Classical solutions. Instantons and solitons.

A very wide and useful class of methods exploited equally fruitfully both in condensed matter and particle physics, are semiclassical methods. They are applicable if, for some reasons, a path integral has saddle point(s) and, in addition, saddle-point calculation is justified by some large (or small) parameter. It happens quite often.

Two cases should be distinguished from the start. Case one: the action has a saddle point. The corresponding solution of the Euler–Lagrange eqns. of motion is generically called an instanton ('t Hooft's term):

$$\mathcal{Z} = \int D\phi(x,t)\, \exp\left\{ - \int dt \int d^d x\, \mathcal{L}(\phi, \dot{\phi}, \partial\phi) \right\}$$

$$\phi^{\text{class}}(x,t): \qquad -\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\phi}} - \frac{\partial}{\partial_i}\frac{\partial\mathcal{L}}{\partial(\partial_i\phi)} + \frac{\partial\mathcal{L}}{\partial\phi} = 0.$$

Case two: the energy functional has a saddle point

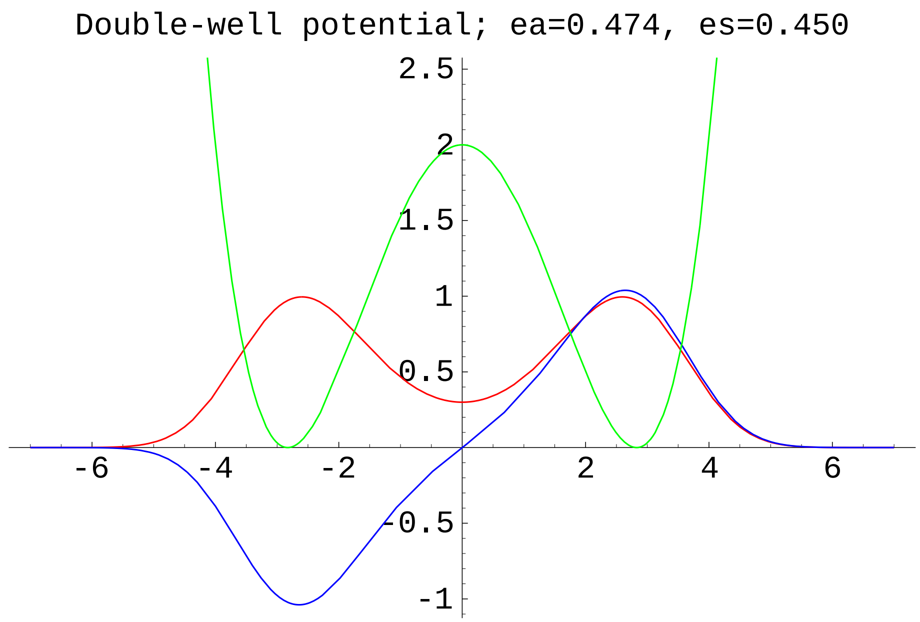
$$\phi^{\text{class}}(x): \qquad -\frac{\partial}{\partial_i}\frac{\partial\mathcal{L}}{\partial(\partial_i\phi)} + \frac{\partial\mathcal{L}}{\partial\phi} = 0.$$

Then the solution is generically called a soliton.

- instanton is a saddle point of the action
- soliton is a saddle point of the energy

An instanton in d dimensions is often a soliton in $d+1$ dimensions. Instantons are more easy to deal with, so let us start from instantons.

Quantum mechanics, double-well potential



All levels are split into two. One can calculate level splitting in two ways:

- Solving Schrödinger eqn
- From instantons [A. Polyakov (1974)]

Schrödinger eqn:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V(x)\,\psi = \epsilon\,\psi,$$

$$V(x) \quad = \quad V_0\left(\frac{x^2}{x_0^2} - 1\right)^2, \quad \left.\frac{d^2V}{dx^2}\right|_{x=\pm x_0} = \frac{8V_0}{x_0^2} = m\omega_0^2.$$

We shall use units $\hbar = m = \omega_0 = 1$. Neglecting the influence of the other well the Schrödinger eqn becomes

$$-\tfrac{1}{2}\frac{d^2\psi}{dx^2} + \tfrac{1}{2}(x \pm x_0)^2\,\psi = \epsilon\,\psi, \qquad \epsilon_n = n + \tfrac{1}{2}.$$

