The Virtual Scaling Function of AdS/CFT

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0901.2747 [hep-th] together with Lisa Freyhult, Uppsala

Outlook

- Gluon Scattering/Wilson Loops
- Integrability
- String Theory

Review: Cusp anomalous dimension Γ_{cusp}

- integral equation
- results at weak and strong coupling

New: Virtual Scaling Function

- What does this mean?
- Katze im Sack
- weak $\longrightarrow$ strong coupling

motivation: AdS/CFT correspondence

- duality at weak/strong coupling: $\mathcal{N} = 4$ super Yang-Mills dual to IIB string theory on $AdS_5 \times S^5$. [Maldacena]
- spectra of both theories should coincide: $E = \Delta$
- hard to test

$\mathcal{N} = 4$ super Yang Mills

- compute anomalous dimension of local gauge invariant operators build from sYM fields e.g.

$$\mathcal{O}(x) = \mathrm{Tr}\,(\mathcal{Y}(x)\mathcal{X}(x)\mathcal{U}(x)\mathcal{Z}(x)\mathcal{F}(x)(\mathcal{D}\mathcal{Z}(x))) + \dots .$$

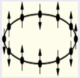
- anomalous dimension in conformal gauge theory defined as eigenvalues of the dilatation generator

$$\mathcal{D}\,\mathcal{O}_\alpha = \Delta_\alpha\,\mathcal{O}_\alpha\,.$$

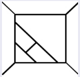
- in planar limit $N \to \infty$ the dilatation generator $\mathcal{D}$ corresponds to a long-ranged integrable super spin chain with $\mathfrak{psu}(2,2|4)$ symmetry
- can be diagonalized with Bethe ansatz [Minahan, Zarembo],[Beisert, Staudacher]

Solve conformal $\mathcal{N} = 4$ super Yang-Mills

- Spectrum of Anomalous Dimension



- Scattering Amplitudes of on-shell states



Use anomalous dimension to "solve" gluon amplitudes

Lance Dixon: *"...gluon scattering amplitudes ... "1/4" solved."*

Virtual Scaling Function

gluon scattering amplitudes "3/8" solved

X-tra

Repeat at strong coupling and compare with string spectrum!

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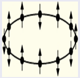

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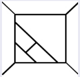

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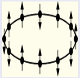

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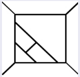

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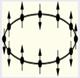

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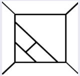

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Gluon amplitudes

- [Bern,Dixon,Smirnov] proposed the following form for n gluon planar, MHV scattering amplitude

$$\log\left(\frac{\mathcal{A}_n}{\mathcal{A}_n^{\rm tree}}\right) = \mathrm{Div}_n + \mathrm{finite}$$

- in dimensional regularization $d = 4 - 2\epsilon$

$$\mathrm{Div}_n = -\sum_{i=1}^{n}\left[\frac{1}{8\epsilon^2} f^{(-2)}\left(\frac{\lambda\mu^{2\epsilon}}{s_{i,i+1}^{\epsilon}}\right) + \frac{1}{4\epsilon} g^{(-1)}\left(\frac{\lambda\mu^{2\epsilon}}{s_{i,i+1}^{\epsilon}}\right)\right]$$

- with the cusp anomalous dimension $f(\lambda)$ and collinear ad $g(\lambda)$

$$\left(\lambda\frac{d}{d\lambda}\right)^2 f^{(-2)}(\lambda) = f(\lambda), \quad \lambda\frac{d}{d\lambda} g^{(-1)}(\lambda) = g(\lambda)$$

Wilson loops

- specified by light-like segments proportional to gluon-momenta with expectation value

$$< W_{\{k_i\}} >= \mathrm{Div}' + \mathrm{finite}$$

- divergent terms from UV divergencies at the cusps of the Wilson lines

Gluon Amp./Wilson loop

- The leading poles $\sim 1/\epsilon^2$ agree. They are related to the anomalous dimension of twist-two operators $f \sim \Delta(\mathrm{Tr}\,(\mathcal{Z}\mathcal{D}^M\mathcal{Z}))$
- the functions $g(\lambda)$ are different! $\rightarrow$ virtual scaling function

