$2d$ models: $O(N), CP(N)$.

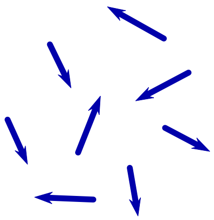
Asymptotic freedom. Restoration of symmetry.

$O(N)$-sigma model (or vector model) in $d = 1 + 1$

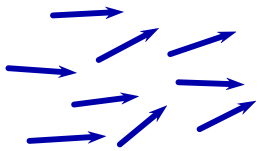
$$
\begin{aligned}
\text{Action} = S &= \int dt dx \left(\frac{1}{2} \partial_\mu \mathbf{\Phi} \partial_\mu \mathbf{\Phi} + V(\mathbf{\Phi}) \right), \\
V(\mathbf{\Phi}) &= \frac{1}{4} \left(\mathbf{\Phi}^2 - v^2 \right)^2, \\
\mathbf{\Phi} &= \text{vector in N}-\text{dimensional space}
\end{aligned}
$$

The model describes, e.g., a $1d$ chain of large spins that can stick in any direction in N dimensions, with spontaneous magnetization whose direction is arbitrary. There are, therefore, $N - 1$ Goldstone particles i.e. massless particles (or 'gapless' excitations, spin waves). We shall be interested in the $T^{\rm o} \rightarrow 0$ limit such that

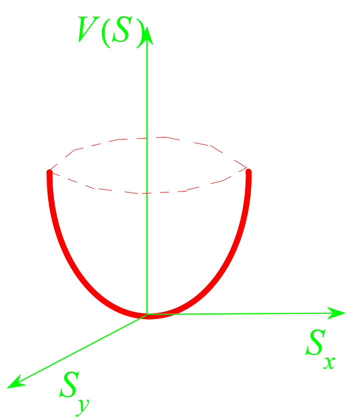
$$\int_0^{1/T^{\rm o}} dt \int dx = \int d^2x.$$

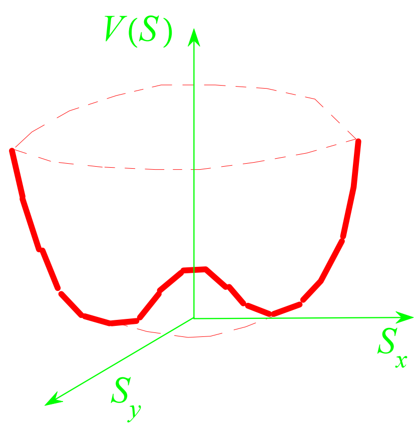


$S = 0$



$S \neq 0$





If one neglects fluctuations of $|\mathbf{\Phi}|$ one arrives to the $O(N)$ sigma-model $(\mathbf{n} = \mathbf{\Phi}/v)$:

$$\mathcal{Z} = \int D\mathbf{n}(x)\, \delta(\mathbf{n}^2 - 1) \exp\left(-\frac{1}{2g^2}\int d^2x\, (\partial_\mu \mathbf{n} \cdot \partial_\mu \mathbf{n})\right) ,$$

$$g^2 = \frac{1}{v^2} = \text{dimensionless coupling constant.}$$

Perturbation theory

Let us suppose that $\mathbf{n}(x)$ fluctuates weakly about some constant direction $\mathbf{n}_0$. We parameterize

$$\mathbf{n} = \mathbf{n}_0\sqrt{1 - g^2\mathbf{m}^2} + g\mathbf{m}, \qquad (\mathbf{m} \cdot \mathbf{n}_0) = 0,$$

$$S = \int d^2x \left(\partial_\mu \mathbf{m} \cdot \partial_\mu \mathbf{m} + g^2 \frac{(\mathbf{m} \cdot \partial_\mu \mathbf{m})(\mathbf{m} \cdot \partial_\mu \mathbf{m})}{1 - g^2\mathbf{m}^2}\right) ,$$

$$\int D\mathbf{n}(x) \quad \delta(\mathbf{n}^2 - 1) = \int \frac{D\mathbf{m}(x)}{\sqrt{1 - g^2\mathbf{m}^2}} .$$

