On timelike and spacelike minimal surfaces in AdS_n and the Alday-Maldacena conjecture

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- Motivation

- General formalism

- Spacelike minimal surfaces

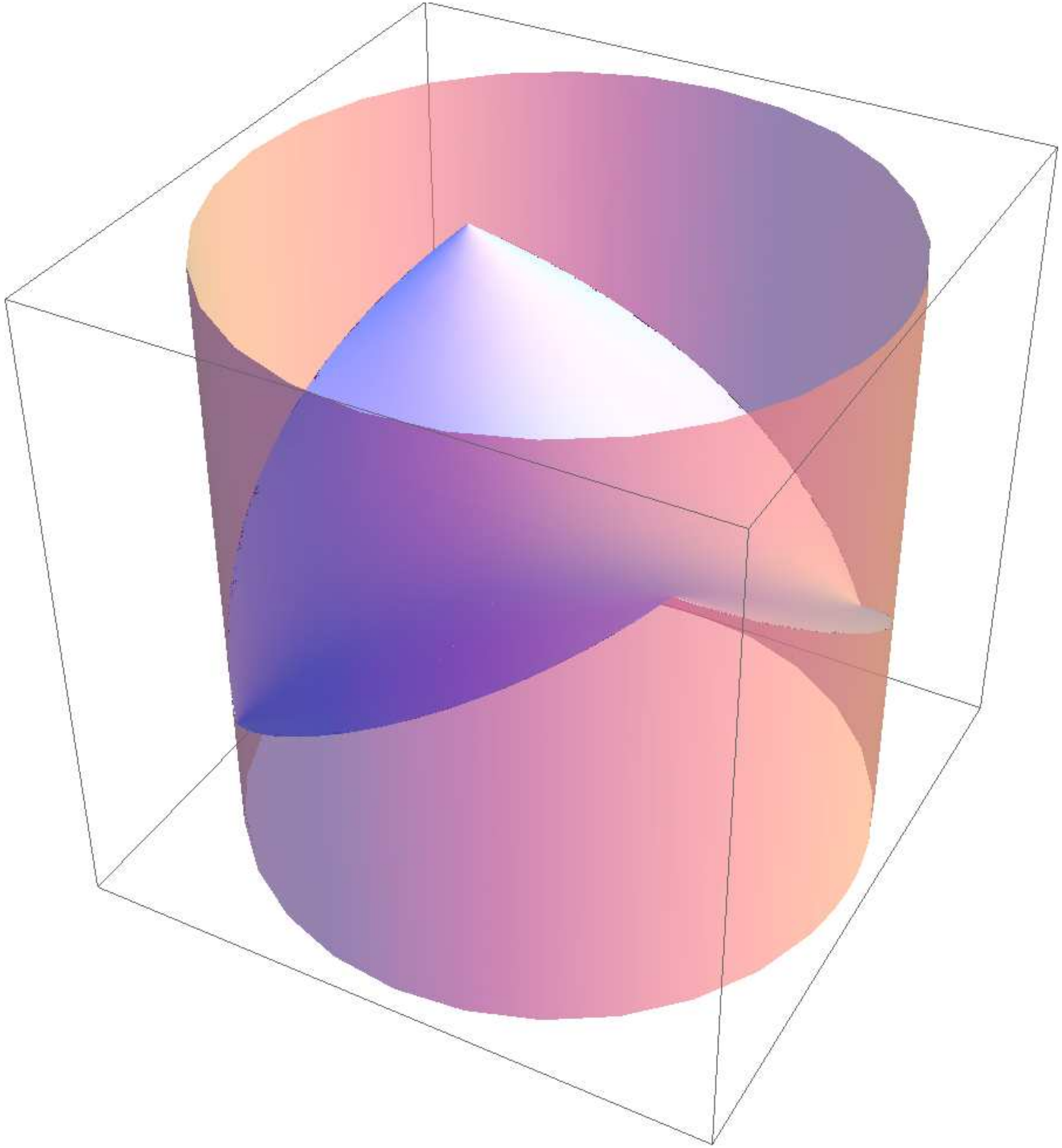
- No flat spacelike minimal surfaces in AdS_n beyond 4-cusp case

Alday, Maldacena :

gluon scatt. ampl. $\Leftrightarrow$ string world surfaces in AdS_5 appr. a lightlike polygon on the boundary

