Skyrmions.
Quantum fluctuations about solitons.

Let us imagine that it is not an action but an energy functional in d dimensions,

$$\mathcal{E} = \int d^d x \, \left(\tfrac{1}{2}\partial_i\phi\partial_i\phi + V(\phi)\right),$$

that has a nontrivial local minimum. The classical solution $\phi^{\rm class}(x)$ of the (nonlinear) Euler–Lagrange eqn,

$$-\Delta\,\phi + \frac{\partial V}{\partial\phi} = 0,$$

is generically called a soliton.

In QFT fields fluctuate but in the case of a soliton they fluctuate not about zero values but rather about $\phi^{\rm cl}(x)$. In this lecture we study how to deal with quantum fluctuations about (static) solitons.

We decompose the general static field,

$$\phi(x) = \phi^{\mathrm{cl}}(x, \xi) + \chi(x)$$

and plug it into the energy functional expanding it to the second order in χ. As in the instanton case, ξ will be a set of collective coordinates characterizing the soliton. Apparently, one of the collective coordinates is a d-dimensional center of the soliton in space. Since the Euler–Lagrange equation is translational-invariant $\phi^{\mathrm{cl}}(x - x_0)$ with any center x_0 is its solution. However, sometimes there are more collective coordinates, e.g. rotation of the soliton as a whole.

We shall consider a particular example of a soliton, called the Skyrmion, derived from Skyrme, a British nuclear physicist who invented it in 1960. Later on we shall see that it is of direct relevance to protons and neutrons but at this point it will be just a mathematical exercise. More recently, skyrmions or its modifications have been used in condensed matter physics.

The field of the model is an $SU(2)$ matrix which is a function of $3d$ coordinates,

$$
\begin{aligned}
U(x) &= \exp(i\pi^A(x)\,\tau^A) = \cos|\pi| + i\,\frac{\pi^A\,\tau^A}{|\pi|}\,\sin|\pi| \\
&= u_4\,\mathbf{1} + i\,u_A\,\tau^A, \quad u_1^2 + \ldots + u_4^2 = 1,
\end{aligned}
$$

where τ^A are three 2×2 Pauli matrices. The action of the Skyrme model is

$$
\begin{aligned}
S &= \int dt \int d^3x \left\{ -\frac{F^2}{4}\,\mathsf{Tr}(U^\dagger\partial_\mu U\; U^\dagger\partial_\mu U) \right. \\
&+ \left. \frac{1}{2g^2}\,\mathsf{Tr}[U^\dagger\partial_\mu U, U^\dagger\partial_\nu U]^2 \right\}, \qquad [F^2] = \mathrm{cm}^{-2}, \quad [g^2] = \mathrm{cm}^0.
\end{aligned}
$$

The energy density $\mathcal{E}$ is $\int d^3x\ldots$ of the same expression but only with spatial derivatives, $\mu = i = 1, 2, 3$.

We wish to find a static soliton which is a local minimum of the energy functional $\mathcal{E}$.

There are in fact three functions $\pi^A(x)$ that have to be varied. We look for the solution using an Ansatz which is as close to spherical symmetry as it is possible:

$$\pi^A(x) = \frac{x^A}{r} P(r), \quad r = |\mathbf{x}|.$$

It is called the hedgehog Ansatz, $P(r)$ being its profile function.

There is an interesting relation to topology here. If $\pi^A(\mathbf{x}) \to 0$ at spatial infinity so that $U(\mathbf{x}) \to \mathbf{1}$ in all directions, one can say that the spatial infinity is just one point. Therefore, our $3d$ space is topologically equivalent to the $3d$ sphere S^3. Mathematicians say that $U(\mathbf{x})$ is a mapping $S^3 \mapsto S^3$ whose homotopy group is denoted as $\pi_3(S^3) = Z$, a group of integers. Topologically non-equivalent mappings $U(\mathbf{x})$, i.e. those which can not be continuously deformed one to another, are classified by their winding number, an integer analytically given by