(BDS conjecture is not valid for $n \geq 6$)

- leading poles are given by cusp anomalous dimension $f(g)$, which in turn is equal to the anomalous dimension of twist-two operators

Twist operators in $\mathcal{N} = 4$ sYM

- can be represented by doping covariant light-cone derivatives $\mathcal{D}$ into protected 1/2-BPS state $\mathrm{Tr}\,(\mathcal{Z}^L)$
- spin-M, twist-L operator takes the form $\mathrm{Tr}\,(\mathcal{D}^{s_1}\mathcal{Z}\mathcal{D}^{s_2}\mathcal{Z}\cdots\mathcal{D}^{s_L}\mathcal{Z})$ with $s_1 + s_2 + \ldots + s_L = M$,
- anomalous dimension are given by energy eigenvalues of a length-L non-compact $\mathfrak{sl}(2)$ spin chain with M magnons underlying a factorized scattering
- use Bethe ansatz to diagonalize spin chain Hamiltonian and compute anomalous dimension

- wrapping effects at order g^{2L+4} not captured by Bethe ansatz
- How can we compute $f(g)$ for $L = 2$ to all orders?

[Belitksy,Gorsky,Korchemsky]

- anomalous dimension of twist operators grow logarithmic with spin M, $\gamma_L(g, M) = f(g) \log M + \ldots$
- scaling function $f(g)$ is independent of the twist L and universal

- that implies $f_2 = f_3 = \ldots = f_\infty$, which can be determined by asymptotic Bethe ansatz
- $f(g)$ is an all loop result

Twist-two $\mathrm{Tr}\,(\mathcal{Z}\mathcal{D}^M\mathcal{Z})$ and Bethe ansatz [Beisert,Eden,Staudacher]

$$\left(\frac{x_k^+}{x_k^-}\right)^2 = \prod_{\substack{j=1\\ j\neq k}}^{M} \frac{x_k^- - x_j^+}{x_k^+ - x_j^-}\,\frac{1-g^2/x_k^+x_j^-}{1-g^2/x_k^-x_j^+}\,\exp(2i\theta(u_k,u_j))\,, \qquad \prod_{k=1}^{M}\frac{x_k^+}{x_k^-}=1\,.$$

leads to integral equation for $f(g)$ in the large spin limit $M\to\infty$

- the scaling function is given by $f(g) = 16g^2\hat{\sigma}(0)$ where

$$\hat{\sigma}(t) = \frac{t}{e^t-1}\left[K(2gt,0) - 4g^2\int_0^\infty dt'\,K(2gt,2gt')\hat{\sigma}(t')\right],$$

the integral kernel $K(t,t') = K_0(t,t') + K_1(t,t') + K_d(t,t')$ is given by

$$K_0(t,t') = \frac{2}{tt'}\sum_{n=1}^{\infty}(2n-1)J_{2n-1}(t)J_{2n-1}(t')\,, \qquad K_1(t,t') = \frac{2}{tt'}\sum_{n=1}^{\infty}(2n)J_{2n}(t)J_{2n}(t')\,,$$

$$K_d(t,t) = 8g^2\int_0^\infty dt''\,K_1(t,2gt'')\,\frac{t''}{e^{t''}-1}\,K_0(2gt'',t')\,.$$

- Fredholm integral equation of 2nd kind, at weak coupling the equation can be solved iteratively by expanding in g
- first few orders are obtained very fast

$$f(g) = 8\,g^2 - \frac{8\pi^2}{3}\,g^4 + \frac{88\pi^4}{45}\,g^6 - \left(\frac{584\pi^6}{315} + 64\zeta_3^2\right)g^8 + \ldots$$