- First lightlike cusp between infinite straight lines in boundary of Poincare patch (= Minkowski$_2$)
- extension to full AdS_3 gives totally symmetric tetragon on $\partial(AdS_3)$
- after isometry trafo in AdS_5 one can reach each lightlike tetragon in Minkowski$_4$

But needed minimal surfaces known only for tetragon, difficult Plateau like problem



Y^N coordinates in $\mathbb{R}^{2,n-1}$,

AdS_n embedded as hyperboloid

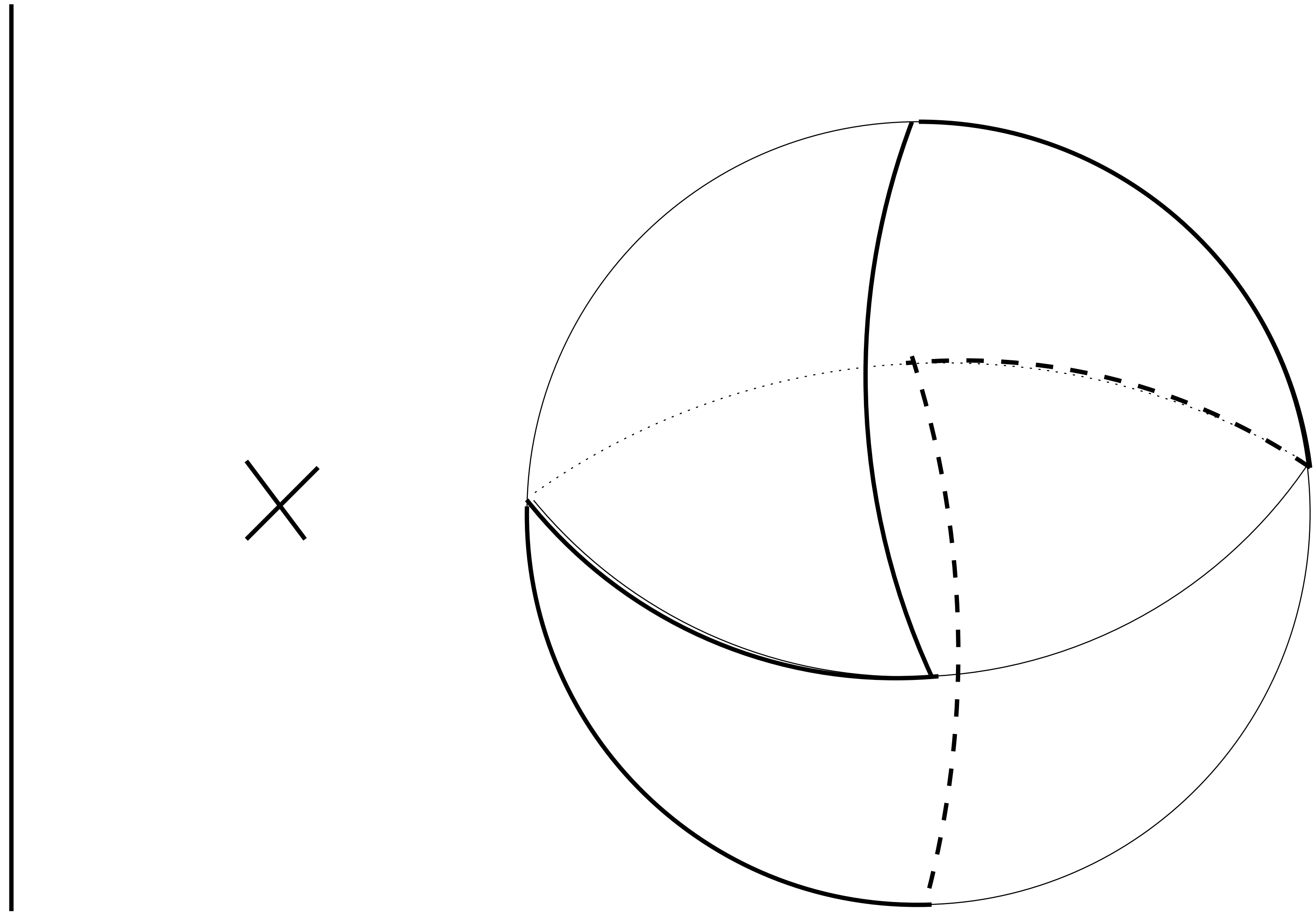
$$(Y^0(X))^2 + (Y^{0'})^2 - (Y^1)^2 - \cdots - (Y^{n-1})^2 \;=\; 1$$

