A Violation Of The Area Law For Fermionic Entanglement Entropy

or: How much can you say in a phone call?

Robert Helling[1] Hajo Leschke[2] and Wolfgang Spitzer[2]

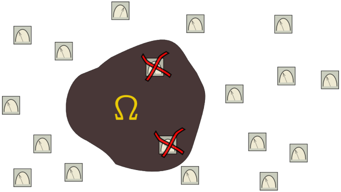
[1]Arnold Sommerfeld Center
Ludwig-Maximilians-Universität München

[2]Institut für Theoretische Physik
Universität Erlangen-Nürnberg

Nordic String Meeting, March 27, 2009

Outline

Entanglement Entropy — the price of ignorance



- Take your favourite QFT model in $\mathbb{R}^n$ and put it in its ground state $|\Psi\rangle$.
- Restrict yourself to measurements outside a compact black-box $\Omega \subset \mathbb{R}^n$
- Outside the black-box the system is described by a density matrix $\varrho_\Omega = \mathrm{tr}_{\mathcal{H}_{\text{black-box}}} |\Psi\rangle\langle\Psi|$
- In general, ϱ_Ω is a mixed state with *entanglement entropy* $S(\Omega) = -\mathrm{tr}(\varrho_\Omega \log \varrho_\Omega) > 0$.

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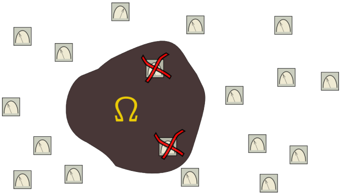

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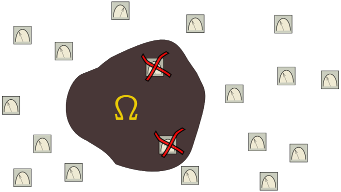

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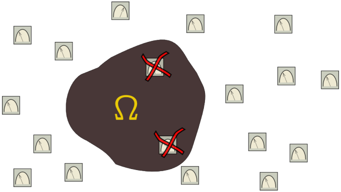

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Entanglement Entropy and the Area Law

- $S(\Omega)$ is a complicated function of Ω and the QFT model.
- Semi-classical asymptotics: Blow up Ω by a factor of R.
- A universal area law $S(R\Omega) \sim R^{n-1}$ was observed for many models: Bosons, Lattice models, CFT's, ... (NB: read R^0 as $\log R$).
- Intuition: Entanglement originates from correlations across the boundary $\partial\Omega$ in a layer of width ξ. (but: CFT).
- Gioev and Klich first observerved that for free fermions $S(R\Omega) \geq O(R^{n-1} \log R)$.
- We improve this including an area type geometric coefficient by proving a special case of Widom's conjecture.

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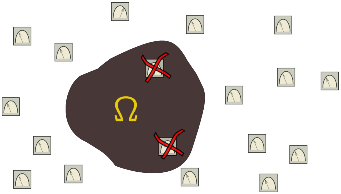

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- Could be mechanism of entropy associated to horizons.
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- Condensed matter: Measure for convergence of density matrix renormalization group method (DMRG)

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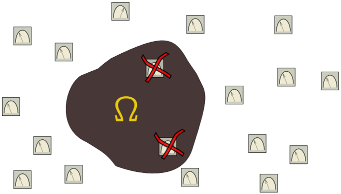

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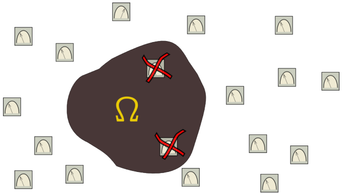

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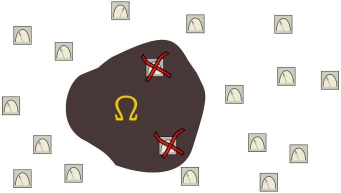

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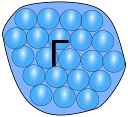
Fermionic Entanglement Entropy