Approximate ground-state wave functions for the double-well potential:

$$\psi_{\mathrm{s,a}}(x) = \frac{\psi(x - x_0) \pm \psi(x + x_0)}{\sqrt{2}}, \qquad \psi(x) \sim \exp(-\tfrac{1}{2}x^2).$$

Let us solve the double-well Schrödinger eqn more accurately. Near $x = -x_0$

$$\psi_{\rm I}(x) = c_{\rm I}\, D_{\epsilon-\frac{1}{2}}(-(x+x_0))$$

where $D_\nu(x)$ is the *parabolic cylinder* function. It decays at $x \longrightarrow -\infty$ but grows at $x \longrightarrow +\infty$. Near $x = x_0$

$$\psi_{\rm II}(x) = c_{\rm II}\, D_{\epsilon-\frac{1}{2}}((x-x_0))$$

It decays at $x \longrightarrow +\infty$ but grows at $x \longrightarrow -\infty$. Under the barrier one writes the WKB semiclassical wave function being a linear combination of rising and falling functions,

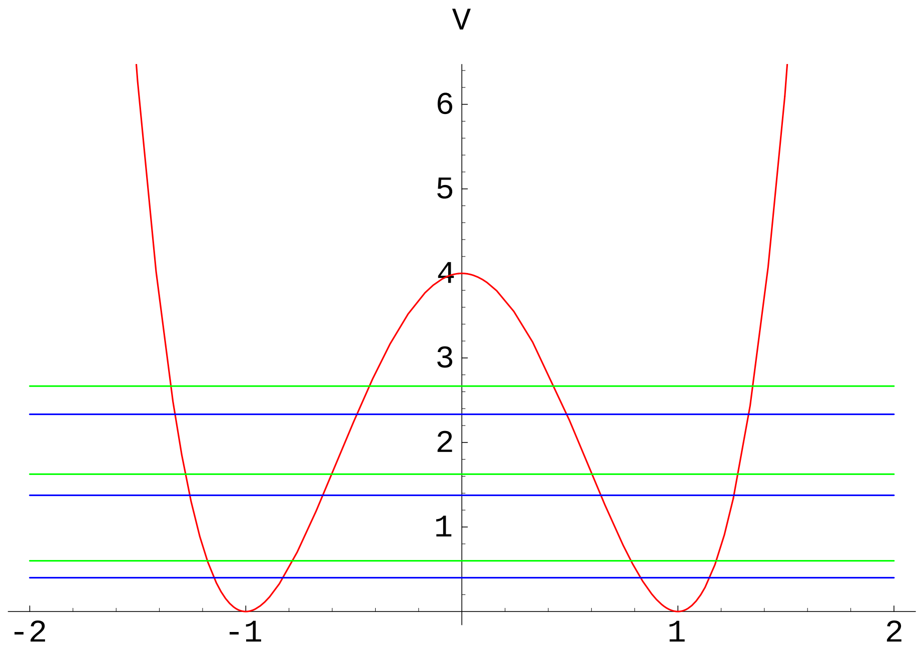
$$\begin{aligned} \psi_{\rm III}(x) &= \frac{c}{\sqrt{p(x)}}\left[\exp\int_0^x p\,dx \pm \exp-\int_0^x p\,dx\right], \\ p(x) &= \sqrt{2\,(V(x)-\epsilon)} \end{aligned}$$

Now one has to find the coefficients $c_{\rm I,II}$ from c by matching the wave functions and their derivatives near the two regions where the wave functions 'dive' under the barrier. That gives the eigenvalues ϵ_n:

$$\begin{aligned}\epsilon_n^{\pm} &= n + \tfrac{1}{2} \pm \tfrac{1}{2}\Delta_n + \left(\text{series in } \frac{1}{V_0}\right), \\ \Delta_n &= \frac{4}{n!}\sqrt{\frac{8V_0}{\pi}}(64V_0)^n \exp\left(-\frac{16}{3}V_0\right)\left(1 + O(1/V_0)\right).\end{aligned}$$

Conclusions:

- Splittings are exponentially small in the barrier hight V_0
- Splittings increase with the level number n.