- it agrees to four-loops with the leading singularities of the gluon amplitudes [Bern,Dixon,Smirnov] [Chachazo,Spradlin,Volovich]
- at five-loops it agrees with Padé approximation estimates [Bern,Czakon,Dixon,Kosower,Smirnov]

weak coupling

f has finite radius of convergence

$r_c = 1/4$

strong coupling and string theory

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much afford was spend to analyze the integral equation at strong coupling
[Lipatov,Kotikov], [Benna,Benvenuti,Klebanov,Scardicchio], [Alday,Arutyunov,Benna,Eden,Klebanov], [Kostov,Serban,Volin], [Basso,Korchemsky,Kotanski], [Kostov,Serban,Volin], ...

strong coupling limit

- strong coupling expansion is given by

$$f(g) = 4g - \frac{3\log 2}{\pi} - \frac{1}{g}\frac{\mathrm{K}}{4\pi^2} - \frac{1}{g^2}\left(\frac{3\mathrm{K}\log 2}{16\pi^3} + \frac{27\zeta_3}{512\pi^3}\right) + \ldots$$

- agrees with known string data [Gubser,Klebanov,Polyakov], [Frolov,Tseytlin], [Roiban,Tirziu,Tseytlin], [Roiban,Tseytlin], first interpolating function

What?

- twist L operators can be represented by $\mathrm{Tr}\,(\mathcal{D}^M \mathcal{Z}^L)$
- the large spin scaling behavior of anomalous dimension is given by

$$\gamma_L(g, M) = f(g)(\log M) + B_L(g) + \dots .$$

- in analogy with the QCD splitting function we call B_L the virtual part
- for finite values of the spin, anomalous dim. can be computed for twist-two and -three operators in terms of harmonic sums analytical by solving the Baxter equation [Kotikov,Rej,Zieme]
- for twist L operators wrapping effects are expected to enter the an. dim. at order g^{2L+4}
- A twist depended quantity can not be computed from the Bethe ansatz correctly to all orders! (?)

Wrapping Contributions

the four-loop anomalous dimension of the Konishi operator, $L = M = 2$, has been computed from

- the asymptotic dilatation generator [Fiamberti,Santambrogio,Sieg,Zanon]
- finite-size corrections to the ABA using Lüscher formulas [Bajnok,Janik]
- 131.015 Feynman diagrams [Velizhanin]

they all agree!

Twist-two general spin M

- using the Lüscher formulas the four-loop anomalous dimension has been computed for general values of the spin M [Bajnok,Janik,Lukowski]
- the result agrees with pole structure predicted by the BFKL equation
- leading transcendental part agrees with *full direct* computation [Velizhanin]

Katze im Sack

- the result is given by

$$\gamma_4^{\mathrm{wrap}} = 128 S_1^2 (2S_{-5} - 2S_5 + 4S_{-2,-3} - 4S_{3,-2} + 4S_{4,1} - 8S_{-2,-2,1} - 5\zeta_5 - 3S_{-2}\zeta_3)$$

- for $M \to \infty$ it behaves like $\gamma_4^{\mathrm{wrap}} \sim \log^2 M / M^2$ [Beccaria,Forini]

Twist-three at five loops [Beccaria,Forini,Lukowski,Zieme]

- γ_5 computed with ABA, max. transcendentality, reciprocity, Lüscher, agrees with Y-System, result
- not related to BFKL, but poles at $M = -2$ can be resumed and agree with NLO conjecture [Kotikov,Lipatov,Rej, Staudacher,Velizhanin]
- again for large spin $\gamma_5^{\mathrm{wrap}} \sim \log^2 M / M^2$

The finite order corrections $\mathcal{O}(M^0)$ can be computed to all orders from the ABA. $B_L(g) = L\, h_1(g) + h_2(g)$

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How do we compute these corrections?