$$N_{\rm wind} \quad = \quad -\frac{1}{24\pi^2}\int d^3\mathbf{x}\, \epsilon_{ijk} \mathrm{Tr}\Big(U^\dagger\partial_i U\Big)\Big(U^\dagger\partial_j U\Big)\Big(U^\dagger\partial_k U\Big)$$

$$= \quad -\frac{1}{\pi}\left[P(r) - \frac{\sin 2P(r)}{2}\right]_0^\infty .$$

If $P(\infty) = 0$ and $P(0) = n\pi$ the skyrmion winds n times. We shall look for a soliton with $N_{\rm wind} = 1$ meaning that the boundary condition is $P(0) = \pi = 3.1459\ldots$.

Let us show that the energy functional must have a minimum with such a boundary condition. Imagine that we have already found the soliton, and its spatial size is some r_0. Then

$$\mathcal{E} = c_1\, r_0 + \frac{c_2}{r_0}$$

which clearly has a minimum in r_0. [This is a quite general dimensional analysis which is useful for checking beforehand if there is a soliton in a nonlinear system.]

The skyrmion profile $P(r/r_0)$ can be easily found from solving numerically the second-order nonlinear diff equation, see G. Adkins et al., *Nucl. Phys.* B233 (1984) 109. However we are interested now in the method of treating the quantum fluctuations about a soliton. The soliton is static but quantum fluctuations may depend on time. We, therefore, present the general unitary-matrix field as, say, a left-hand shift in the $SU(2)$ group:

$$U(x,t) = L(x,t)U^{\rm cl}(x), \quad L = \mathbf{1} + il - \tfrac{1}{2}\, l^2 + \ldots$$

where $l(x,t)$ is an (infinitesimal) traceless hermitian 2×2 matrix; then U belongs to $SU(2)$. First, we put it into $\mathcal{E}$ and expand it to the second order in $l(x,t)$. The zeroth order is the classical energy of the soliton $\mathcal{E}^{\rm cl}$, the linear term is absent because the soliton is found from the Euler–Lagrange eqn following from the requirement that the first variation is zero. The quadratic term in the fluctuations has the form

$$\begin{aligned}
\mathcal{E}^{(2)} &= \int d^3x \,\mathsf{Tr}\, l \left(-\Delta(U^{\rm cl})\right) \, l + O(l^3), \\
- & \;\; \Delta(U^{\rm cl}) = -\Delta - [U^{\rm cl}\partial_i\, U^{\rm cl\,\dagger}, \partial_i] + \ldots
\end{aligned}$$

We now decompose the general fluctuation field $l(x,t)$ in the complete set of eigenfunctions of this second-order diff operator:

$$-\Delta(U^{\rm cl})y_n(x) = \omega_n^2\, y_n(x), \quad \int y_m^\dagger\, y_n = \delta_{mn},$$

$$\begin{aligned} l(x,t) &= \sum_n c_n(t)\, y_n(x), \\ \mathcal{E}^{(2)} &= \tfrac{1}{2}\sum_n \omega_n^2\, c_n^2(t). \end{aligned}$$

Plugging the expansion into the kinetic energy term of the action, containing time derivatives ∂_0 one abtains the action of an infinite number of harmonic oscillators,

$$S^{(2)} = \int dt \sum_n \left(\tfrac{1}{2}\dot{c}_n^2 - \tfrac{1}{2}\omega_n^2 c_n^2\right)$$

Everyone knows how to quantize that: one replaces

$$\dot{c}_n \to -i\frac{\partial}{\partial c_n}$$

writes the Hamiltonian

$$\mathcal{H} = \sum_n \left(-\tfrac{1}{2}\frac{\partial^2}{\partial\, c_n^2} + \tfrac{1}{2}\omega_n^2\, c_n^2\right)$$

and gets Schrödinger's eqn for an infinite number of harmonic oscillators,

$$\begin{aligned} \mathcal{H}\,\Psi(c_1, c_2, \ldots) &= \mathcal{E}\,\Psi(c_1, c_2, \ldots), \qquad (1) \\ \left(-\tfrac{1}{2}\frac{\partial^2}{\partial\, c_n^2} + \tfrac{1}{2}\omega_n^2\, c_n^2\right)\psi(c_n) &= \epsilon_n\,\psi(c_n). \end{aligned}$$