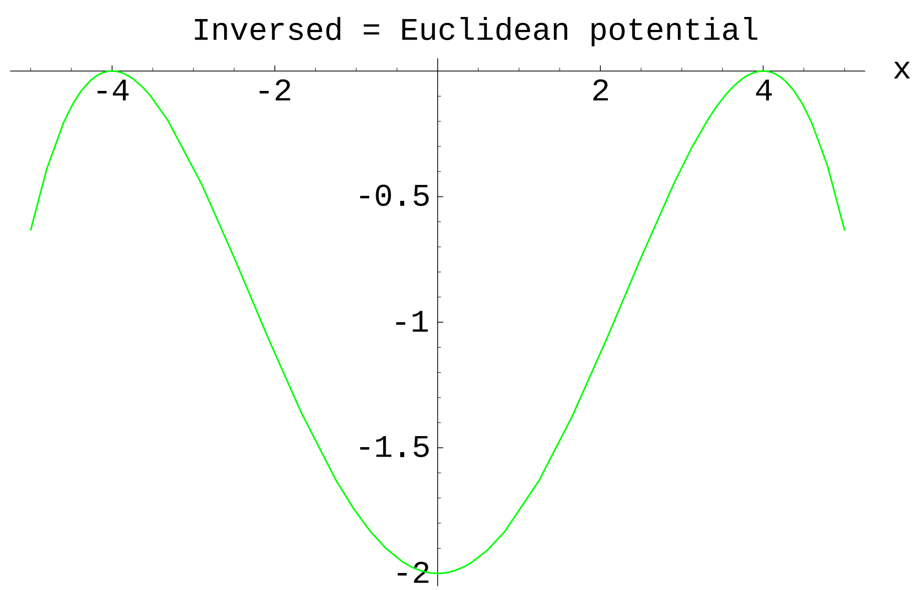


Instanton

We shall now derive the same result for level splittings using quantum field theory. Recall

Feynman's path integral:

$$\langle x_0 | e^{-HT} | -x_0 \rangle = \int Dx(t) \, \exp\left(- \int_0^T dt \left[\frac{\dot{x}^2}{2} + V(x) \right] \right),$$
$$V_E(x) = -V(x).$$



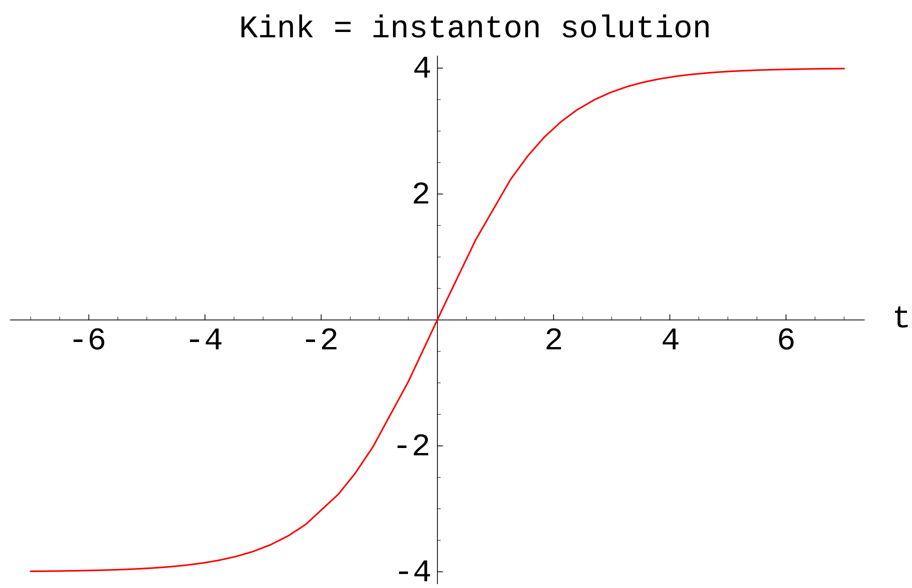
Classical equation of motion

$$-\ddot{x} - V_E'(x) = 0 \quad \text{or, since } E = 0,$$

$$\dot{x} \quad = \quad \sqrt{-2V_E(x)} = \frac{\sqrt{2V_0}}{x_0^2}(x^2 - x_0^2), \quad 8V_0 = x_0^2.$$