- tricky from the Bethe ansatz
- we start from the non-linear integral equation (NLIE) of $\mathfrak{sl}(2)$ for the counting function that takes into account 'hidden' degrees of freedom [Freyhult,Rej,Staudacher] (roots of the transfer matrix)
- 'off-shell' version of one-loop Bethe equations

From the NLIE one can derive an integral equation including $\mathcal{O}(M^0)$ corrections

$$\hat{\sigma}(t) = \frac{t}{e^t - 1}\Big[K(2gt,0)(\log M + \gamma_{\mathrm{E}} - (L-2)\log 2) - \frac{L}{8g^2 t}(J_0(2gt) - 1)$$
$$+\frac{1}{2}\int_0^\infty dt' \left(\frac{2}{e^{t'}-1} - \frac{L-2}{e^{t'/2}+1}\right)(K(2gt,2gt') - K(2gt,0))$$
$$-4g^2\int_0^\infty dt'\, K(2gt,2gt')\hat{\sigma}(t')\Big],$$

- anomalous dimension given by $\gamma_L(g,M) = 16g^2\,\hat{\sigma}(0)$
- $\gamma_L(g,M) = f(g)(\log M + \gamma_{\mathrm{E}} - (L-2)\log 2) + B_L(g) + \ldots$
- at weak coupling easy to expand and to solve iteratively
- to solve at strong coupling face the same trouble as before, try to map it to functions that determine $f(g)$ ▸ solution
- drop the part $\sim f(g)$, which has already been analyzed

Strong Coupling Expansion

- virtual part is fiven by $B_L(g) = 16g^2\gamma_1^{(1)}(g)$ B_2

$$\gamma_1^{(1)}(g) = \frac{1}{16g^2}(L-2)\epsilon_1(g) + \gamma_1^{(0)}(L-2)\log 2 - \gamma_1^{(0)}(\gamma_{\mathrm{E}} + \log g)$$
$$- \frac{1}{4g}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{2k}\Gamma_{2k} - \frac{1}{4g}\sum_{k=0}^{\infty}\frac{(-1)^{k+1}}{2k-1}\Gamma_{2k-1}$$

- 1st generalized scaling function $\epsilon_1(g) = -1 + \mathcal{O}(e^{-\pi g})$
  [Basso,Korchemsky], [Fioravanti,Grinza,Rossi]
- Γ_k are functions that determine the cusp anomalous dimension
- sums can be performed

two-cusp spinning string $L = 2$

- with strong coupling $f(g) = 4g - 3\log 2/\pi$ and $g = \sqrt{\lambda}/4\pi$ we predict string energy up to one-loop

$$E - S = L + \gamma_L\left(\frac{\sqrt{\lambda}}{4\pi}, S\right)\Big|_{L=2} = \left(\frac{\sqrt{\lambda}}{\pi} - \frac{3\log 2}{\pi}\right)\log\frac{4\pi\, S}{\sqrt{\lambda}} + \frac{\sqrt{\lambda}}{\pi}(\log 2 - 1) + 1 + \frac{6\log 2}{\pi} - \frac{3(\log 2)^2}{\pi},$$

- and determine the constant c of [Beccaria,Forini,Tirziu,Tseytlin] to be $c = 6\log 2 + \pi$ in agreement with algebraic curve approach [Gromov]

Proof of $B_L(g) = L\, h_1(g) + h_2(g)$

To take the strong coupling limit, we had to resum *all* orders at weak coupling!

more strong coupling, where $c_1 = \frac{3\log 2}{4\pi}$

$$B_2(g+c_1) = (\log \tfrac{2}{g} - \gamma_{\mathrm{E}}) f(g+c_1) - 4g - 1 + \frac{1}{g}\frac{\mathrm{K}}{2\pi^2} - \frac{1}{g^2}\frac{9\zeta(3)}{2^8\pi^3}$$
$$+ \frac{1}{g^3}\Big(\frac{9\,\beta(4)}{2^7\pi^4} - \frac{\mathrm{K}^2}{2^7\pi^4}\Big) - \frac{1}{g^4}\Big(\frac{6831\,\zeta(5)}{2^{18}\pi^5} - \frac{423\,\mathrm{K}\,\zeta(3)}{2^{13}\pi^5}\Big) + \mathcal{O}(1/g^5)$$

more spin

use explicit holes (roots of transfer matrix) to derive from NLIE

$$\gamma_2(g,M) = f(g)\Big(\log M + \gamma_{\mathrm{E}} + \frac{f(g)}{2}\frac{\log M + \gamma_{\mathrm{E}}}{M}$$
$$+ \frac{1 + B_2(g)}{2\,M}\Big) + B_2(g) + \dots\,,$$

agrees with string theory [Beccaria,Forini,Tirziu,Tseytlin] and reciprocity [Basso,Korchemsky]

