Yang–Mills theory in 2,3,4 dimensions

It has been noticed that God has not been tremendously inventive: having once discovered successful dynamics He tends to use it again and again under various circumstances. One of such dynamical systems is, undoubtedly, quantum YM theory. Apart from being the basis of the Standard Model of fundamental interactions, non-Abelian gauge symmetry is found in condensed matter systems, e.g. in superfluid $^3\mathrm{He}$ [see G. Volovik et al., cond-mat/9809125]. Quantum Gravity can be also thought of as a non-Abelian gauge theory.

According to the study by Jackson and Okun into the history [Rev. Mod. Phys. **73** (2001) 663, `hep-ph/0012061`], gauge symmetry in classical electrodynamics was discovered by a Danish physicist Ludvig Lorenz (1840's), and in quantum mechanics by Vladimir Fock [Zeit. f. Physik (1926)]. The term 'gauge invariance' has been introduced by Hermann Weyl (1929). The non-Abelian gauge symmetry was invented by Yang and Mills (1954), although there are rumors that Pauli new about it long before but didn't publish his work as he thought there were no massless gauge fields in our world...

$SU(N)$ gauge group

YM theory is based on semi-simple continuous Lie groups. We'll restrict ourselves to a particular case of $SU(N)$ groups, which is sufficient for most practical purposes. $SU(N)$ groups are formed by unitary ($UU^\dagger = U^\dagger U = \mathbf{1}$) unimodular ($\det\, U = 1$) $N \times N$ matrices having $N^2 - 1$ degrees of freedom, i.e. described by $N^2 - 1$ functions of d variables where d is the dimension of space-time. A possible parametrization of $SU(N)$ group elements is

$$U(x) = e^{i\omega^a(x)\, t^a}$$

where t^a are $N^2 - 1$ traceless hermitian $N \times N$ matrices. We shall assume the normalization

$$\mathrm{Tr}\, t^a t^b = \tfrac{1}{2}\delta^{ab}, \qquad t^a t^a = \frac{N^2-1}{2N}\,\mathbf{1}.$$

For example, in $SU(2)$ one can choose the basis of Pauli matrices,

$$t^1 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \; t^2 = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \; t^3 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

In $SU(3)$ one can choose 8 Gell-Mann matrices,

$$\begin{aligned}
t^1 &= \frac{1}{2}\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad t^2 = \frac{1}{2}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\
t^3 &= \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad t^4 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \\
t^5 &= \frac{1}{2}\begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad t^6 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},
\end{aligned}$$

$$t^7 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad t^8 = \frac{1}{2\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

The group generators t^a close under commutation relations,

$$\left[t^a\, t^b\right] = i\, f^{abc}\, t^c$$

where the totally antisymmetric f^{abc} are called the structure constants of the group. For the $SU(N)$ group

$$f^{abc}\, f^{abd} = N\, \delta^{cd}.$$

In $SU(2)$ $f^{abc} = \epsilon^{abc}$.

The fundamental representation of $SU(N)$ is formed by N-spinors; they transform by the $N \times N$ matrices:

$$\psi^\alpha \to U^\alpha_\beta\, \psi^\beta, \qquad \alpha, \beta = 1...N.$$

Complex-conjugate N-spinors (which should be presented by a row) transform by the hermitian-conjugated matrix,

$$\psi^{\dagger}_{\alpha} \longrightarrow \psi^{\dagger}_{\beta}\, U^{\dagger\,\beta}_{\alpha}.$$

It is clear that the combination $\psi^{\dagger}_{\alpha}\psi^{\alpha}$ is an $SU(N)$ invariant,

$$\psi^{\dagger}_{\alpha}\psi^{\alpha} \longrightarrow \psi^{\dagger}_{\beta}\, U^{\dagger\,\beta}_{\alpha}\, U^{\alpha}_{\gamma}\, \psi^{\gamma} = \psi^{\dagger}\psi.$$

There are also quantities transforming according to the $N^2 - 1$-dimensional adjoint representation,

$$\begin{aligned} A^a &\longrightarrow O^{ab}\, A^b, \qquad O^{ab} = \tfrac{1}{2}\mathrm{Tr}(U^{\dagger} t^a U t^b), \\ O^{ab} O^{ac} &= \delta^{bc}. \end{aligned}$$

To verify the last eqn. you'll need the Fiertz identity,

$$(t^a)^{\alpha}_{\beta}\; (t^a)^{\gamma}_{\delta} = -\frac{1}{2N}\delta^{\alpha}_{\beta}\,\delta^{\gamma}_{\delta} + \frac{1}{2}\delta^{\alpha}_{\delta}\,\delta^{\gamma}_{\beta}$$