The expansion coefficients c_n play the role of the coordinates of the harmonic oscillators; there are infinitely many of them, as it should be in field theory. The energy of the k-th excited state is given by

$$\epsilon_n^k = (k + \tfrac{1}{2})\,\omega_n, \quad k \geq 0.$$

The lowest state corresponds to taking all values of $k = 0$, and is described by a product of the oscillator wave functions,

$$\Psi(c_1, c_2, \ldots) = \prod_n \psi_n^{k=0}(c_n) = \prod_n \exp\left(-\tfrac{1}{2}\omega_n\, c_n^2\right),$$

while the aggregate energy of zero-point oscillations about the soliton is given by $\mathcal{E} = \frac{1}{2}\sum_n \omega_n$. [If the n-th mode is excited one has to add the corresponding excitation energy ω_n.]

The sum of ω's is, of course, badly divergent. To make it convergent, one has to subtract the aggregate energy of the zero-point oscillations in the empty space, without the soliton. It is physically motivated since we are interested in the energy of the soliton, as compared to no-soliton. In the empty space the diff operator of the quadratic form is just a Laplacian $-\Delta$ and its eigenfunctions $y_n^{(0)}$ are plane waves. It is convenient to discretize the spectrum $\omega_n^{(0)}$ by putting the system into a box of finite size.

The full energy of the soliton is

$$\mathcal{E} = \mathcal{E}^{\mathrm{cl}} + \tfrac{1}{2}\sum_n (\omega_n - \omega_n^{(0)}).$$

Soliton zero modes

It is clear that some of the eigenfunctions y_n of the quadratic form have zero eigenvalues, $\omega_i = 0$, $i = 1...p$ They are called zero modes and are always related to the symmetry of the problem at hand. For example, a translated skyrmion, $U(x) \to U(x - x_0) \approx U(x) - x_{0i}\partial_i U$ is also a solution. If we pick a quantum fluctuation which is in fact an infinitesimal translation of the soliton as a whole, it will have zero energy and hence it will be a zero mode.

Three normalized zero modes corresponding to three translations are

$$y_{1,2,3}(x) = -i\,\partial_{1,2,3}U\,U^\dagger \left(\int d^3x\,\mathrm{Tr}\,\partial_i U \partial_i U^\dagger\right)^{-\frac{1}{2}}.$$

[One can verify that it is a zero mode of the quadratic form $-\Delta(U^{\rm cl})$ but it is clear without calculations.] The Schrödinger eqn,

$$-\tfrac{1}{2}\frac{\partial^2}{\partial\, c_i^2}\,\psi(c_i) = \epsilon\,\psi(c_i),$$

has a plane-wave solution,

$$\psi = e^{i(P\cdot x_0)}, \quad x_{0i} = c_i\left(\int \mathrm{Tr}\,\partial_i U\,\partial_i U^\dagger\right)^{\frac{1}{2}},$$

corresponding to moving the center of gravity of the skyrmion with momentum P.

More interesting are rotations of the skyrmion.

A general statement is that if the 'chiral' field $U_{\rm cl}(\mathbf{x})$ minimizes the energy functional, a field corresponding to rotated spatial axes, $x_i \longrightarrow O_{ij}x_j$, or to a unitary-rotated matrix in 'flavor' space, $U_{\rm cl} \longrightarrow RU_{\rm cl}R^\dagger$, has obviously the same energy. This is because the energy functional is isotropic both in flavor and ordinary spaces.