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Gluon Amplitudes/Wilson loops

- leading poles identical, subleading singularities are different
- they differ by $B_2(g)$

[Dixon,Magnea,Sterman]

$$G = G_{eik} + B_2$$

- consistent with available data [Dixon et al], [Korchemsky et al], two-loop $n = 4, 5$ cusp Wilson loop
- gluon amplitudes '$\frac{3}{8}$-solved'

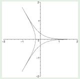
Summary

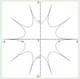
- determined finite order correction to large spin asymptotic of anomalous dimension of twist operators at weak and strong coupling
- at strong coupling and twist $L = 2$ we reproduce string theory result
  - also reproduce $\log M$, $1/M$ and $\log M/M$ string results, they depend on $f(g)$ and $B_2(g)$ in agreement with reciprocity
  - obtained a further interpolating function of AdS/CFT
- determined the difference between subleading poles of gluon amplitudes and Wilson loops
- gluon amplitudes '$\frac{3}{8}$-solved'

Outlook

String Theory

- compute higher orders in $1/g$?
- compute subleading spin correction of spiky string configurations





Gluon Amplitudes/Wilson loops

- Which operator hides behind G_{eik} in $\mathcal{N} = 4$ sYM?
- Can we solve gluon amplitudes at least to '1/2'?

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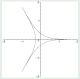

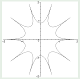

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$$\gamma_{10}^{\mathrm{ABA}} = 136S_9 + 368S_{1,8} + 2832S_{2,7} + 4272S_{3,6} + 848S_{4,5} - 3024S_{5,4} - 2736S_{6,3} - 1168S_{7,2}$$
$$-496S_{8,1} - 5376S_{1,1,7} - 12352S_{1,2,6} - 8832S_{1,3,5} + 1600S_{1,4,4} + 3968S_{1,5,3} - 64S_{1,6,2}$$
$$-1344S_{1,7,1} - 12352S_{2,1,6} - 13760S_{2,2,5} - 2112S_{2,3,4} + 4288S_{2,4,3} - 960S_{2,5,2} - 5440S_{2,6,1}$$
$$-9088S_{3,1,5} - 2432S_{3,2,4} + 5120S_{3,3,3} + 2688S_{3,4,2} - 4160S_{3,5,1} + 1280S_{4,1,4} + 5824S_{4,2,3}$$
$$+6400S_{4,3,2} + 2112S_{4,4,1} + 5120S_{5,1,3} + 6208S_{5,2,2} + 5312S_{5,3,1} + 3904S_{6,1,2} + 3904S_{6,2,1}$$
$$+1728S_{7,1,1} + 21504S_{1,1,1,6} + 22784S_{1,1,2,5} + 5632S_{1,1,3,4} - 1280S_{1,1,4,3} + 6912S_{1,1,5,2}$$
$$+11520S_{1,1,6,1} + 22784S_{1,2,1,5} + 9088S_{1,2,2,4} - 1024S_{1,2,3,3} + 6784S_{1,2,4,2} + 17152S_{1,2,5,1}$$
$$+5504S_{1,3,1,4} - 3456S_{1,3,2,3} - 1536S_{1,3,3,2} + 7680S_{1,3,4,1} - 4480S_{1,4,1,3} - 6272S_{1,4,2,2}$$
$$-3584S_{1,4,3,1} - 3840S_{1,5,1,2} - 3840S_{1,5,2,1} + 768S_{1,6,1,1} + 22784S_{2,1,1,5} + 9088S_{2,1,2,4}$$
$$-1024S_{2,1,3,3} + 6784S_{2,1,4,2} + 17152S_{2,1,5,1} + 9088S_{2,2,1,4} - 2688\,S_{2,2,2,3} + 640\,S_{2,2,3,2}$$
$$+13440S_{2,2,4,1} - 3456S_{2,3,1,3} - 7040S_{2,3,2,2} - 768S_{2,3,3,1} - 4480S_{2,4,1,2} - 4480S_{2,4,2,1}$$
$$+2816S_{2,5,1,1} + 6272S_{3,1,1,4} - 2944S_{3,1,2,3} - 1536S_{3,1,3,2} + 7936S_{3,1,4,1} - 2944S_{3,2,1,3}$$
$$-7296S_{3,2,2,2} - 768S_{3,2,3,1} - 6656S_{3,3,1,2} - 6656S_{3,3,2,1} - 1024S_{3,4,1,1} - 3968S_{4,1,1,3}$$
$$-6528S_{4,1,2,2} - 3584S_{4,1,3,1} - 6528S_{4,2,1,2} - 6528S_{4,2,2,1} - 4864S_{4,3,1,1} - 5376S_{5,1,1,2}$$
$$-5376S_{5,1,2,1} - 5376S_{5,2,1,1} - 4608S_{6,1,1,1} - 32768S_{1,1,1,1,5} - 10240S_{1,1,1,2,4} - 3072S_{1,1,1,3,3}$$
$$-17920S_{1,1,1,4,2} - 30720S_{1,1,1,5,1} - 10240S_{1,1,2,1,4} - 8704S_{1,1,2,3,2} - 24064S_{1,1,2,4,1}$$