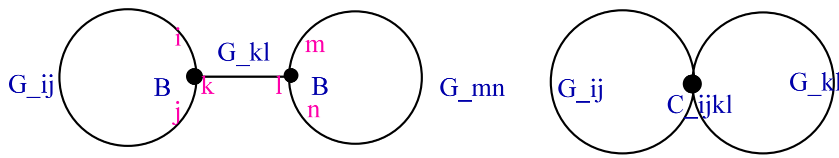
"Perturbation theory" in the usual integral:

$$
\begin{aligned}
\mathcal{Z} &= \int_{-\infty}^{\infty} d\phi \, \exp\left(-\frac{A\phi^2}{2} - B\,g\,\phi^3 - C\,g^2\,\phi^4 + \ldots\right) \\
&= \int d\phi \, \exp\left(-\frac{A\phi^2}{2}\right)\left[1 - B\,g\,\phi^3 + \frac{1}{2}(Bg\phi^3)^2 - C\,g^2\,\phi^4 + \ldots\right] \\
&= \sqrt{\frac{2\pi}{A}}\left[1 + \frac{1}{2}B^2\,g^2\frac{1\cdot 3\cdot 5}{A^3} - C\,g^2\frac{1\cdot 3}{A^2} + \ldots\right]
\end{aligned}
$$

Generalization to multi-dimensional integral:

$$
\begin{aligned}
\mathcal{Z} &= \int d\phi_i \, \exp\left(-\frac{\phi_i A_{ij}\phi_j}{2} - g\,B_{ijk}\phi_i\phi_j\phi_k - g^2\,C_{ijkl}\phi_i\phi_j\phi_k\phi_l + \ldots\right) \\
&= \sqrt{\frac{2\pi}{\det A}}\left[1 + \frac{15}{2}\,g^2\,B_{ijk}\,A^{-1}_{ij}\,A^{-1}_{kl}\,B_{mnl}\,A^{-1}_{mn} - 3g^2\,C_{ijkl}\,A^{-1}_{ij}\,A^{-1}_{kl} + \ldots\right]
\end{aligned}
$$

$$\text{"propagator"} = G_{ij} \equiv < \phi_i \phi_j > = A^{-1}_{ij} = \frac{\int d\phi_i \, \phi_i \phi_j \, \exp\left(-\frac{1}{2}\, \phi A \phi\right)}{\int d\phi_i \, \exp\left(-\frac{1}{2}\, \phi A \phi\right)}$$

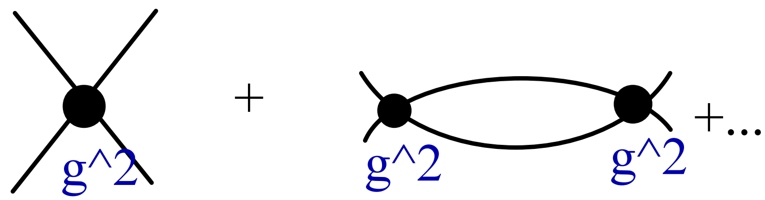


In our field theory case, the free propagator is

$$\begin{aligned} G^{ab}(x-y) &\equiv < m^a(x) m^b(y) > = \frac{\int D\mathbf{m}\, m^a(x) m^b(y) \, \exp\left(-\int d^2x \, \frac{1}{2}\, \partial_\mu m^a \partial_\mu m^a + \ldots\right)}{\int D\mathbf{m} \, \exp\left(-\int d^2x \, \frac{1}{2}\, \partial_\mu m^a \partial_\mu m^a + \ldots\right)} \\ &= \delta^{ab} \int \frac{d^2p}{(2\pi)^2} \, e^{i\,\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} \, \frac{1}{p^2} = -\frac{\delta^{ab}}{\pi} \, \ln(x-y)^2 \end{aligned}$$

– describes the massless correlation function of two ‘spins’.

Integrating out high momenta $\mu_1 < |k| < \mu_2$ we find that the dimensionless coupling constant is renormalized:



$$\frac{1}{g^2(\mu_1)} = \frac{1}{g^2(\mu_2)} - \frac{N-2}{2\pi} \ln \frac{\mu_2}{\mu_1}, \qquad g^2(\mu_1) > g^2(\mu_2) \text{ !}$$

This is called **asymptotic freedom**: at large momenta the interaction is weak, whereas at low momenta it blows up.

Momentum Λ where the coupling constant formally becomes infinite:

$$0 = \frac{1}{g^2(\mu)} - \frac{N-1}{2\pi} \ln \frac{\mu}{\Lambda}, \qquad \text{or} \qquad \Lambda = \mu \, \exp\left(-\frac{2\pi}{N-2} \frac{1}{g^2(\mu)}\right).$$

μ is the UV cutoff, e.g. inverse lattice spacing, $\mu = 1/a$; $g^2(\mu)$ is the coupling constant *defined at the cutoff momentum*. In condensed matter physics $1/\Lambda$ is called the **correlation length**: at this distance the propagator $< \mathbf{n}(x)\mathbf{n}(y) >$ deviates very strongly from the free one; something happens there.

