String Quantization in d dimensions

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Nambu-Goto action: Worldsheet Σ is a critical point of $S = \int d^2 vol =$ area of a (2-dim) surface immersed into $(\mathbb{R}^d, \eta)$ with η Euclidean or Minkowskian metric $\Rightarrow$ Σ is a geometric object.

Parametrization

$$\mathrm{x} : \mathbb{R} \times [0, 2\pi) \to \mathbb{R}^d .$$
$$(\tau, \sigma) \mapsto \mathrm{x}_\mu(\tau, \sigma) \ , \quad \mu \in \{0, \dots, d-1\}$$

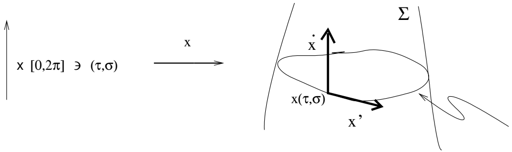
For η Minkowski, choose x such that

$$\dot{\mathrm{x}} := \partial_\tau \mathrm{x} \in \mathbb{R}^d \quad \text{is timelike,}$$
$$\mathrm{x}' := \partial_\sigma \mathrm{x} \in \mathbb{R}^d \quad \text{is spacelike.}$$

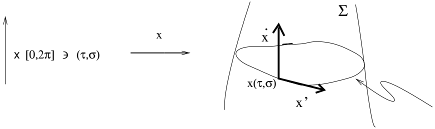
Lagrangian density for S is $L = \sqrt{-\det g}$, g (induced metric on Σ)

Momentum $p = \frac{\partial L}{\partial \dot{\mathrm{x}}}$ In conformal coordinates $p = \dot{\mathrm{x}}$.

This talk: Σ tube-shaped (closed string).



curve of constant τ, $\sigma \mapsto \mathrm{x}(\tau, \sigma)$



curve of constant τ,
$c : \sigma \mapsto \mathrm{x}(\tau, \sigma)$

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Hamiltonian Formalism: Canonical Poisson structure.

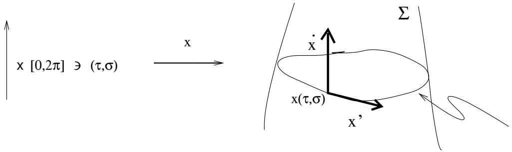
Constraints $p^2 + (\mathrm{x}')^2 \approx 0$ (lapse), $p\,\mathrm{x}' \approx 0$ (shift) are **first class**
$\rightarrow$ **Gauge symmetry**

Constraints are infinitesimal generators of **reparametrizations**.

Reparametrization invariant objects Poisson-**commute** with the constraints.

Reparametrization invariant integrals

Right/Left mover $u^{\pm} := p \pm \mathrm{x}'$, i.e. $u^{\pm}(\tau,\sigma) \in T_{x(\tau,\sigma)}\Sigma \simeq \mathbb{R}^d$.
Constraints: $(u^{+})^2 \approx 0$ and $(u^{-})^2 \approx 0$



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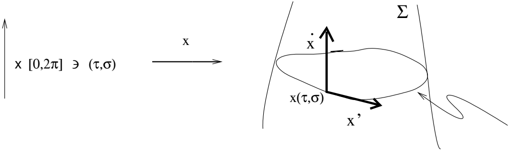

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Define for any $\sigma \in [0,2\pi)$, any $n \in \mathbb{N}$, any $\mu_1,..,\mu_n \in \{0,..,d-1\}$ the iterated integral along c, starting in $c(\sigma)$

$$R^{\pm}_{\mu_1\cdots\mu_n}(\tau,\sigma) = \int_{\sigma}^{\sigma+2\pi} d\sigma_1 \int_{\sigma}^{\sigma_1} d\sigma_2 \cdots \int_{\sigma}^{\sigma_{n-1}} d\sigma_n \; u^{\pm}_{\mu_1}(\tau,\sigma_1)\cdots u^{\pm}_{\mu_n}(\tau,\sigma_n)$$

where $u^{\pm}_{\mu}$, $\mu \in \{0,\ldots,d-1\}$ are the d components of $u^{\pm} \in \mathbb{R}^d$.