(summation over a is understood here). Please check that if the quantity A^a transforms according to the adjoint representation the following two combinations are invariants of the $SU(N)$ rotations:

$$A^a A^a = \mathrm{inv}, \qquad A^a\,\psi^\dagger\,t^a\,\psi = \mathrm{inv}.$$

The essence of gauge invariance is the requirement that one should be able to perform $SU(N)$ rotations locally, i.e. the parameters of the $SU(N)$ matrices can depend on space-time. This causes a difficulty since any kinetic-energy term for fields has derivatives which differentiate the parameters of the $SU(N)$ rotation. To overcome the difficulty, one introduces the Yang–Mills covariant derivative,

$$\nabla_\mu \stackrel{d}{=} \partial_\mu\,\mathbf{1} - iA_\mu^a t^a,$$

where A_μ^a is called the YM field or YM potential. It is similar to the electro-magnetic potential but has $(N^2 - 1)\cdot d$ components. Under gauge transformation it transforms according to

$$A_\mu = A_\mu^a\, t^a \to U\, A_\mu\, U^\dagger + iU\, \partial_\mu U^\dagger$$

leading to the transformation of the covariant derivative

$$\nabla_\mu \to U\, \nabla_\mu\, U^\dagger.$$

Consider the commutator of two ∇'s:

$$[\nabla_\mu \nabla_\nu] = -iF_{\mu\nu},$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu A_\nu] = F_{\mu\nu}^a t^a,$$

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + f^{abc}\, A_\mu^b\, A_\nu^c.$$

$F_{\mu\nu}^a = -F_{\nu\mu}^a$ is called the YM field strength or YM curvature; it has $(N^2-1)\cdot d(d-1)/2$

components, but not all of them are independent as they are expressed through $(N^2-1)\cdot d$ potentials A^a_μ. The fact that not all components of $F_{\mu\nu}$ are independent is expressed by the Bianchi identity following from the definition,

$$[\nabla_\lambda F_{\mu\nu}] + [\nabla_\mu F_{\nu\lambda}] + [\nabla_\nu F_{\lambda\mu}] = 0.$$

We can now construct a perfect invariant under local $SU(N)$ rotations,

$$\mathsf{Tr}\, F_{\mu\nu}F_{\mu\nu} = \tfrac{1}{2}F^a_{\mu\nu}F^a_{\mu\nu}.$$

This is the generalization of the EM lagrangian, $E^2 - B^2$, to the non-Abelian gauge group. It contains terms quadratic, cubic and quartic in the YM field. Thus, YM theory is self-interacting even without fermions, as it follows from the requirement of gauge invariance.

The Euclidean YM partition function is

$$\mathcal{Z} = \int DA_\mu \, \exp\left(-\frac{1}{2g^2}\int d^dx\, \mathsf{Tr}\, F_{\mu\nu}F_{\mu\nu}\right)$$

where g^2 is the gauge coupling constant; $[g^2] = [m]^{4-d}$. In $4d$ it is dimensionless.

The very simply-written YM action in fact encodes enormously rich dynamics. First, it is believed that the theory leads to the confinement of color – a unique phenomenon having no analogs in the history of physics. Second, it is believed that the originally massless YM fields A_μ disappear from the physical spectrum. Instead, gauge-invariant or 'colorless' bound states must appear, which have a nonzero mass. Third, the whole rich realm of strong (or nuclear) interactions, from π's to ^{238}U is, in principle, deducible from Quantum Chromodynamics (=QCD) being nothing but YM theory based on the $SU(3)$ gauge group.

($\$1 \cdot 10^6$ prize..)

YM theory on the lattice [K. Wilson (1974), A. Polyakov (1975)]

Lattice-regularized partition function

$$\mathcal{Z}(\beta) = \int \prod_{\text{links}} dU_{\text{link}} \ \exp\left(\sum_{\text{plaq}} \beta \ \frac{\mathsf{Tr}\ U_{\text{plaq}} + \text{c.c.}}{2\ \mathsf{Tr}\ 1}\right)$$

$$\to \int DA_\mu \ \exp\left(-\frac{1}{2g_d^2}\int d^d x \ \mathsf{Tr}\ F_{\mu\nu}^2\right),$$

$$\beta = \frac{2N}{a^{4-d} g_d^2}$$

the continuum limit is obtained at $a \to 0, \quad \beta \to \infty$ and

$$\begin{cases} g_2^2 = \frac{2N}{a^2\beta} = \text{fixed}, & d = 2, \\ g_3^2 = \frac{2N}{a\beta} = \text{fixed}, & d = 3, \\ \Lambda = \frac{1}{a}\exp\left(-\frac{12\beta\pi^2}{11N^2}\right) = \text{fixed}, & d = 4. \end{cases}$$

