Quantum anomalies.

Quantum anomalies are among the most interesting and far reaching phenomena in QFT. The phenomenon is the following: Let the action (or Hamiltonian) possess, at the classical level, invariance under certain continuous transformations. It means that there is a corresponding conserved Nöther current, $\partial_\mu J_\mu^{\rm Noether} = 0$.

Now we switch in quantum fluctuations of the field. Fluctuations may have infinitely high Fourier components (= momenta), and one has to regularize the theory somehow. One tries to implement the regularization so as not to spoil the symmetries of the theory. For example, it is preferable not to violate the translational and rotational invariance (but it is violated by lattice regularization.) In gauge theories, invariance under gauge transformations is sacred (but it is violated by cutting off momenta $|k| > k_{\rm max}$). These symmetries are preserved by dimensional regularization (but it is not applicable to chiral fermions). One of the best regularizations is by proper time, see L10 (but it is not universally applicable either).

Q.: What symmetry is broken by Pauli–Villars large-mass regularization?

Whatever the regularization, the hope is that when one takes the cutoff to ∞ symmetry is restored, and quantum corrections do not violate the conservation of the classical Nöther currents. However, there are cases when it is impossible, as a matter of principle, – not because of our limited wisdom. If a symmetry is violated at the quantum level, it is called a quantum anomaly.

Most anomalies occur in theories with fermions, although there are examples of purely bosonic anomalies. We shall work with relativistic fermions but in Euclidean space-time:

$$S_{\rm Dirac} = \int d^d x\, \psi^\dagger (\gamma_\mu \partial_\mu + im)\psi,$$

where the Dirac matrices γ_μ obey the algebra

$$\gamma_\mu^\dagger = \gamma_m u, \quad \gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = \delta_{\mu\nu}.$$

In $d = 1 + 1$ and $d = 2 + 1$ γ_μ are 2×2 matrices which can be chosen to be Pauli matrices (they satisfy the algebra)

$$\begin{aligned} d = 2: \quad \gamma_1 &= \sigma_1, \quad \gamma_2 = \sigma_2, \\ d = 3: \quad \gamma_1 &= \sigma_1, \quad \gamma_2 = \sigma_2, \quad \gamma_3 = \sigma_3. \end{aligned}$$

In $d = 3 + 1$ γ_μ are 4×4 matrices which can be chosen as

$$d = 4: \quad \gamma_i = \begin{pmatrix} 0 & -i\sigma_i \\ i\sigma_i & 0 \end{pmatrix}, \quad \gamma_4 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

In even number of dimensions a special role is played by a matrix which we shall generically denote by γ_5.

$$\left.\begin{array}{ll} d = 2: & \text{``}\gamma_5\text{''} = -i\gamma_1\gamma_2 \\ d = 4: & \gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4 \end{array}\right\} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$\gamma_\mu\gamma_5 + \gamma_5\gamma_\mu = 0, \qquad \gamma_5^2 = 1.$$

For those who are more familiar with Minkowski notations, the glossary for passing to the Euclidean space is

$$\begin{aligned} t_E &= it_M, \quad A_{E4} = -iA_{M0}, \quad \psi_E^\dagger = i\bar{\psi}_M, \\ \gamma_{E4} &= \gamma_{M0}, \quad \gamma_{Ei} = -i\gamma_{Mi}, \quad \gamma_{E5} = \gamma_{M5}. \end{aligned}$$

We shall consider the gauge theory where fermions interact with the gauge potential A_μ via the covariant derivative, (see L5):

$$\partial_\mu \to \nabla_\mu \stackrel{d}{=} \partial_\mu \mathbf{1} - iA_\mu^a t^a,$$

where A_μ^a are N^2-1 gauge fields of the $SU(N)$ gauge group, and t^a are the generators of the group. In the case of QED there is just one gauge field A_μ, the photon. The fermion action is

$$S_{\text{ferm}} = \int d^d x\, \psi^\dagger \left(\gamma_\mu \nabla_\mu + im\right) \psi.$$

The classical eqns of motion obtained from varying the action in respect to ψ, $\psi^\dagger$ independently, are the Dirac eqns in the external gauge field,

$$\begin{aligned} (\gamma_\mu \nabla_\mu + im)\psi &\equiv (\not\nabla + im)\psi = 0, \\ \psi^\dagger(\overleftarrow{\not\nabla} - im) &\equiv \psi^\dagger(\gamma_\mu(\overleftarrow{\partial} + iA_\mu^a t^a) - im) = 0. \end{aligned}$$