Specifically for the hedgehog Ansatz any spatial rotation is equivalent to a flavor rotation. Indeed, the space-rotating 3×3 matrix O_{ij} can be written as

$$O_{ij} = \frac{1}{2}\mathrm{Tr}(S\tau_i S^\dagger \tau_j)$$

where S is an $SU(2)$ 2×2 matrix and τ_i are the three Pauli matrices. One can immediately check that O_{ij} are real orthogonal 3-parameter matrices with $O_{ij}O_{kj} = \delta_{ik}$ and $O_{ij}O_{ik} = \delta_{jk}$.

We now rotate the skyrmion both in 'flavor' and ordinary spaces,

$$R\, U^{\mathrm{cl}}(O\mathbf{x})\; R^\dagger \quad = \quad \tilde{R}\, U_{\mathrm{cl}}(\mathbf{x})\; \tilde{R}^\dagger,$$

$$\tilde{R} = R\, S.$$

Therefore, if one considers the hedgehog Ansatz rotated *both* in flavor and usual spaces, the latter can be completely absorbed into the former one. For that reason it is sufficient to consider only one kind of rotations.

The general strategy is to consider a slowly rotating Ansatz,

$$\tilde{U}(\mathbf{x},t) = R(t)\, U^{\mathrm{cl}}(\mathbf{x})\, R^\dagger(t)$$

and to expand the action in 'right' (Ω_A) and 'left' ($\tilde{\Omega}_A$) angular velocities,

$$\begin{aligned} \Omega_A &= -i\mathrm{Tr}(R^\dagger \dot{R} \tau^A), \\ \tilde{\Omega}_A &= -i\mathrm{Tr}(\dot{R} R^\dagger \tau^A), \\ \Omega^2 &= \tilde{\Omega}^2 = 2\mathrm{Tr}\dot{R}^\dagger \dot{R}. \end{aligned}$$

One gets the following rotation lagrangian:

$$L^{\mathrm{rot}} = \frac{I}{2}\Omega^2$$

where I is the moment of inertia of the skyrmion; it is a functional of the skyrmion profile $P(r)$. This is the lagrangian for the spherical top: the two sets of angular velocities have the meaning of those in the 'lab frame' and 'body fixed frame'. One can use the canonical quantization procedure to quantize it. One has to introduce two sets of angular momenta,

S_i (canonically conjugate to Ω_i) and T_A (conjugate to $\tilde{\Omega}_A$). Both sets of operators act on the coordinates of the spherical top, say, the Euler angles. It will be more convenient for us to say that the coordinates of the spherical top are just the entries of the unitary matrix R defining its finite-angle rotation.

The angular momenta operators J_i, T_A act on R as generators of right (left) multiplication,

$$
\begin{aligned}
e^{i(\alpha J)}\, R\, e^{-i(\alpha J)} &= R\, e^{i(\alpha\sigma)}, \\
e^{i(\alpha T)}\, R\, e^{-i(\alpha T)} &= e^{-i(\alpha\tau)}\, R,
\end{aligned}
$$

and satisfy the commutation relations

$$
\begin{aligned}
[T_A, T_B] &= i\, \epsilon_{ABC}\, T_C, \\
[J_i, J_j] &= i\, \epsilon_{ijk}\, J_k, \\
[T_A, J_i] &= 0, \qquad T_A^2 = J_i^2.
\end{aligned}
$$

A possible realization of these operators is

$$\begin{aligned} J_i &= R_{pk} \left(\frac{\sigma_i}{2}\right)_{kq} \frac{\partial}{\partial R_{pq}}, \\ T_A &= -\left(\frac{\tau_A}{2}\right)_{pk} R_{kq} \frac{\partial}{\partial R_{pq}}. \end{aligned}$$

The rotational hamiltonian is

$$\begin{aligned} H^{\mathrm{rot}} &= \Omega_i J_i - L^{\mathrm{rot}} = \tilde{\Omega}_A T_A - L^{\mathrm{rot}} \\ &= \frac{J_i^2}{2I_1} = \frac{T_A^2}{2I_1}. \end{aligned}$$

Comparing the definition of the generators with the Ansatz one observes that T_A is the flavor operator and J_i is the spin operator, since the former acts to the left from R and the latter acts to the right.