In $O(N = 2)$ model there is no renormalization – why? Because the theory is 'free':

$$\mathbf{n} = (\cos\alpha, \sin\alpha), \qquad \partial_\mu \mathbf{n} \partial_\mu \mathbf{n} = \partial_\mu \alpha \partial_\mu \alpha.$$

A more general view on the renormalization

The nonlinear $O(N)$ sigma model describes long wavelength fluctuations of the vector order parameter $\mathbf{n}(x)$. It is a particular example of a wider class of 'nonlinear sigma models' with the action

$$S = \tfrac{1}{2}\int d^2x\, g_{ab}(w)\partial_\mu w^a \partial_\mu w^b$$

with the integration measure

$$D\mu[w] = \prod_x \sqrt{\det g_{ab}}$$

In our case the 'metric tensor' $g_{ab}(w)$ is

$$g_{ab} \quad = \quad \left(\delta_{ab} + \frac{w^a w^b}{\frac{1}{g^2} - \mathbf{w}^2}\right), \qquad \sqrt{\det g_{ab}} = \frac{1}{\sqrt{1 - g^2\,\mathbf{w}^2}}, \qquad a = 1, \ldots N-1.$$

It describes fluctuations of the order parameter on a sphere S^{N-1} of radius $1/g$. More general sigma models describe fluctuations of the order parameter belonging to some more general space, whose metric is given by $g_{ab}(w)$.

Integrating out high-momenta fluctuations, $\mu_1 < |k| < \mu_2$, the metric is renormalized: [D. Friedan, *Ann. Phys.* 163 (1985) 318]

$$\frac{dg_{ab}}{d\ln\,\mu} = -\frac{1}{4\pi}R_{ab} - \frac{1}{8\pi^2}R_{acde}R_b^{\ cde} - \ldots$$

Here R_{abcd} and R_{ab} are the Riemann and Ricci (respectively) tensors of the curved manifold (to which the order parameter belongs) and are defined as follows. First, one defines the Christoffel symbols (or connection coefficients),

$$\Gamma^a_{bc} = \tfrac{1}{2}g^{ad}\left(\frac{\partial}{\partial w^b}g_{cd} + \frac{\partial}{\partial w^c}g_{bd} - \frac{\partial}{\partial w^d}g_{bc}\right), \quad \Gamma^a_{bc} = \Gamma^a_{cb},$$

and the covariant derivative in curved space,

$$(\nabla_b)^a_c = \frac{\partial}{\partial w^b}\delta^a_c + \Gamma^a_{bc}.$$

Second, one defines the Riemann tensor as the commutator

$$R^a_{bcd} = [\nabla_c \nabla_d]^a_b \,.$$

One can use the contravariant metric tensor g^{ab} such that $g^{ac}g_{cb} = \delta^a_b$ to rise and contract the indices. The rank-2 Ricci tensor is the partial contraction of the rank-4 Riemann tensor, while the full contraction is the scalar curvature:

$$R_{ab} = R^c_{\ acb}, \qquad R = g^{ab}R_{ab} = g^{ab}g^{cd}R_{acbd}.$$

Thus, the renormalization is determined by geometry of the internal space of the order parameter. In the simplest case of the $O(3)$ model corresponding to the geometry of the sphere one obtains

$$R_{ab} = g^2\, g_{ab}, \quad R_{acde}R_b^{\ cde} = 2g^4\, g_{ab}, \quad R = 2g^2,$$

so that the **form** of the metric does not change from integrating over high momenta. Therefore, Friedan's eqn. comes to the renormalization of the scalar curvature or of the radius of the sphere on which the fields live, or of the coupling constant, which we know already:

$$\frac{d\left(\frac{1}{g^2(\mu)}\right)}{d\ln\ \mu} = -\frac{1}{2\pi} - \frac{g^2}{2\pi^2} - \ldots$$

However, there are models where the geometry of the order parameter space changes when one integrates out fluctuations with smaller and smaller momenta: it becomes either more curved, meaning going more and more into the strong coupling regime, or more flat, meaning critical behavior in the limit of large wavelengths. See V. Fateev et al., *Nucl. Phys.* B406 (1993) 521.