Tetragon surface in AdS_3:

$$Y^0 \;=\; \cosh\sigma \;\; \cosh\tau \;, \qquad Y^{0'} \;=\; \sinh\sigma \;\; \sinh\tau \;,$$
$$Y^1 \;=\; \sinh\sigma \;\; \cosh\tau \;, \qquad Y^2 \;=\; \cosh\sigma \;\; \sinh\tau \;.$$

- induced metric: $g_{\mu\nu} = \delta_{\mu\nu}$ $\quad\Rightarrow\quad$ minimal, spacelike, *flat*
- Is there also such surface in max. symmetric situation for AdS_4 or AdS_5?

Most symmetric polygon for AdS_4



Boundary of AdS_4 = R x S^2

spacelike tetragon surface is double Wick rotation of rigid rotating string

$$\tau \mapsto i\tau, \quad Y^2 \mapsto iY^{0'}, \quad Y^{0'} \mapsto iY^2$$

$\exists$ flat timelike minimal surface in AdS_5:

rigid string with 2 independent rotations (Frolov, Tseytlin)

$$(A\cos\kappa\tau, A\sin\kappa\tau, B\sin\sigma\cos\omega\tau, B\sin\sigma\sin\omega\tau, B\cos\sigma\cos\omega\tau, B\cos\sigma\sin\omega\tau)$$

with $g_{\mu\nu} \;=\; B^2\;\eta_{\mu\nu}$

$$\& \qquad A^2 - B^2 = 1, \quad \omega^2 = 1 + \kappa^2\;, \quad B^2 = \frac{\kappa^2}{2}$$

But now to compensate $\tau \mapsto i\tau$, one has to Wick rotate $Y^{0'}, Y^2, Y^4$

$\Rightarrow$ ends up $\underline{not}$ in AdS_5.

min. surface (all mean curvatures = 0, stationarity of volume functl.)

$$g^{\mu\nu}\Big(\nabla_\mu \partial_\nu X^k(z) \; + \; \partial_\mu X^j \partial_\nu X^l \; \Gamma^k_{jl}(X(z))\Big) \;=\; 0 \;,$$

AdS_n as a hyperboloid in $\mathbb{R}^{2,n-1}$, conformal coord. on the surface

$$\partial\bar\partial Y^N(X(z)) \; - \; \partial Y^K \bar\partial Y_K \; Y^N \;=\; 0 \;.$$

- $\partial = \partial_\sigma + \partial_\tau, \quad \bar\partial = \partial_\sigma - \partial_\tau$ timelike surfaces
- $\partial = \partial_\sigma - i\partial_\tau, \quad \bar\partial = \partial_\sigma + i\partial_\tau$ spacelike surfaces

complete the vectors $Y, \partial Y, \bar\partial Y$ to a basis in $\mathbb{R}^{2,n-1}$ (Pohlmeyer, de Vega/Sanchez, Jevicki et al ....)

$$\{e_N\} \;=\; \{Y, \partial Y, \bar\partial Y, B_4, \ldots, B_{n+1}\} \;.$$

timelike surfaces $\Rightarrow$ normal space Euclidean

spacelike surfaces $\Rightarrow$ normal space Lorentzian

Move the basis along the surface

$$\partial\, e_N \;=\; A_N{}^K\, e_K\;, \qquad \bar\partial\, e_N \;=\; \bar A_N{}^K\, e_K\;. \qquad (*)$$

- find a suitable parametrization of the dyn. (geom.) degrees of freedom in the matrices A and $\bar A$
- derive diff. eqs. for the corr. functions from the eq. of motion (minimal surface condition) and the integrability condition
- after solving these diff. eqs., the surface has to be reconstructed by integrating $(*)$

$$\alpha(\sigma,\tau) \;=\; \log(\partial Y, \bar\partial Y)$$

$$u_a(\sigma,\tau) \;=\; (B_a, \partial\partial Y)\;, \quad \bar u_a(\sigma,\tau) \;=\; (B_a, \bar\partial\bar\partial Y)$$

$$
\begin{aligned}
\partial Y &= \qquad\qquad \partial Y \\
\partial\partial Y &= \qquad\quad \partial\alpha\partial Y \qquad\qquad\qquad + u^b B_b \\
\partial\bar{\partial} Y &= e^{\alpha} Y \\
\partial B_a &= \qquad\qquad\qquad - e^{\alpha}\ u_a \bar{\partial} Y \ + \ A_a{}^b B_b \ ,
\end{aligned}
$$

as well as the eqs. generated by replacements $\partial \leftrightarrow \bar{\partial}$, $u_a \to \bar{u}_a$, $A_a{}^b \to \bar{A}_a{}^b$.