Λ has the dimension on mass and gives the scale in the continuum theory. It is known from experiment. YM theory in $4d$ is asymptotically free, with the 'running coupling constant' given by

$$\frac{8\pi^2}{g^2(\mu)} = \frac{11\,N}{3}\,\ln\frac{\mu}{\Lambda}.$$

Wilson's criteria of confinement: the area behaviour of the Wilson loop

$$\begin{aligned} W_{js} &= \prod_{\text{links}} \mathrm{Tr}_{js}(UUU.....U) \\ &\to \mathrm{Tr}\,\mathrm{P}\,\exp i\oint dx_\mu\,A^a_\mu\,t^a_{js} \end{aligned}$$

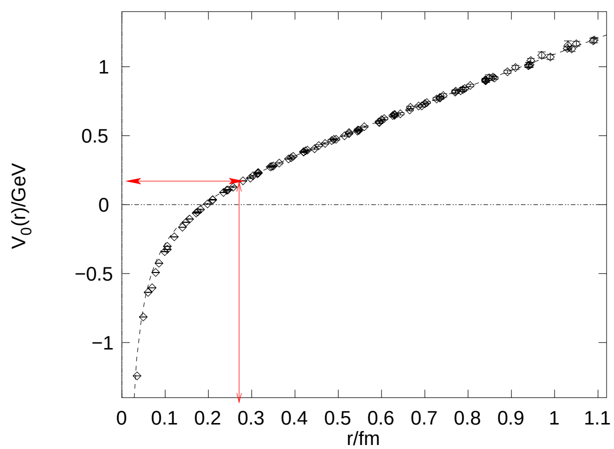
$$\langle W\rangle = \exp\left[-V(R)\,T\right] \qquad \text{at } T\to\infty.$$

$V(R)$ is the potential between quark and antiquark at separation R.
The area law, $W \sim \exp(-\sigma\,\mathrm{Area})$, means

$$V(R) = \sigma\, R,$$

i.e. the linear rising potential. The 'string tension' must be

$$\begin{aligned} \sigma &\simeq g_2^2 \quad &\text{in} \quad d = 2, \\ \sigma &\simeq g_3^4 \quad &\text{in} \quad d = 3, \\ \sigma &\simeq \Lambda^2 \quad &\text{in} \quad d = 4. \end{aligned}$$



The potential energy of two infinitely-heavy quarks, as function of their separation, simulated on an $SU(2)$ lattice. The units come from setting $\sqrt{\sigma} = 420\,\mathrm{MeV}$ [G. Bali et al. (1995)].

Dual transformation on the lattice, $SU(2)$

Insert a unity for every plaquette:

$$1 = \prod_{\text{plaquettes}} \int dU_{\text{plaq}} \delta(U_{\text{plaq}}\,,\; U_1 U_2 U_3^\dagger U_4^\dagger)$$

$$\delta(U, V) = \sum_{J=0,\frac{1}{2},1,\frac{3}{2},\ldots} (2J+1)\; D^J_{m_1 m_2}(U^\dagger) D^J_{m_2 m_1}(V),$$

$$V = U_1 U_2 U_3^\dagger U_4^\dagger.$$

$D^J_{mn}(\alpha, \beta, \gamma)$ are Wigner finite-rotation matrices and depend on Euler angles α, β, γ. They are $(2J+1)^2$-fold degenerate eigenfunctions of the angular momentum operator,

$$\mathbf{J}^2\, D^J_{mn} = J(J+1)\, D^J_{mn}$$

$$J = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$$
$$-J \leq m \quad , \quad n \leq J.$$

They are ortho-normalized,

$$\int dU\, D^{J_1}_{kl}(U^\dagger)\, D^{J_2}_{mn}(U) = \frac{1}{2J+1} \delta_{J_1 J_2}\, \delta_{kn}\, \delta_{lm}.$$

Integration over plaquette variables factorizes into:

$$\int dU \exp\left(\beta \frac{\mathrm{Tr}\ U + \mathrm{Tr}\ U^\dagger}{2\ \mathrm{Tr}\ 1}\right) D^J_{mn}(U^\dagger)$$
$$= \delta_{mn}\, \frac{2}{\beta} I_{2J+1}(\beta)$$
$$\frac{2}{\beta} I_{2J+1}(\beta) = \frac{2}{\beta} I_1(\beta)\ T_J(\beta),$$