In the massless fermion limit ($m=0$) the fermion action in even number of dimensions is invariant under axial rotation,

$$\begin{aligned} \psi &\to e^{i\alpha\gamma_5}\,\psi, \quad \psi^\dagger \to \psi^\dagger\, e^{i\alpha\gamma_5}, \\ e^{i\alpha\gamma_5} &= \cos\alpha\,\mathbf{1} + i\gamma_5\,\sin\alpha. \end{aligned}$$

The corresponding Nöther current, called the axial current,

$$J_{\mu 5} \stackrel{d}{=} \psi^\dagger\,\gamma_\mu\gamma_5\,\psi$$

is conserved at the classical level, since

$$\partial_\mu J_{\mu 5} = \psi^\dagger \overleftarrow{\not\nabla} \gamma_5 \psi - \psi^\dagger \gamma_5 \not\nabla \psi = 0$$

owing to the Dirac eqns at zero m. However, Dirac eqn is a classical equation of motion, whereas the fluctuating quantum fields ψ, $\psi^\dagger$ need not be always at the saddle point.

IN CLASSICAL THEORY ONE DIFFERENTIATES
IN QUANTUM THEORY ONE INTEGRATES

Let is consider the quantum average of $\partial_\mu J_{\mu 5}$:

$$< \partial_\mu J_{\mu 5} >= \frac{\int D\psi D\psi^\dagger \left[\psi^\dagger \overleftarrow{\not\nabla} \gamma_5 \psi - \psi^\dagger \gamma_5 \not\nabla \psi\right] e^{S_{\rm ferm}}}{\int D\psi D\psi^\dagger e^{S_{\rm ferm}}}.$$

We notice that

$$\begin{aligned} \not\nabla \psi(x)\, e^{S_{\rm ferm}} &= \frac{\delta}{\delta\psi^\dagger(x)} e^{S_{\rm ferm}}, \\ \psi^\dagger(x) \overleftarrow{\not\nabla}\, e^{S_{\rm ferm}} &= -\frac{\delta}{\delta\psi(x)} e^{S_{\rm ferm}}, \end{aligned}$$

["-" in the second line is because ψ, $\psi^\dagger$ anticommute.] We formally integrate by parts using $\delta\psi(y)/\delta\psi(x) =$

$\delta^{(4)}(x-y)$ (we are now in four dimensions) and get

$$< \partial_\mu J_{\mu 5} >= 2\delta^{(4)}(0)\,\mathrm{Tr}\,\gamma_5 = 2\,\mathrm{Sp}\,\gamma_5.$$

This is an indefinite expression as $\mathrm{Tr}\,\gamma_5 = 0$ but $\delta(0) = \infty$. Let us regularize it to see if it is zero, infinite or maybe finite.

$$\begin{aligned}
2\,\mathrm{Sp}\,\gamma_5 &= \lim_{M\to\infty} 2\int \frac{d^4p}{(2\pi)^4}\,\mathrm{Tr}\,\gamma_5\, e^{-ipx} \exp\left(\frac{\nabla\!\!\!/^2}{M^2}\right) e^{ipx}, \\
\nabla\!\!\!/^2 &= \gamma_\mu\gamma_\nu\,\nabla_\mu\nabla_\nu = \left(\tfrac{1}{2}\{\gamma_\mu\gamma_\nu\} + \tfrac{1}{2}[\gamma_\mu\gamma_\nu]\right)\nabla_\mu\nabla_\nu \\
&= \mathbf{1}\cdot\nabla_\mu^2 - \frac{i}{2}\sigma_{\mu\nu}\,F_{\mu\nu}, \qquad [\nabla_\mu\,\nabla_\nu] = -iF_{\mu\nu}.
\end{aligned}$$

Tr here is the usual matrix trace but both in Dirac and YM indices. We drag $e^{i(p\cdot x)}$ through the operator; as a result it cancels but the differentiation operator is shifted, $\nabla_\mu \to \nabla_\mu + ip_\mu$. We integrate over p:

$$\int \frac{d^4p}{(2\pi)^4} e^{-p^2/M^2} = \frac{M^4}{16\pi^2}.$$

Now we have to expand the exponent to the order where $\mathrm{Tr}\,\gamma_5 ... \neq 0$. It happens in the second order, since

$$\mathrm{Tr}\,\gamma_5\,\sigma_{\alpha\beta}\,\sigma_{\mu\nu} = -4\epsilon_{\alpha\beta\mu\nu} \neq 0\,!$$

Here $\epsilon_{\alpha\beta\mu\nu}$ is the $4d$ antisymmetric tensor, $\epsilon_{1234} = 1$. Expanding the exponent to the second order we get $-F^{\alpha\beta}F^{a}_{\mu\nu}/(8M^4)$ so that $1/M^4$ cancels with M^4 from the p integration!