integrability $\Rightarrow$

$$
\begin{aligned}
\partial\bar{\partial}\alpha - e^{-\alpha} u^b \bar{u}_b - e^{\alpha} &= 0 \\
\partial\bar{u}_a - \ A_a{}^b \bar{u}_b &= 0 \ , \qquad\qquad \bar{\partial} u_a - \ \bar{A}_a{}^b u_b \ = \ 0 \ , \\
e^{\alpha}\left(\bar{u}_a u^b - u_a \bar{u}^b\right) &= \partial\bar{A}_a{}^b - \bar{\partial} A_a{}^b + \bar{A}_a{}^c A_c{}^b - A_a{}^c \bar{A}_c{}^b \ .
\end{aligned}
$$

a, b, c run from $4, \ldots, (n+1)$

Gauss, Codazzi-Mainardi, Ricci

- scalar curvature of surface: $R \;=\; -2\, e^{-\alpha}\, \partial\bar{\partial}\alpha$
- Gauss eq. $M \subset N$: $R^{M}_{\alpha\beta\mu\nu} \;-\; R^{N}_{\alpha\beta\mu\nu} \;=\; \left(l^{a}_{\alpha\mu}\, l^{b}_{\beta\nu} \;-\; l^{a}_{\alpha\nu}\, l^{b}_{\beta\mu}\right) h_{ab}$
- second fund. forms $l^{a}_{\mu\nu} \;= h^{ab}\,(B_b, \nabla_\mu \partial_\nu X) \;=\; h^{ab}\,(B_b, \partial_\mu \partial_\nu Y)$

minimal: $g^{\mu\nu}\, l^{c}_{\mu\nu} \;=\; 0\;, \quad \forall c$

timelike minimal $\quad u = a + b,\; \bar{u} = a - b\;:\quad l^{c} \;=\; \frac{1}{2}\begin{pmatrix} a^c & b^c \\ b^c & a^c \end{pmatrix}$

spacelike minimal $\quad u = a + ib,\; \bar{u} = a - ib\;:\quad l^{c} \;=\; \frac{1}{2}\begin{pmatrix} a^c & -b^c \\ -b^c & -a^c \end{pmatrix}$

always either $g_{\mu\nu}$ or h_{ab} indefinite

The bar now implies complex conjugation !

By a conformal (holomorphic) trafo $z \mapsto \zeta(z)$, $\bar{z} \mapsto \overline{\zeta(z)}$: $\quad u^c u_c \;=\; 1$

with $u_c = a_c + i b_c \quad \Rightarrow \quad a^c\, a_c \,-\, b^c\, b_c \;=\; 1\;, \quad a^c\, b_c \;=\; 0$

After suitable gauge trafo:

$$
\begin{array}{lcll}
\text{spacelike I} & (b^c\, b_c \;>\; 0)\;, & u^c \;=\; (0, i \sinh \beta/2, \cosh \beta/2) \\
\text{spacelike II} & (-1 \;\leq\; b^c\, b_c \;<\; 0)\;, & u^c \;=\; (i \sin \beta/2, \cos \beta/2, 0) \\
\text{spacelike III} & (b^c\, b_c \;= 0)\;, & u^c \;=\; (1 + i\beta, 1 + i\beta, 1)\;.
\end{array}
$$

covariant description of subdivision I - III:

$$T \;=\; \frac{1}{8\ |\mathrm{det}g|}\ \epsilon^{\alpha\beta}\ \epsilon^{\mu\nu}\ \mathrm{tr}(F_{\alpha\beta}F_{\mu\nu})$$

in conformal coordinates:

$$T \;=\; \frac{1}{2}\, e^{-2\alpha}\ \mathrm{tr}\ F^2 \;=\; e^{-4\alpha}\Big((\bar{u}_a u^a)^2 \;-\; (\bar{u}_a \bar{u}^a)(u_b u^b)\Big)$$

with Gauss and $C = (\bar{u}_a \bar{u}^a)(u_b u^b)$

$$R \;+\; 2 \;\pm\; 2\ e^{-2\alpha}\ \sqrt{C \;+\; e^{4\alpha}\ T} \;=\; 0$$

Exceptional cases: $C = 0$

Non-exceptional cases: after completely fixing coordinates $C = 1$

$$e^{-4\alpha} \;=\; \frac{(R+2)^2}{4} \;-\; T\ , \qquad \alpha \text{ expressed by invariants !!}$$