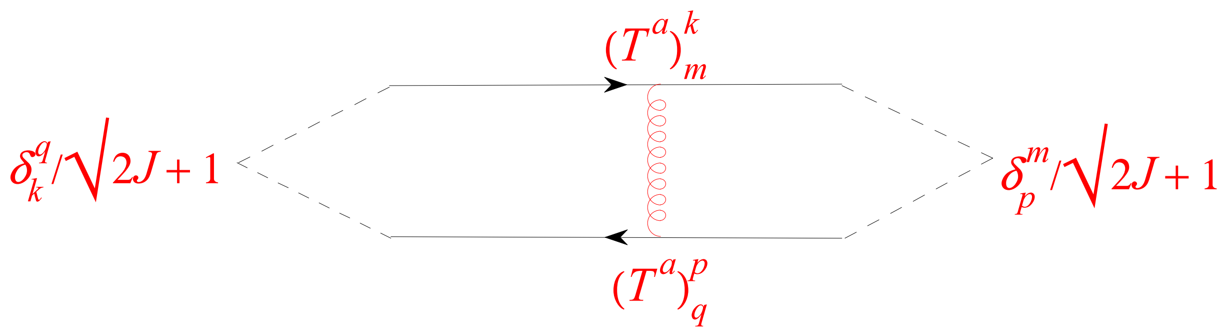
$$T_J(\beta) \to \exp\left[-\frac{2J(J+1)}{\beta}\right]$$

NB: continuum limit: $J \sim \sqrt{\beta} \gg 1$!

In $2d$ because of the orthogonality of D-functions all plaquettes have the same J outside the loop and the same J' inside the loop, with $|J - j_s| \le J' \le J + j_s$ where j_s is the 'color spin' of the source along the loop. The average Wilson loop is, therefore, exactly computable in $2d$:

$$\begin{aligned} \langle W_{j_s}(S)\rangle &= \frac{\sum_J \left[T_J(\beta)\right]^{\frac{V}{a^2}} \sum_{J'=|J-j_s|}^{J+j_s} \left[T_{J'}(\beta)\,/\,T_J(\beta)\right]^{\frac{S}{a^2}}}{\sum_J \left[T_J(\beta)\right]^{\frac{V}{a^2}}} \\ &\to \left[T_{j_s}(\beta)\right]^{\frac{S}{a^2}} \to \exp\left[-\frac{g_2^2}{2} j_s(j_s+1)\,S\right], \end{aligned}$$

– the needed area behavior (with the 'Casimir' string tension) $\Rightarrow$ confinement in $d = 1{+}1$!



$$\mathrm{Tr}(T^a\,T^a) = J(J+1)(2J+1)$$

$$-\frac{\partial^2}{\partial\,x^2}\phi = g^2\delta(x) \Rightarrow \phi = \frac{g^2}{2}\,|x|$$

$$V = \frac{g^2}{2}\,J(J+1)\,|x-y|$$

is the Coulomb energy of a quark and an antiquark with ‘isospin’ J in $2d$.

3jm symbols (≈ Clebsch–Gordan coeff's):

$$\int dU \; D^{J_1}_{a_1 b_1}(U) \; D^{J_2}_{a_2 b_2}(U) \; D^{J_3}_{a_3 b_3}(U)$$

$$= \begin{pmatrix} J_1 & J_2 & J_3 \\ a_1 & a_2 & a_3 \end{pmatrix} \begin{pmatrix} J_1 & J_2 & J_3 \\ b_1 & b_2 & b_3 \end{pmatrix}$$

6j symbols = a contraction of four 3jm's:

$$\sum_{klmnop} (-1)^{j_4+n+j_5+o+j_6+p} \begin{pmatrix} j_1 & j_2 & j_3 \\ k & l & m \end{pmatrix} \begin{pmatrix} j_1 & j_5 & j_6 \\ k & o & -p \end{pmatrix}$$
$$\times \begin{pmatrix} j_4 & j_2 & j_6 \\ -n & l & p \end{pmatrix} \begin{pmatrix} j_4 & j_5 & j_3 \\ n & -o & m \end{pmatrix} = \begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{Bmatrix}$$

9j symbols = a contraction of six 3jm's:

$$\sum \begin{pmatrix} j_1 & j_2 & j_3 \\ k & l & m \end{pmatrix} \begin{pmatrix} j_4 & j_5 & j_6 \\ n & o & p \end{pmatrix} \begin{pmatrix} j_7 & j_8 & j_9 \\ q & r & s \end{pmatrix} \begin{pmatrix} j_1 & j_4 & j_7 \\ k & n & q \end{pmatrix}$$

$$\times \begin{pmatrix} j_2 & j_5 & j_8 \\ l & o & r \end{pmatrix} \begin{pmatrix} j_3 & j_6 & j_9 \\ m & p & s \end{pmatrix} = \begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \\ j_7 & j_8 & j_9 \end{Bmatrix}$$