Therefore, the result is finite:

$$< \partial_\mu J_{\mu 5} >= \frac{1}{16\pi^2} F^a_{\mu\nu} \tilde{F}^a_{\mu\nu} = \frac{1}{4\pi^2} (\mathbf{E}^a \cdot \mathbf{B}^a)$$

where $\tilde{F}^a_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}F^a_{\alpha\beta}$ is called the dual field strength. [Electric field $\mathbf{E}$ is dual to the magnetic field $\mathbf{B}$ and $vice\ versa$.]

This is the famous axial anomaly: taking into account quantum fluctuations of the fermion field, the axial current is not conserved. The original symmetry of the classical theory in respect to axial rotations is explicitly broken by quantum corrections. It is *not* spontaneous breaking of symmetry, but explicit: the theory just does not have the symmetry at all.

Notice that, were fermions massive, the axial current would not be conserved even classically,

$$\partial_\mu J_{\mu 5}|_{\rm normal} = 2i\,\psi^\dagger\, m\, \psi$$

The anomalous piece is something one gets in addition, and it is present even in the massless limit.

I have presented probably the most economic derivation of the anomaly but there are other ways to derive it, for example, from Feynman graphs:

... but it is hard work.

J. Steinberger (1949), J. Schwinger (1951), S. Adler (1969), J. Bell and R. Jackiw (1969) ... For a pedagogical review see M. Shifman, *Phys. Reports* 209 (1991) 341. An interesting review stressing mathematical aspects of anomalies: A. Morozov, *Sov. Phys. Uspekhi* 150 (1986) 337.

Anomaly in $2d$ QED

To get a better insight in this remarkable phenomenon let us study a simpler case: a $2d$ massless electrodynamics, named the Schwinger model:

$$S = \int d^2x \left(\frac{1}{4e^2} F_{\mu\nu}^2 + \psi^\dagger \not\nabla \psi \right), \quad \nabla = \partial - iA.$$

Again, there is a symmetry under axial rotations and hence a (would-be) conserved axial current, $J_{\mu 5} =$

$\bar\psi^\dagger\gamma_\mu\gamma_5\psi$. Classically, its divergence is zero owing to the Dirac eqns of motion. However, in QFT one integrates over the fields rather than varying in respect to the fields.

We proceed exactly as in the case of $4d$ QCD. The role of γ_5 is played in $2d$ by the Pauli matrix σ_3 but I, nevertheless, denote it by γ_5:

$$
\begin{aligned}
< \partial_\mu J_{\mu 5} > &= \lim_{M\to\infty} 2\int \frac{d^2p}{(2\pi)^2} \\
&\cdot \quad \mathrm{Tr}\,\gamma_5\, e^{-ipx} \exp\left(\frac{\not\nabla^2}{M^2}\right) e^{ipx}, \\
\not\nabla^2 &= \nabla_\mu^2 - \frac{i}{2}\sigma_{\mu\nu}\, F_{\mu\nu}.
\end{aligned}
$$

We drag $e^{i(p\cdot x)}$ through the operator; as a result it cancels but the differentiation operator is shifted, $\nabla_\mu \to \nabla_\mu + ip_\mu$. We integrate over p:

$$
\int \frac{d^2p}{(2\pi)^2} e^{-p^2/M^2} = \frac{M^2}{4\pi}.
$$

Now we have to expand the exponent to the order where $\mathrm{Tr}\,\gamma_5 ... \neq 0$. It happens already in the first order, since

$$
\mathrm{Tr}\,\gamma_5\,\sigma_{\mu\nu} = 2i\,\epsilon_{\mu\nu}
$$

M^2 cancels, and we obtain a finite result:

$$< \partial_\mu J_{\mu 5} >= \frac{1}{2\pi}\, \epsilon_{\mu\nu}\, F^a_{\mu\nu}.$$

This is the axial anomaly for the $2d$ QED.