$\Rightarrow$ for *all* minimal surfaces in $AdS_n,\ n \geq 4$

$$\frac{(R+2)^2}{4} - T \geq 0 \ , \quad \text{saturation by the except. cases}$$

timelike case: $T \leq 0 \Rightarrow$ no subdivision

spacelike case:

$$\begin{aligned} \text{case I} &: \quad 0 \leq T < \frac{(R+2)^2}{4} \ , \\ \text{case II} &: \quad T \leq 0 \ , \\ \text{case III} &: \quad T = 0 \ , \qquad \text{not all} \ \ F_a{}^b = 0 \end{aligned}$$

case spacelike I

$$A_5{}^6 = -\frac{i}{2}\,\partial\beta\ , \quad A_4{}^5 = \rho\cosh\frac{\beta}{2}\ , \quad A_4{}^6 = i\rho\sinh\frac{\beta}{2}$$

$$\partial\bar\partial\alpha - e^{-\alpha}\cosh\beta - e^{\alpha} = 0\ ,$$

$$\partial\bar\partial\beta + (e^{-\alpha} + \rho\bar\rho)\sinh\beta = 0\ ,$$

$$(\bar\rho\partial\beta - \rho\bar\partial\beta)\ \sinh\frac{\beta}{2} + (\partial\bar\rho - \bar\partial\rho)\ \cosh\frac{\beta}{2} = 0\ ,$$

$$(\bar\rho\partial\beta + \rho\bar\partial\beta)\ \cosh\frac{\beta}{2} + (\partial\bar\rho + \bar\partial\rho)\ \sinh\frac{\beta}{2} = 0\ .$$

case spacelike II

$$A_4{}^5 \;=\; \frac{i}{2}\,\partial\beta\;, \quad A_4{}^6 \;=\; \rho\cos\frac{\beta}{2}\;, \quad A_5{}^6 = i\rho\sin\frac{\beta}{2}\;.$$

$$\partial\bar\partial\alpha \;-\; e^{-\alpha}\;\cos\beta \;-\; e^{\alpha} \;=\; 0\;,$$

$$\partial\bar\partial\beta \;+\; (e^{-\alpha} \;+\; \rho\bar\rho)\;\sin\beta \;=\; 0\;,$$

$$(\bar\rho\partial\beta - \rho\bar\partial\beta)\;\sin\frac{\beta}{2} \;-\; (\partial\bar\rho - \bar\partial\rho)\;\cos\frac{\beta}{2} \;=\; 0\;,$$

$$(\bar\rho\partial\beta + \rho\bar\partial\beta)\;\cos\frac{\beta}{2} \;+\; (\partial\bar\rho + \bar\partial\rho)\;\sin\frac{\beta}{2} \;=\; 0\;.$$

case spacelike III

$$\partial\bar\partial\alpha - 2\cosh\alpha \;=\; 0\;,$$

....

....

On flat surface always $\exists$ coordinates with metric $\eta_{\mu\nu}$ or $\delta_{\mu\nu}$.

Here the diffeom. freedom fixed already, $\Rightarrow$ to analyze <u>all</u> $\partial\bar{\partial}\alpha = 0$

<u>AdS_3</u>: From sinh-Gordon $\quad\Rightarrow\quad \alpha = 0$

$$A_N^{\ K} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \end{pmatrix}, \qquad \bar{A}_N^{\ K} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & i & 0 & 0 \end{pmatrix}.$$

Reconstruction of surface: Solve linear first order eqs. for frame $\{e_N\}$

$$e_N(\sigma,\tau) = \mathcal{M}_N^{\ K}\, e_K(0,0)\,, \quad \mathcal{M}_N^{\ K} = \Big(\exp\big(\frac{\sigma+i\tau}{2}\,A\big)\ \exp\big(\frac{\sigma-i\tau}{2}\,\bar{A}\big)\Big)_N^{\ K}.$$