Its solution with zero Euclidean ‘energy’ is a kink:

$$x(t) = x_0 \tanh \frac{t}{2}.$$



It is the instanton of the $0+1$ dimensional field theory. The action along the tunneling trajectory is

$$\begin{aligned} S &= \int dt \left[\frac{\dot{x}^2}{2} + V(x)\right] = \int dt\, \dot{x}^2 = \frac{x_0^2}{4}\int \frac{dt}{\cosh^2 \frac{t}{2}} \\ &= \frac{2}{3}x_0^2 = \frac{16}{3}V_0 \gg 1 \end{aligned}$$

Tunneling amplitude

$$\mathcal{A} \sim \exp(-S) = \exp(-\frac{16}{3}V_0) \ll 1.$$

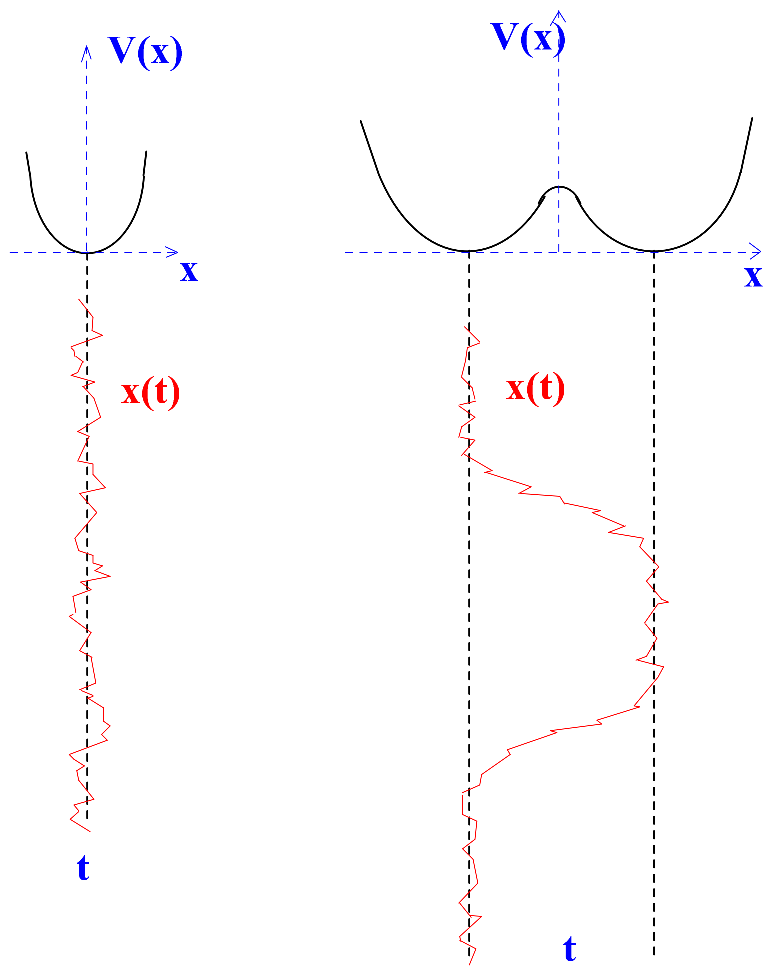
Quantum fluctuations about the instanton

What is $\int Dx(t)$? We write a general trajectory $x(t)$ as a sum of a classical trajectory and of presumably small quantum fluctuations about it,

$$x(t) = x_{\rm cl}(t,\xi) + y(t), \qquad |y(t)| \ll |x_{\rm cl}(t)|,$$

$$x_{\rm cl}(t,\xi) = x_0 \tanh\frac{t-t_0}{2}, \qquad \xi = t_0.$$

ξ is the set of 'collective coordinates' characterizing the classical solution, of which the action is independent. In this simple case there is only one collective coordinate, the time t_0 when the tunneling from one well to another happens. It is also called the 'instanton center'.