A General Relativity package "GRTensor" for `Maple` and `Mathematica` can be found at `http://grtensor.phy.queensu.ca/`

Nonperturbative method: $1/N$ expansion

Imagine, the spin field $\mathbf{n}(x)$ is a multi-dimensional vector, $N \gg 1$. Then the theory can be easily solved exactly! One introduces an auxiliary integration over a field called the Lagrange multiplier, to get rid of the restriction $\mathbf{n}^2 = 1$:

$$\prod_x \delta(\mathbf{n}^2(x) - 1) = \int D\lambda(x)\ \exp \int d^2x \left(-\frac{\lambda}{2g^2}\right)(\mathbf{n}^2 - 1).$$

Integration over λ is along the imaginary axis but we shall look for a saddle point in λ which is real.

$$\mathcal{Z} = \int D\lambda \int d\mathbf{n}\ \exp \int d^2x \left[-\frac{1}{2g^2}(\partial_\mu \mathbf{n} \partial_\mu \mathbf{n} + \lambda \mathbf{n}^2) + \frac{\lambda}{2g^2}\right].$$

Now the integral over $\mathbf{n}$ is gaussian (as for the free field) with λ playing the role of a mass^2 term. The path integral is similar to that for the harmonic oscillator (see L1).

$$\int d\mathbf{n}\ \exp -\frac{1}{2g^2} \int d^2x\, (\partial_\mu \mathbf{n} \partial_\mu \mathbf{n} + \lambda \mathbf{n}^2)$$

$$= \text{const.} \frac{1}{\sqrt{\det[-\partial^2 + \lambda]}}$$

where det... is a functional determinant.

Formal manipulations with functional determinants:

$$\mathcal{D} = -\partial^2 + \lambda, \qquad \mathcal{D}_0 = -\partial^2.$$

$$\sqrt{\frac{\det \mathcal{D}_0}{\det \mathcal{D}}} = \exp\left[-\tfrac{1}{2}\mathrm{Sp}\ (\ln\ \mathcal{D} - \ln\ \mathcal{D}_0)\right]$$

$$= \exp\left(-\frac{N}{2} \int d^2x \int \frac{d^2k}{(2\pi)^2} \ln \frac{k^2+\lambda}{k^2} + \text{gradients of } \lambda\right).$$

Let us try to find a saddle point in λ putting $\partial/\partial\lambda \ldots \to 0$:

$$\int d^2x \left(\frac{1}{2g^2} - \frac{N}{2} \int \frac{d^2k}{(2\pi)^2} \frac{1}{k^2 + \lambda} \right) = 0.$$

The k integral is logarithmically divergent at large k. Let us cut it at some large momentum $|k| = \mu \gg \lambda$ implying that the bare coupling constant $g^2(\mu)$ is defined at that value of the cut-off momentum. We get

$$\frac{1}{g^2(\mu)} = N \int \frac{d^2k}{(2\pi)^2} \frac{1}{k^2 + \lambda} = \frac{N}{4\pi} \ln \frac{\mu^2}{\lambda}.$$

Hence, the Lagrange multiplier λ develops a nonzero expectation value

$$< \lambda >= \mu^2 \exp\left(-\frac{4\pi}{N g^2(\mu)} \right) = \Lambda^2.$$

Discussion

- In this derivation, we have neglected gradients of $\lambda(x)$. They are suppressed as $1/N$ hence the conclusion that the $\mathbf{n}$ field gets a mass $\sqrt{<\lambda>}$ is, strictly speaking, justified only at $N \gg 1$, However, the exact solution [P. Wiegmann, *Phys. Lett.* B152 (1985) 209] shows that $<\lambda> \neq 0$ is correct even for the $O(N=3)$ model. One has to replace $N \to N-2$.

- $<\lambda>$ is proportional to the UV cutoff μ^2, however the above combination of μ and $g^2(\mu)$ is in fact cutoff-independent! (Because when one changes μ one changes $g^2(\mu)$ according to the renormalization eqn.) This phenomenon is called transmutation of dimensions: it's the only way how a dimensionfull quantity (here: the mass gap $\sqrt{\lambda}$) can appear in a theory where the coupling constant g^2 is dimensionless.

- The appearance of the mass $\sqrt{\lambda}$ means that instead of $N-1$ massless Goldstone particles of the perturbation theory one obtains N massive particles. The broken symmetry (spontaneous magnetization) is restored!

Further reading (books):

A. Polyakov, *Field Theories and Strings*, Harwood Academic (1988)
E. Fradkin, *Field Theories of Condensed Matter Physics*, Addison–Wesley (1991)
A. Tsvelik, *Quantum Field Theory in Condensed Matter Physics*, Cambridge U. Press (1995)

Problem:

Derive charge renormalization in the $O(N)$ model from Friedan's equation.