$$+1024S_{1,1,3,1,3} + 2560S_{1,1,3,2,2} - 4096S_{1,1,3,3,1} - 512S_{1,1,4,1,2} - 512S_{1,1,4,2,1} - 10240S_{1,1,5,1}$$
$$-10240S_{1,2,1,1,4} - 8704S_{1,2,1,3,2} - 24064S_{1,2,1,4,1} + 3072S_{1,2,2,2,2} - 6656S_{1,2,2,3,1}$$
$$+512S_{1,2,3,1,2} + 512S_{1,2,3,2,1} - 10752S_{1,2,4,1,1} + 1024S_{1,3,1,1,3} + 3072S_{1,3,1,2,2} - 3584S_{1,3,1,3}$$
$$+3072S_{1,3,2,1,2} + 3072S_{1,3,2,2,1} - 2560S_{1,3,3,1,1} + 3072S_{1,4,1,1,2} + 3072S_{1,4,1,2,1} + 3072S_{1,4,2,}$$
$$+3072S_{1,5,1,1,1} - 10240S_{2,1,1,1,4} - 8704S_{2,1,1,3,2} - 24064S_{2,1,1,4,1} + 3072S_{2,1,2,2,2}$$
$$-6656S_{2,1,2,3,1} + 512S_{2,1,3,1,2} + 512S_{2,1,3,2,1} - 10752S_{2,1,4,1,1} + 3072S_{2,2,1,2,2} - 6656S_{2,2,1,3}$$
$$+3072S_{2,2,2,1,2} + 3072S_{2,2,2,2,1} - 5632S_{2,2,3,1,1} + 3072S_{2,3,1,1,2} + 3072S_{2,3,1,2,1} + 3072S_{2,3,2,}$$
$$+3072S_{2,4,1,1,1} + 3072S_{3,1,1,2,2} - 4096S_{3,1,1,3,1} + 3072S_{3,1,2,1,2} + 3072S_{3,1,2,2,1} - 2560S_{3,1,3,}$$
$$+3072S_{3,2,1,1,2} + 3072S_{3,2,1,2,1} + 3072S_{3,2,2,1,1} + 4608S_{3,3,1,1,1} + 3072S_{4,1,1,1,2} + 3072S_{4,1,1,}$$
$$+3072S_{4,1,2,1,1} + 3072S_{4,2,1,1,1} + 3072S_{5,1,1,1,1} + 16384S_{1,1,1,1,3,2} + 32768S_{1,1,1,1,4,1}$$
$$+8192S_{1,1,1,2,3,1} + 4096S_{1,1,1,3,1,2} + 4096S_{1,1,1,3,2,1} + 20480S_{1,1,1,4,1,1} + 8192\,S_{1,1,2,1,3,1}$$
$$+12288S_{1,1,2,3,1,1} + 8192S_{1,2,1,1,3,1} + 12288S_{1,2,1,3,1,1} + 8192S_{2,1,1,1,3,1} + 12288S_{2,1,1,3,1,1}$$
$$-16384S_{1,1,1,1,3,1,1} + \zeta_3\,(896S_6 - 2304S_{1,5} - 1792S_{2,4} - 768S_{3,3} - 1792S_{4,2} - 2304S_{5,1}$$
$$+2560S_{1,1,4} + 512S_{1,2,3} + 1536S_{1,3,2} + 3584S_{1,4,1} + 512S_{2,1,3} + 1536S_{2,3,1} + 512S_{3,1,2}$$
$$+512S_{3,2,1} + 2560S_{4,1,1} - 2048S_{1,1,3,1} - 2048S_{1,3,1,1}) + 1280\,\zeta_5\,(S_{1,3} + S_{3,1} - S_4)$$