To make the above decomposition unique one has to impose conditions on the small fluctuations $y(t)$: they must be orthogonal, in the Hilbert space sense, to changing the collective coordinates. The partition function is, loosely speaking,

$$\mathcal{Z} = \int Dx(t)\, e^{-S[x(t)]}.$$

We insert a unity *à la* Faddeev–Popov:

$$\begin{aligned} 1 &= \int d\xi \int Dy(t)\, \delta\left(x(t) - x_{\rm cl}(t) - y(t)\right) \\ &\cdot \quad \delta\left(\int dt\, \psi(t,\xi)\, y(t)\right)\, \Phi\left[x(t)\right] \end{aligned}$$

where $\Phi[x(t)]$ is a functional fixed from the requirement that the r.h.s. is indeed unity, identically. Therefore, we get by definition,

$$\frac{1}{\Phi\left[x_{\rm cl}(t,\xi)\right]} = d\xi' \int Dy'(t)\, \delta\left(x_{\rm cl}(t,\xi) + y(t) - \right.$$

$$- \quad x_{\rm cl}(t,\xi') - y'(t)) \ \cdot \delta\left(\int dt\, \psi(t,\xi')\, y'(t)\right).$$

We expect to find from the δ-functions:

$$\xi' = \xi, \qquad y'(t) = y(t),$$

so we can expand

$$x_{\rm cl}(t,\xi') = x_{\rm cl}(t,\xi) + \frac{\partial x_{\rm cl}}{\partial \xi}(\xi' - \xi) + \ldots$$

The functional δ-function becomes

$$\delta\left(y(t) - y'(t) - \frac{\partial x_{\rm cl}}{\partial \xi}(\xi' - \xi)\right).$$

We use this δ-function to integrate $\int Dy'(t)$ and get using $\int da\, \delta(a\, b) = \frac{1}{b}$

$$
\begin{aligned}
\Phi^{-1} &= \int d\xi'\, \delta \left(\int dt \left[\psi(t,\xi) + \frac{\partial \psi}{\partial \xi}(\xi' - \xi) \right] \right. \\
&\cdot \left. \left[y(t) - \frac{\partial x_{\rm cl}}{\partial \xi}(\xi' - \xi) \right] \right)
\end{aligned}
$$

where we have substituted $\psi(t,\xi')$ by the first [...] and $y'(t)$ by the second [...] We need not an abstract Φ but the one inside the partition function which includes the $\delta\left(\int dt\, \psi(t,\xi) y(t)\right)$. Hence the leading term is zero, and we obtain

$$
\begin{aligned}
\Phi^{-1} &= \int d\xi'\, \delta \left\{ \int dt \left[\psi \frac{\partial x_{\rm cl}}{\partial \xi} - \frac{\partial \psi}{\partial \xi}\, y \right] (\xi' - \xi) \right\} \\
&= \left(\int dt \left[\psi(t,\xi) \frac{\partial x_{\rm cl}(t,\xi)}{\partial \xi} - \frac{\partial \psi(t,\xi)}{\partial \xi}\, y(t) \right] \right)^{-1} .
\end{aligned}
$$

Finally, the well-defined partition function is written as an integral over p collective coordinates ξ_i and a path integral over small oscillations $y(t)$ about the classical solution, subject to a constraint that they are orthogonal to p functions $\psi_i(t,\xi)$:

$$\begin{aligned} \mathcal{Z} &= \int Dy(t) \prod_{i=1}^{p} \int d\xi_i \, \delta\left(\int dt \, \psi_i(t,\xi) \, y(t) \right) \\ &\cdot \quad \text{det}_{\{ij\}} \int dt \left[\psi_i \frac{\partial x_{\rm cl}}{\partial \xi_j} - \frac{\partial \psi_i}{\partial_{\xi_j}} y \right] \\ &\cdot \quad \exp\left(-S\left[x_{\rm cl}(t,\xi) + y(t)\right]\right) \end{aligned}$$

This expression is in fact independent of the choice of the functions $\psi_i(t,\xi)$: the only restriction is that

$$\text{det}_{\{ij\}} \int dt \, \psi_i \frac{\partial x_{\rm cl}}{\partial \xi_j} \neq 0.$$

Now we have to expand the action about the saddle point $x_{\rm cl}$:

$$\begin{aligned} S[x_{\rm cl}+y] &= S[x_{\rm cl}] + y\,\frac{\delta S}{\delta x}\bigg|_{x=x_{\rm cl}} + \frac{y^2}{2}\,\frac{\delta^2 S}{\delta x^2} + \ldots \\ &= S_0 + \tfrac{1}{2}\int dt\,\left[\dot{y}^2 + V''(x_{\rm cl}(t))\,y^2\right]. \end{aligned}$$