The normalized eigenfunctions of the mutually commuting operators J_3, T_3 and

$J^2 = T^2$ with eigenvalues $J_3,\ T_3$ and $J(J+1) = T(T+1)$ are

$$\Psi^{(J=T)}_{T_3\, J_3}(R) = \sqrt{2J+1}(-1)^{T+T_3} D^{(J=T)}_{-T_3\, J_3}(R)$$

where $D(R)$ are Wigner's finite-rotation matrices. For example, in the $J = T = 1/2$ representation $D^{1/2}_{pq}(R) = R_{pq}$, i.e. coincides with the unitary matrix itself.

The rotational energy is thus

$$E^{\mathrm{rot}} = \frac{J(J+1)}{2I_1} = \frac{T(T+1)}{2I_1}$$

and is $(2J+1)^2 = (2T+1)^2$-fold degenerate.

Rotations are compact; for that reason the quantization of rotations leads to a discrete spectrum, whereas translations are non-compact, and the spectrum is continuous.

When applied to the nucleons (later) the above wave functions describe at $J = T = 1/2$ four nucleon states (proton, neutron, spin up, spin down) and at $J = T = 3/2$ the sixteen states of the Δ-resonance.

Conclusions:

Quantization of translational zero modes of a soliton is trivial [plane waves of a soliton as a whole]. If there are other zero modes related to the symmetry of the problem [like rotations of the soliton as a whole] their quantization may be quite amusing.

Problems

1. Show that for $U = \exp[i\,(\mathbf{n}\cdot\tau)\,P(r)]$ the winding number is

$$N_{\rm wind} := -\frac{1}{24\pi^2}\int d^3\mathbf{x}\,\epsilon_{ijk}\,\mathsf{Tr}\Big(U^\dagger\partial_i U\Big)\Big(U^\dagger\partial_j U\Big)\Big(U^\dagger\partial_k U\Big)$$

$$= -\frac{1}{\pi}\left[P(r) - \frac{\sin 2P(r)}{2}\right]_0^\infty .$$

2. Show that $O_{ij} = \frac{1}{2}\mathsf{Tr}(S\tau_i S^\dagger \tau_j)$ where S is an $SU(2)$ 2×2 unitary matrix and τ_i are the three Pauli matrices, is an $SO(3)$ real orthogonal matrix: $OO^T = O^T O = \mathbf{1}$ or $O_{ij}O_{kj} = \delta_{ik}$ and $O_{ij}O_{ik} = \delta_{jk}$.

Wigner D-functions

Wigner D-functions are eigenfunctions of the square of the angular momentum operator (written in terms of, say, three Euler angles $\alpha,\ \beta,\ \gamma$),

$$\mathbf{J}^2 D^J_{mn}(\alpha,\beta,\gamma) = J(J+1)D^J_{mn}(\alpha,\beta,\gamma),$$
$$J \quad = \quad 0,\ \frac{1}{2},\ 1,\ \frac{3}{2}...,\qquad -J \le m,n \le +J,$$

and can be said to be eigenfunctions of a spherical top; they are $(2J+1)^2$-fold degenerate. The 'magnetic' quantum numbers $m,\ n$ have the meaning of the projections of the angular momentum of a spherical top on the third axes in the 'body-fixed' and 'lab' frames. One can parametrize a 2×2 unitary matrix by Euler angles as

$$U = \exp(i\alpha\tau^3)\exp(i\beta\tau^2)\exp(i\gamma\tau^3).$$

There two sets of angular momenta operators,

$$S_3 \quad = \quad -i\frac{\partial}{\partial\alpha},\quad S_1 = -i\frac{\partial}{\partial...},\quad S_2 = -i\frac{\partial}{\partial...}$$