- Quasi-free fermions at finite density are characterized by the Fermi sea $\Gamma \subset \mathbb{R}^n$ of forbidden states with $E < \epsilon_F$ in momentum space.
- At zero temperature, all states with $E < \epsilon_F$ are occupied and the system is in the state $\varrho(\mathbf{x}, \mathbf{y}) = P_\Gamma = \widehat{\chi_\Gamma}(\mathbf{x} - \mathbf{y})$.
- One-particle Hilbert space $\mathcal{H} = L^2(\Omega) \oplus L^2(\mathbb{R}^n \setminus \Omega)$, Fock spaces $\mathcal{F}(\mathcal{H}) = \mathcal{F}(L^2(\Omega)) \otimes \mathcal{F}(L^2(\mathbb{R}^n \setminus \Omega))$
- Reduced many particle state ϱ_Ω is characterised by one-particle state $\rho = Q_\Omega P_\Gamma Q_\Omega$ with projection Q_Ω given by multiplication by characteristic function $\chi_\Omega(\mathbf{x})$.
- Entanglement entropy $S(\Omega, \Gamma) = -\operatorname{tr}(\rho \log \rho + (1 - \rho) \log(1 - \rho))$.

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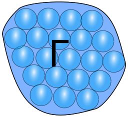

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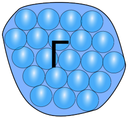

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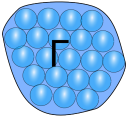

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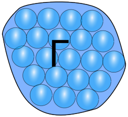

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Not the Strict Area Law

- We find (up to lower order terms in R)

$$S(R\Omega, \Gamma) \geq \left(\frac{R}{2\pi}\right)^{n-1} \frac{\ln(R)}{4\pi^2} \int_{\partial\Omega\times\partial\Gamma} dA(\mathbf{x})dA(\mathbf{p})|\mathbf{n}_\mathbf{x} \cdot \mathbf{n}_\mathbf{p}|.$$

  This is our main result.
- There are strong indications that this inequality is in fact the correct asymptotic scaling
- Violation of area law…
- …but simple surface expression, e.g. no curvature terms.

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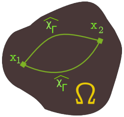

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$$\operatorname{tr}(Q_\Omega P_\Gamma Q_\Omega)^k = \int_\Omega d\mathbf{x}_1 \cdots \int_\Omega d\mathbf{x}_k \widehat{\chi_\Gamma}(\mathbf{x}_1 - \mathbf{x}_2)\widehat{\chi_\Gamma}(\mathbf{x}_2 - \mathbf{x}_3) \cdots \widehat{\chi_\Gamma}(\mathbf{x}_k - \mathbf{x}_1)$$

with $\widehat{\chi_\Gamma}(\mathbf{x}) = \int_\Gamma \frac{d\mathbf{p}}{(2\pi)^{n/2}} e^{i\mathbf{x}\cdot\mathbf{p}}$.

- Asymptotic integrals using steepest descent
- Cancelations and localiazations like

$$\int_1^R dx\, \frac{e^{i\alpha x}}{x} = \delta_{\alpha,0} \ln(R) + O(1)$$

- Crucial observation: for $\mathbf{p} \in \partial\Gamma$ and $\mathbf{v} \in \mathbb{R}^n$, asymptotically

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independent of Ω or Γ.

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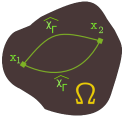

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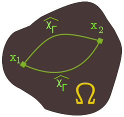

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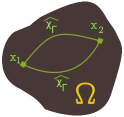

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- In $n = 1$ dimension, Ω and Γ are the unions of m and l intervals respectively.
- Direct computation leads to asymptotically

$$S(R\Omega, \Gamma) = \log(R)ml/\pi^2$$

independent of the size and relative positions of the intervals (as long as they do not touch).
- This measures the amount of information that can be transmitted in a phone call over time interval Ω and frequency band Γ.

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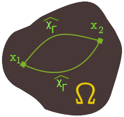

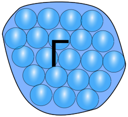

$$S(R\Omega,\Gamma) \to \left(\frac{R}{2\pi}\right)^{n-1} \frac{\ln(R)}{4\pi^2} \int_{\partial\Omega\times\partial\Gamma} dA(\mathbf{x})\,dA(\mathbf{p})\,|\mathbf{n}_{\mathbf{x}} \cdot \mathbf{n}_{\mathbf{p}}|.$$