To integrate over $y(t)$ we expand it over eigenfunctions of an analogous Schrödinger eqn, where ‘time’ plays the role of the coordinate:

$$\left[-\frac{d^2}{dt^2} + V''(x_{\rm cl}(t))\right] y_n(t) = \lambda_n\, y_n(t),$$

$$y(t) = \sum c_n y_n(t), \qquad \int dt\, y_m y_n = \delta_{mn}.$$

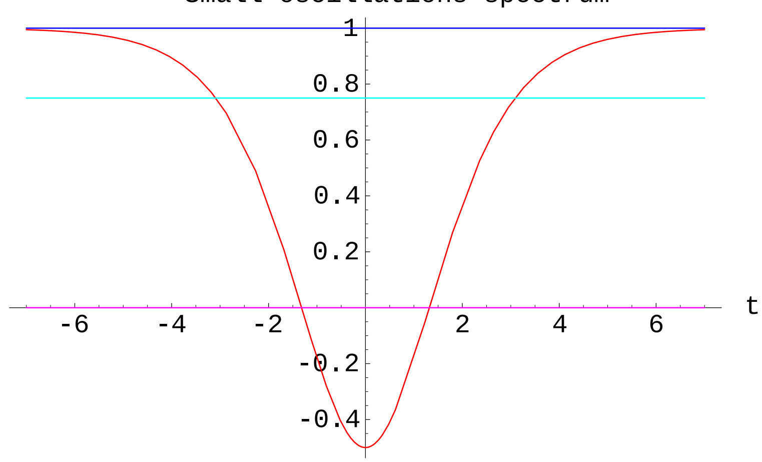
Then

$$\int Dy(t)\, e^{-S} = \prod_n \int \frac{dc_n}{\sqrt{2\pi}}\, e^{-\frac{\lambda_n c_n^2}{2}} = \prod_n \frac{1}{\sqrt{\lambda_n}}.$$

In our case the ‘potential’ is

$$V''(x_{\rm cl}(t)) \quad = \quad 1 - \frac{3}{2}\frac{1}{\cosh^2\frac{t}{2}}.$$

Small oscillations spectrum



Main features of the spectrum: it is continuous but a couple of discrete levels. The bound state with $\lambda_1 = \frac{3}{4}$ accidental but the zero mode with $\lambda_0 = 0$ could be anticipated, since the action evaluated on a shifted kink is the same,

$$\begin{aligned} S\left[x_{\rm cl}(t,\xi+\delta)\right] &= S[x_{\rm cl}(t,\xi)] + \left.\frac{\delta S}{\delta x}\right|_{x=x_{\rm cl}} \frac{\partial x_{\rm cl}}{\partial \xi}\,\delta \\ &+ \tfrac{1}{2}\left[\frac{\delta S}{\delta x}\,\frac{\partial^2 x_{\rm cl}}{\partial \xi^2} + \frac{\partial x_{\rm cl}}{\partial \xi}\,\frac{\delta^2 S}{\delta x^2}\,\frac{\partial x_{\rm cl}}{\partial \xi}\right]\delta^2 + \dots \end{aligned}$$

hence the last term is zero! Moreover, we have in fact found the zero mode eigenfunction, up to a normalization constant,

$$\begin{aligned} y_0(t) &= \text{const.}\,\frac{\partial x_{\rm cl}(t,\xi)}{\partial \xi} = \text{const.}\,\frac{\partial \tanh \frac{t-t_0}{2}}{\partial t_0} \\ &= \frac{\text{const.}}{\cosh^2 \frac{t-t_0}{2}}. \end{aligned}$$

The zero mode is, by construction, normalized to unity:

$$y_0(t) = C\,\dot{x}_{\rm cl}, \qquad C = \left(\int dt\, \dot{x}_{\rm cl}^2\right)^{-\frac{1}{2}} = S_0^{-\frac{1}{2}}.$$