$$
\begin{aligned}
T_3 &= -i\frac{\partial}{\partial\gamma}, \quad T_1 = -i\frac{\partial}{\partial\ldots}, \quad T_2 = -i\frac{\partial}{\partial\ldots} \\
\mathbf{S}^2 &= \mathbf{T^2} \equiv \mathbf{J}^2 \\
&= -\frac{\partial^2}{\partial\beta^2} - \cot\beta\frac{\partial}{\partial\beta} - \frac{1}{\sin^2\beta}\left(\frac{\partial^2}{\partial\alpha^2} - 2\,\cos\beta\frac{\partial^2}{\partial\alpha\partial\gamma} + \frac{\partial^2}{\partial\gamma^2}\right)
\end{aligned}
$$

such that

$$
[S_i S_j] = i\epsilon^{ijk} S_k, \quad [T_i T_j] = i\epsilon^{ijk} T_k, \quad [S_i T_j] = 0.
$$

It is convenient to use the unitary matrix U as a formal argument of the D-functions. Their main properties are:

- Multiplication law (summation over repeated indices understood):

$$
D^J_{kl}(U_1 U_2) = D^J_{km}(U_1) D^J_{ml}(U_2).
$$

- Unitarity ("*" denotes complex conjugate):

$$D^J_{kl}(U^\dagger) = \left(D^J_{lk}(U)\right)^*.$$

- Phase condition:

$$\left(D^J_{lk}(U)\right)^* = (-1)^{l-k} D^J_{-l,-k}(U), \quad D^J_{kl}(1) = \delta^{(2J+1)}_{kl}.$$

- Orthogonality and normalization:

$$\int dU \, D^{J_1}_{kl}(U^\dagger) D^{J_2}_{mn}(U) = \frac{1}{2J_1+1} \delta_{J_1 J_2} \, \delta_{kn} \, \delta_{lm}.$$

Integration here is over the Haar measure:

$$\int dU \ldots = \int d(SU) \ldots = \int d(US) \ldots; \qquad \int dU = 1.$$

- Completeness (the δ-function is understood in the Haar measure sense):

$$\delta(U,V) = \sum_J (2J+1) D^J_{kl}(U^\dagger) D^J_{lk}(V).$$

- Matrix element:

$$\begin{aligned} &\int dU\, D^{J_1}_{a_1 b_1}(U) D^{J_2}_{a_2 b_2}(U) D^{J_3}_{a_3 b_3}(U) \\ = \quad & \begin{pmatrix} J_1 & J_2 & J_3 \\ a_1 & a_2 & a_3 \end{pmatrix} \begin{pmatrix} J_1 & J_2 & J_3 \\ b_1 & b_2 & b_3 \end{pmatrix}, \end{aligned}$$

where $(\ldots)$ denote $3jm$ symbols.

- Decomposition of a direct product of irreps:

$$\begin{aligned} D^{J_1}_{a_1 b_1}(U) D^{J_2}_{a_2 b_2}(U) &= \sum_J (2J+1) \\ \cdot \quad \begin{pmatrix} J & J_1 & J_2 \\ -c & a_1 & a_2 \end{pmatrix} & \begin{pmatrix} J & J_1 & J_2 \\ -d & b_1 & b_2 \end{pmatrix} (-1)^{d-c} D^J_{cd}(U). \end{aligned}$$

A "practical" definition of the $6j$ symbol $\{\ldots\}$ is via a contraction over projections in **three** $3jm$ symbols:

$$\sum_{klm}(-1)^{j_4-k+j_5-l+j_6-m}\begin{pmatrix} j_5 & j_1 & j_6 \\ l & p & -m \end{pmatrix}\begin{pmatrix} j_6 & j_2 & j_4 \\ m & q & -k \end{pmatrix}$$

$$\times\begin{pmatrix} j_4 & j_3 & j_5 \\ k & r & -l \end{pmatrix} = \begin{pmatrix} j_1 & j_2 & j_3 \\ -p & -q & -r \end{pmatrix}\left\{\begin{matrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{matrix}\right\}.$$

The summation over projections k, l, m is such that $p = m - l$, $q = k - m$ and $r = l - k$ are kept fixed.