Brute force exponentiation by Mathematica

Using isomorphy $SO(2,2)$ to $SU(1,1)\times SU(1,1)$ more elegant, application to solitonic solutions of sinh-Gordon (i.e. non constant α) (Jevicki, Jin, Kalousios, Volovich)

$$\mathcal{M}_N^K = \begin{pmatrix} C_\sigma C_\tau & i\ \bar{U}_{\sigma,\tau} & -i\ U_{\sigma,\tau} & S_\sigma S_\tau \\ -i\ U_{\sigma,\tau} & C_\sigma C_\tau & -i\ S_\sigma S_\tau & \bar{U}_{\sigma,\tau} \\ i\ \bar{U}_{\sigma,\tau} & i\ S_\sigma S_\tau & C_\sigma C_\tau & U_{\sigma,\tau} \\ S_\sigma S_\tau & U_{\sigma,\tau} & \bar{U}_{\sigma,\tau} & C_\sigma C_\tau \end{pmatrix}$$

with

$$C_\sigma = \cosh\frac{\sigma}{\sqrt{2}}\ , S_\sigma = \sinh\frac{\sigma}{\sqrt{2}}\ , U_{\sigma,\tau} = \frac{1+i}{2\sqrt{2}}\Big(\sinh\frac{\sigma+\tau}{\sqrt{2}} + i\ \sinh\frac{\sigma-\tau}{\sqrt{2}}\Big)$$

string position $Y(\sigma,\tau)$ is the first vector of the frame $\{e_N\}$

second and third vector not orthonormalized, take corr. linear comb.

- surface can be read off from first row of matrix $\mathcal{M}_N^K$
- result is four cusp surface used by Alday-Maldacena
- freedom of overall $SO(2,2)$ trafo encoded in choice of starting frame $\{e_N(0,0)\}$

AdS_5: form of α-equations $\Rightarrow$ no flat surface in cases spacelike I, III

case spacelike II: $\partial\bar{\partial}\alpha = 0$ $\Rightarrow$ $\cos\beta = -e^{2\alpha}$

assume first $\sin\beta \neq 0$ $\Rightarrow$ $\partial\bar{\partial}\beta \;=\; \frac{4e^{2\alpha}}{\sin\beta}\Big(1 - \frac{\cos\beta\ e^{2\alpha}}{\sin^2\beta}\Big)\ \partial\alpha\bar{\partial}\alpha$

insert in β-eq. $\Rightarrow$ $4e^{2\alpha}\Big(1 + \frac{e^{4\alpha}}{\sin^2\beta}\Big) + \Big(e^{-\alpha} + \rho\bar{\rho}\Big)\sin^2\beta \;=\; 0$

$\Rightarrow$ contradiction (note: $\rho\bar{\rho}$ not pos. def. in timelike case)

$\sin\beta \;=\; 0$: $\Rightarrow$ (since $\cos\beta < 0$) $\cos\beta = -1$

$\rho, \bar{\rho}$-equations degenerate to $\partial\bar{\rho} + \bar{\partial}\rho = 0$

$$A_N^{\ K} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & i\rho \\ 0 & 0 & 0 & 0 & -i\rho & 0 \end{pmatrix} , \quad \bar{A}_N^{\ K} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i & 0 & 0 \\ 0 & i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -i\bar{\rho} \\ 0 & 0 & 0 & 0 & i\bar{\rho} & 0 \end{pmatrix}$$

- Both matrices are block diagonal, exponentials, too
- new degrees of freedom relative to the AdS_3 case, encoded in the lower right blocks with ρ and $\bar{\rho}$, have no influence on first row

$\Rightarrow$ All flat spacelike minimal surfaces in AdS_5 are realized in a subspace $AdS_3 \subset AdS_5$ and are of type just discussed.

Straightforward extension to AdS_n has been done, too.

- Studied Pohlmeyer type reduction and geometric language for AdS_n

- Analyzed differences timelike versus spacelike, subclasses of spacelike

- Exist flat timelike minimal surfaces beyond those in an AdS_3 subspace

- Exist no flat spacelike minimal surfaces beyond those in AdS_3 subspaces

- Did also classification of all flat timelike surfaces in AdS_5.
  All flat timelike minimal surfaces can be parameterized by two arbitrary free chiral functions $\phi(z),\ \bar{\phi}(\bar{z}), \quad \alpha = \phi + \bar{\phi}$.

- Reduction for generic n with full gauge structure for rotations/Lorentz trafo gives interesting relations to WZW models