In the gaussian approximation the functional integral is

$$\begin{aligned} \mathcal{Z} &= e^{-S_0} \int dt_0 \int \frac{dc_0}{\sqrt{2\pi}}\, \delta\left(\int dt\, \psi \cdot (c_0\, C\, \dot{x}_{\rm cl} + \ldots)\right) \\ &\cdot \left(\int dt\, \psi\, \dot{x}_{\rm cl}\right) \prod_{n\neq 0} \frac{1}{\sqrt{\lambda_n}} \\ &= \int dt_0\, e^{-S_0} \sqrt{\frac{S_0}{2\pi}} \prod_{n\neq 0} \frac{1}{\sqrt{\lambda_n}}. \end{aligned}$$

This product of eigenvalues is divergent, just as the product of harmonic oscillator's eigenvalues is divergent. To give sense to the product, we **normalize** it to the product of harmonic oscillator eigenvalues. The fortunate point here is that the Schrödinger eqn for the potential $1/\cosh^2 t$ is known exactly, see L. Landau and E. Lifshits, Quantum mechanics.

For the concrete potential $V(x) = V_0(x^2/x_0^2 - 1)^2, \quad x_0^2 = 8V_0$, the (normalized) product of nonzero eigenvalues is [see V. Novikov et al., The ABC of instantons]

$$\prod_{n\neq 0} \frac{1}{\sqrt{\lambda_n}} = \sqrt{\frac{3}{2}} \cdot \, e^{-\frac{1}{2}T},$$

the last factor being the product of eigenvalues for the harmonic oscillator at $T \to \infty$, see L1.

The contribution of one instanton (here: kink) to the double-well partition function is thus

$$\mathcal{Z} = \int dt_0 \, e^{-S_0} \sqrt{\frac{S_0}{2\pi}} \sqrt{\frac{3}{2}} \, e^{-\frac{1}{2}T} \equiv \int dt_0 \, \left(\tfrac{1}{2}\Delta\right) \, e^{-\frac{1}{2}T}.$$

$\frac{1}{2}\Delta$ is called the instanton weight, it gives the probability amplitude for the instanton (that is for tunneling) to occur.

Instanton gas

Tunneling (i.e. instantons) may occur at any time t_0 and many times, [figure] and one has to sum over all possibilities.

Summing over many kinks and anti-kinks:

$$\begin{aligned} \langle -x_0 | e^{-HT} | x_0 \rangle &= e^{-\frac{1}{2}T} \sum_{n=\mathrm{odd}} \frac{(\frac{1}{2}\Delta\, T)^n}{n!} \\ &= e^{-\frac{1}{2}T} \sinh\left(\tfrac{1}{2}\Delta T\right) , \end{aligned}$$

$$\begin{aligned} \langle x_0 | e^{-HT} | x_0 \rangle &= e^{-\frac{1}{2}T} \sum_{n=\mathrm{even}} \frac{(\frac{1}{2}\Delta\, T)^n}{n!} \\ &= e^{-\frac{1}{2}T} \cosh\left(\tfrac{1}{2}\Delta T\right) , \end{aligned}$$

$$\tfrac{1}{2}\Delta = \sqrt{\frac{3}{2}}\sqrt{\frac{S_0}{2\pi}}e^{-S_0}.$$

On the one hand, $\frac{1}{2}\Delta$ gives the probability amplitude of the instanton. On the other hand, Δ is the level splitting,

$$\Delta = 4\sqrt{\frac{8V_0}{\pi}}e^{-\frac{16}{3}V_0},$$

– the same result as we have obtained from solving directly the Schrödinger eqn for the double-well potential.

Some lessons

- Instanton is a classical tunneling trajectory evolving in imaginary (Euclidean) time.
- Tunneling amplitude is $\sim \sqrt{\mathrm{S}}\,\exp(-\mathrm{S})\left(1 + O\left(\frac{1}{\mathrm{S}}\right)\right)$ where S is the action along the instanton trajectory.
- Instantons have zero modes related to the variation which does not change the action.
- One has to integrate over collective coordinates connected to zero modes.

Problems

1. Find analytically the instanton (kink) trajectory $x_{\rm cl}(t)$ for the periodic potential $V(x) = V_0 \cos(2\pi\, x/x_0)$.

2. Show that the zero mode $y_0(t) = \dot{x}_{\rm cl}(t)$ is indeed the zero-eigenvalue solution of the Schrödinger equation for small oscillations.