Another definition of the $6j$ symbol is via the full contraction of projections in **four** $3jm$ symbols:

$$\sum_{klmnop}(-1)^{j_4+n+j_5+o+j_6+p}\begin{pmatrix} j_1 & j_2 & j_3 \\ k & l & m \end{pmatrix}\begin{pmatrix} j_1 & j_5 & j_6 \\ k & o & -p \end{pmatrix}$$

$$\times \begin{pmatrix} j_4 & j_2 & j_6 \\ -n & l & p \end{pmatrix} \begin{pmatrix} j_4 & j_5 & j_3 \\ n & -o & m \end{pmatrix} = \begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{Bmatrix}.$$

Since the three j's of any $3jm$ symbol satisfy the triangle inequalities, e.g. $|j_1 - j_2| \leq j_3 \leq j_1 + j_2$, etc., the following four triades of the $6j$ symbols have to satisfy the triangle inequalities: $(j_1 j_2 j_3)$, $(j_1 j_5 j_6)$, $(j_2 j_4 j_6)$ and $(j_3 j_4 j_5)$; otherwise, the $6j$ symbol is zero.

The $6j$ symbols are symmetric under permutation of any of two columns and under interchange of the upper and lower arguments simultaneously in any two columns.

A full contraction of **six** $3jm$ symbols yields the $9j$ symbol:

$$\sum \begin{pmatrix} j_1 & j_2 & j_3 \\ k & l & m \end{pmatrix} \begin{pmatrix} j_4 & j_5 & j_6 \\ n & o & p \end{pmatrix} \begin{pmatrix} j_7 & j_8 & j_9 \\ q & r & s \end{pmatrix} \begin{pmatrix} j_1 & j_4 & j_7 \\ k & n & q \end{pmatrix}$$

$$\cdot \quad \begin{pmatrix} j_2 & j_5 & j_8 \\ l & o & r \end{pmatrix} \begin{pmatrix} j_3 & j_6 & j_9 \\ m & p & s \end{pmatrix} = \begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \\ j_7 & j_8 & j_9 \end{Bmatrix}.$$

A convenient reference book on D-functions, $3jm$, $6j$ and $9j$ symbols is D. Varshalovich,

A. Moskalev and V. Khersonskii, *Quantum Theory of Angular Momenta*, World Scientific (1988)

The trace of the D^J-function is called the character of the representation J,

$$
\begin{aligned}
\chi^J(U) &\stackrel{d}{=} \sum_{m=-J}^{J} D^J_{mm}(U) = \frac{\sin(J+\frac{1}{2})\omega}{\sin\frac{1}{2}\omega}, \\
\mathsf{Tr}U &= 2\cos\tfrac{1}{2}\omega = \cos\frac{\beta}{2}\cos\frac{\alpha+\gamma}{2}.
\end{aligned}
$$

For example,

$$
\begin{aligned}
&\int dU \exp\left(\beta\frac{\mathsf{Tr}\ U + \mathsf{Tr}\ U^\dagger}{4}\right) D^J_{mn}(U) \\
= \quad &\delta_{mn}\ \frac{1}{\pi}\int_0^{2\pi} d\omega\ \sin^2\frac{\omega}{2}\ \frac{\sin(J+\frac{1}{2})\omega}{\sin\frac{1}{2}\omega}\ e^{\beta\cos\frac{1}{2}\omega}
\end{aligned}
$$

$$= \quad \delta_{mn} \, \frac{2}{\beta} I_{2J+1}(\beta)$$

where $I_{2J+1}(\beta)$ is the modified Bessel function with the index $2J + 1$. Its asymptotics at large β is:

$$\begin{aligned}
\frac{2}{\beta} I_{2J+1}(\beta) \quad &= \quad \frac{2}{\beta} I_1(\beta) \; T_J(\beta), \\
T_J(\beta) \quad &\xrightarrow{\beta \to \infty} \quad \exp\left[-\frac{2J(J+1)}{\beta} \right]
\end{aligned}$$

NB: continuum limit: $J \sim \beta \gg 1$!