$$\gamma^{wrap} = -64\,S_1^2\,(35\zeta_7 - 40S_2\zeta_5 + (-8S_4 + 16S_{2,2})\zeta_3 + 2S_7 - 4S_{2,5} - 2S_{3,4} - 4S_{4,3}$$
$$-2S_{6,1} + 8S_{2,2,3} + 4S_{3,3,1})\,.$$

ENOUGH

- decompose into parity even/odd parts, introduce a dummy index,

$$\int_0^\infty \frac{dt}{t}\left[\frac{\gamma_+(t,j)}{1-e^{-t/2g}} - \frac{\gamma_-(t,j)}{e^{t/2g}-1}\right] J_{2n}(t) = \frac{jL}{8ng} + jh_{2n}\,,$$
$$\int_0^\infty \frac{dt}{t}\left[\frac{\gamma_-(t,j)}{1-e^{-t/2g}} + \frac{\gamma_+(t,j)}{e^{t/2g}-1}\right] J_{2n-1}(t) = \frac{1-j}{2}\delta_{n,1} + jh_{2n-1}$$

with $h_n = h_n(g)$ given by

$$h_n(g) = \int_0^\infty \frac{dt}{4}\left(\frac{2}{e^t-1} - \frac{L-2}{e^{t/2}+1}\right)\left(\frac{J_n(2gt)}{gt} - \delta_{n,1}\right).$$

For $j = 0$ one recovers the solution of the BES equation, while $j = 1$ leads to the system of equations that determines $B_L(g)$.

how to solve...

- choose some reference j' and multiply both sides of the system with $(2n)\gamma_{2n}(g,j')$ and $(2n-1)\gamma_{2n-1}(g,j')$, respectively
- sum over all $n \geq 1$, make use of the expansion formulas of the even/odd parts $\gamma_+(t) = 2\sum_{n\geq 1}(2n)J_{2n}(t)\gamma_{2n}$
- two equations for the even/odd parts $\gamma_\pm(t,j')$, subtract the even from odd part $\longrightarrow$ integral kernel invariant under $j \leftrightarrow j'$ as such should be its solution, can be used to obtain $B_L(g) = 16g^2\gamma_1(g,1)$

$$B_L(g) = 4g^2 \int_0^\infty dt \Big[\frac{2}{e^t - 1} - \frac{L-2}{e^{t/2}+1}\Big]$$
$$\times \Big[\frac{\gamma_-^{(0)}(2gt) - \gamma_+^{(0)}(2gt)}{gt} - 2\gamma_1^{(0)}(g)\Big] - 4gL\int_0^\infty \frac{dt}{t}\gamma_+^{(0)}(2gt)$$

- $B_L(g) = -8g^4(7-2L)\zeta(3) + \mathcal{O}(g^6)$ agrees with max. transc. part of QCD splitting function [Moch,Vermaseren,Voigt]

for twist L=2 one finds

$$B_2(g) = (-\gamma_{\mathrm{E}} - \log g) f(g) - 4g(1 - \log 2) \\ -(1 - \frac{6 \log 2}{\pi} + \frac{3(\log 2)^2}{\pi}) + \mathcal{O}(1/g)\,.$$

B_L