Explanation of the anomaly

We write the 2-component spinor field as

$$\begin{aligned}
\psi &= \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}, \\
\psi_L &= \frac{1+\gamma_5}{2}\,\psi, \quad \psi_R = \frac{1-\gamma_5}{2}\,\psi, \\
\gamma_5 &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},
\end{aligned}$$

and call the components $\psi_{L(R)}$ left- (right-) polarized (or 'handed') particles. The vector and axial currents are the sum and the difference of the currents of left- and right-handed particles,

$$\begin{aligned}
J_\mu &= \psi_L^\dagger \gamma_\mu \psi_L + \psi_R^\dagger \gamma_\mu \psi_R, \\
J_{\mu 5} &= \psi_L^\dagger \gamma_\mu \psi_L - \psi_R^\dagger \gamma_\mu \psi_R.
\end{aligned}$$

The conservation of the vector current means that the following combination does not change in time:

$$Q = \int dx\, J_{05} = (N_L + N_R) - (N_{\bar{L}} + N_{\bar{R}}) = N - \bar{N} = \text{const.}$$

Here $N_{L(R)}$ is the number of left- (right-) handed particles, and $N_{\bar{L}(\bar{R})}$ is the number of left- (right-) handed antiparticles or holes, in the language of solid state physics. The conservation of the axial current would mean that $(N_L - N_R) - (N_{\bar{L}} - N_{\bar{R}})$ is time-independent. However, the anomaly means that it is changed by the amount

$$\Delta Q_5 = \Delta(N_L - N_R - N_{\bar{L}} + N_{\bar{R}}) = \frac{1}{2\pi} \int dt \int dx\, \epsilon_{\mu\nu} F_{\mu\nu}$$

Let us choose a physical gauge A_0 or, better in the Euclidean space, $A_2 = 0$. We have $\epsilon_{\mu\nu} F_{\mu\nu} = 2\epsilon_{\mu\nu}\partial_\mu A_\nu = -2\partial_2 A_1 = -2\dot{A}_1$. Therefore, we can write

$$\Delta Q_5 = -2\,\Delta\left(\int dx\, \frac{A_1}{2\pi}\right)$$

Let us consider a simple example: we put a system in a box $[0, L]$ and change A_1 adiabatically in time starting from $A_1 = 0$. Let it remain a constant in the space interval $[0, L]$. We see that when it reaches the value of $A_1 = \frac{2\pi}{L}$ the axial charge of the fermion system changes by two units! How can it happen?

It is easy to solve the Dirac equation in a such a simple case:

$$\left[\gamma_2 \frac{\partial}{\partial t} + \gamma_1 \left(\frac{\partial}{\partial x} - i A_1 \right) \right] \psi = 0,$$

$$\psi(x,t) = e^{-iEt} \phi(x),$$

$$\sigma_3 \left(i \frac{\partial}{\partial x} + A_1 \right) \phi = E \phi$$

This is a two-component eqn: they differ by the sign of E. We impose an anti-periodic boundary condition for the fermion wave functions, $\phi(L) = -\phi(0)$. For the upper component (the left-handed fermions) the wave functions are

$$\phi_L = \exp \left[i \left(k + \tfrac{1}{2}\right) \frac{2\pi}{L} x \right], \qquad E_{Lk} = -\left(k + \tfrac{1}{2}\right) \frac{2\pi}{L} + A_1.$$

For the lower component (the right-handed fermions) the wave function is the same but the energy has the opposite sign (for given) k. The spectrum is equidistant, it stretches from $-\infty$ to $+\infty$, and the levels move adiabatically as we slowly change A_1.
see figure

According to the Dirac theory the vacuum corresponds to filling in all states with negative energies, both for left-handed and right-handed fermions. When

A_1 reaches the value of $\frac{2\pi}{L}$ all energy levels assume their original positions, however their fillings change! A right-handed fermion is now having a positive energy, while a left-handed fermion goes down and leaves one level with negative energy unoccupied. If we compare with the original vacuum at $A_1 = 0$, one right-handed fermion is created and one left-handed one is annihilated! Note that the vector charge is conserved but the axial charge is changed exactly by two units, as predicted by the anomaly!

