops (say, lying in the $z=0$ plane), one can use the saddle-point method and find the 'best' fields $\overrightarrow{B}$, ψ minimizing the energy together with the surface source:

$$B_i(x,y,z) = \delta_{iz}\, B(z), \qquad \psi = \psi(z),$$

$$
\begin{aligned}
\frac{1}{g^2}B(z) \quad &+ \quad \frac{i}{4\pi}\frac{d\psi}{dz} = \frac{i}{2}\,\delta(z), \\
i\frac{dB}{dz} \quad &- \quad 8\pi\zeta\sin\psi = 0,
\end{aligned}
$$

for x, y inside the contour. The solution:

$$
\begin{aligned}
B \quad &= \quad i\,\frac{g^2\mu}{\pi}\,\frac{e^{-\mu|z|}}{1+e^{-2\mu|z|}}, \\
\psi \quad &= \quad 4\,\mathrm{sign}(z)\,\mathrm{atan}\left(e^{-\mu|z|}\right).
\end{aligned}
$$

The solution corresponds to a purely imaginary double layer of monopoles around the surface.

String tension

$$
< W > \quad = \quad \exp\left(-\sigma\,\mathrm{Area}\right),
$$

$$\sigma = \frac{g^2 \mu}{2\pi^2} = \frac{2g}{\pi}\sqrt{2\zeta},$$

proportional to the *square root* of the mean monopole density $N/V = 2\zeta$.

No massless states are left in the theory. For example, consider the correlation function of the magnetic field (in momentum space):

$$< B_i^3(p) B_j^3(-p) >$$

$$= g^2 \left(\delta_{ij} - \frac{p_i p_j}{p^2} \right) + (4\pi)^2 \frac{N}{V} \frac{p_i p_j}{p^4} - (4\pi)^2 \frac{N}{V} \frac{\mu^2}{p^2 + \mu^2} \frac{p_i p_j}{p^4}$$

$$= g^2 \left(\delta_{ij} - \frac{p_i p_j}{p^2 + \mu^2} \right), \qquad \text{Debye mass}: \ \mu^2 = 2\zeta \left(\frac{4\pi}{g} \right)^2.$$

Best dreams fulfilled! [A. Polyakov, Nucl. Phys. B (1977)]. However,
i) $d = 2 + 1$ and ii) the gauge group $SU(2)$ is explicitly broken by the Higgs field down to $U(1)$ where the *dual* photon gets the mass.

More recent achievement: $\mathcal{N} = 2, \quad d = 3 + 1$ supersymmetric theory, softly broken to $\mathcal{N} = 1$ supersymmetric theory. It is also shown to possess confinement and mass gap – more or less due to the same mechanism (of monopole condensation) [Seiberg and Witten (1994), Douglas and Shenker (1995)].