The crucial detail in this story [V. Gribov (1981)] is that we are counting states only above certain threshold, the cutoff. A left-handed fermion has dipped under the cutoff while a right-handed fermion emerged from under the cutoff. Without an explicit UV regularization we would not be able to see the effect, but QFT's are not defined without a regularization.

Another important detail is that a normal-looking field (here: A_1) affects very deep-lying levels. It is counter-intuitive: usually particles with very high momenta do not 'notice' the external disturbances and move as free. However, in some specific cases particles with any momenta are affected by the presence of the external field: when the Dirac sea moves as a whole as one adiabatically changes the external field – this is the case of the anomaly.

Quantum anomaly has two faces: the "ultraviolet face" and the "infrared" one. On the one hand one needs an UV regularization to treat the problem accurately. On the other hand, one observes the appearance and dissapearance of soft levels. This is because the Dirac sea moves as a whole. The IR side of the anomaly has an important consequences related to fermion zero modes. That topic will be studied in L21.

Interpetation of the axial anomaly in QCD

The interpetation of the axial current in $4d$ theory is the same as in the $2d$ Schwinger model:

$$\int d^3x\, J_{05} = Q_5 = (N_L - N_R) - (N_{\bar{L}} - N_{\bar{R}})$$

is the axial charge of the fermion system. Classically, it is conserved but quantum effects lead to its non-conservation:

$$\Delta Q_5 = \frac{1}{16\pi^2} \int dt \int d^3x\, F^a_{\mu\nu}\, \tilde{F}^a_{\mu\nu}.$$

An important point is that the integrand is a full derivative:

$$\frac{1}{16\pi^2} F^a_{\mu\nu}\, \tilde{F}^a_{\mu\nu} = 2\,\partial_\mu\, K_\mu,$$

$$K_\mu = \frac{1}{16\pi^2}\, \epsilon_{\mu\alpha\beta\gamma} \left(A^a_\alpha\, \partial_\beta\, A^a_\gamma + \frac{1}{3} f^{abc}\, A^a_\alpha\, A^b_\beta\, A^c_\gamma \right).$$

Assuming that the YM field A_μ^a is decaying fast enough at spatial infinity we obtain

$$\Delta Q_5 = 2\int_{-\infty}^{+\infty} dt\,\frac{\partial}{\partial t}\,X, = 2\,\Delta\,X, \qquad X \stackrel{d}{=} \int d^3x\,K_0.$$

The functional $X[A]$ is a famous quantity: it is called the Chern–Simons number. It has the following remarkable property: Let us perform a large gauge transformation with the help of a time-independent unitary matrix $S(\mathbf{x})$:

$$A_i \to S^\dagger\,A_i\,S + i\,S^\dagger\,\partial_i\,S,$$

with $S(\mathbf{x})$ tending to the unity matrix at spatial infinity. Then the Chern–Simons number shifts by

$$X \to X + \frac{1}{24\pi^2}\int d^3\mathbf{x}\,\epsilon_{ijk}\mathrm{Tr}\left(S^\dagger\partial_i S\right)\left(S^\dagger\partial_j S\right)\left(S^\dagger\partial_k S\right).$$

This is exactly the same integral we have encountered in L8: it gives the winding number of the mapping $S^3 \mapsto S^3$ determined by $S(\mathbf{x})$. Therefore, under gauge transformations the Chern–Simons number X either does not change at all (if the mapping $S(\mathbf{x})$ can be continuously deformed to $S(\mathbf{x}) = \mathbf{1}$) or is shifted by an integer.

Imagine, we start from $A_i = 0$ and adiabatically change it with time in such a way that finally we reach a field which is a "pure gauge",

$$A_i = i\, S^\dagger\, \partial_i\, S,$$

with the unitary matrix $S(\mathbf{x})$ having $N_{\rm wind} = 1$. It means that we change X from 0 to 1, so that $\Delta\, X = 1$. The anomaly says that, in such a case, $\Delta\, Q_5 = 2$: one left-handed fermion has been created and one right-handed fermion has been annihilated during this process. (figure)

An interesting quantum anomaly in superfluid $^3{\rm He}-{\rm A}$ and $^3{\rm He}-{\rm B}$ has been found theoretically and confirmed experimentally – see V. Eltsov, M. Krusius and G. Volovik, "Superfluid $^3{\rm He}$: a Laboratory Model System for Quantum Field Theory", e-print cond-mat/9809125.