[Pohlmeyer, Pohlmeyer+Rehren since 80's]

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Define $Z^{\pm}_{\mu_1 \cdots \mu_n} := R^{\pm}_{\mu_1 \cdots \mu_n} + R^{\pm}_{\mu_2 \cdots \mu_n \mu_1} + \cdots + R^{\pm}_{\mu_n \mu_1 \cdots \mu_{n-1}}$ (cyclic sum)

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Theorem Cyclic sums Z are independent of the parametrization x,
Poisson-commute with the constraints.
In particular, $\partial_\sigma Z = const\ \partial_\tau Z = 0$.

Method: Nambu-Goto string = integrable system.

Reconstruction of worldsheet from set of all Z (complete system).

[Pohlmeyer, Pohlmeyer+Rehren since 80's]

Canonical framework

Fourier decomposition of the string's parametrization x
$\rightarrow$ re-interpretation as **conformal field theory**.
Difficulty: restore reparametrization invariance (only consistent in critical dimension, i.e. $d = 26$)

In general, a polynomial in Fourier modes α is not invariant under reparametrizations. But: the invariant cyclic sums $Z_{\mu_1 \ldots \mu_n}$ can be written as linear combinations of infinite sums of polynomials in Fourier modes, e.g. for $Z_{\mu_1 \ldots \mu_4}$

$$\sum_{n,m>0} \frac{1}{nm} \alpha_{\mu_1}^{-n} \alpha_{\mu_2}^{-m} \alpha_{\mu_3}^{n} \alpha_{\mu_4}^{m}\,, \qquad \sum_{n,m>0} \frac{1}{nm} \alpha_{\mu_1}^{-n} \alpha_{\mu_2}^{-m} \alpha_{\mu_3}^{n+m} \alpha_{\mu_4}^{0} \qquad \text{and more}$$

Poisson-commute with constraints, e.g. classical BRST charge...

BRST approach

BRST Charge Ω_0, an infinite sum of 2-fold products of Fourier modes $\sum \alpha\alpha$ with the property, that for any infinite sum of products of Fourier modes $X = \sum \alpha \cdots \alpha$,we have

$$\{\Omega_0, X\} = 0 \quad \text{iff } X \text{ is reparametrization invariant}$$

Observe: identification of reparametrization invariance (= physical relevance) via an **inner derivation**!

Canonical quantization: replace positive Fourier modes by annihilation operators, negative Fourier modes by creation operators on Fock space, replace polynomials by normal ordered counterparts (creation operators on the left).

Problem: concept not invariant under changes of the parametrization: in general, a reparametrization mixes positive and negative modes.

BRST approach II

One considers the BRST operator Ω on Fockspace, a nilpotent operator $\Omega =: \Omega_0 : + \text{more}$, with $: \Omega_0 :$ denoting the canonically quantized counterpart of Ω_0.

Idea: the physically relevant subspace in Fockspace is $\ker \Omega$ and physically relevant operators on Fock space commute with Ω on $\ker \Omega$.

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Prop [Bahns] Canonically quantized cyclic sums $Z_{\dots}$ of the iterated integrals $R_{\dots}$ do not commute with Ω on $\ker \Omega$ regardless of the dimension.

Proof: obstructions appear for any $Z_{\mu_1 \dots \mu_n}$ with $n \geq 3$. ($n \leq 2$ is ok)

Observe: in usual approach one considers infinite sums of at most two-fold products of operators.

What does this mean?

Canonical approach corresponds to a choice of representation (Fock space)

Alternative: Invariant Quantization [Pohlmeyer's programme]

Classical Poisson algebra, bracket $\{Z..., Z...\}$ = linear comb. of Z... and shuffle multiplication on set of Z. Deform this Poisson algebra to an associative quantum algebra. As a last step, study **representation theory** of the quantum algebra.

Attempts at representation/quantization (in one step)

- Fock space: no representation space for algebra of invariants [B].
- Thiemann: lqg framework (no critical dimension)
- Schreiber: DDF operators (problem with Lorentz covariance)

The shuffle algebra

Results on the invariants Z due to Pohlmeyer, Rehren and collaborators since 1986. Reformulated within framework of the shuffle algebras over a (finite) alphabet [Bahns 08] $R_{\mu_1 \cdots \mu_n} \to$ word $\mu_1 \cdots \mu_n$.

The shuffle algebra

- Let $d \in \mathbb{N}$, consider the alphabet $A = \{0, \ldots, d-1\}$ and the set of words A^*. Let W_n denote the free module over $\mathbb{C}$ with basis given by all words $\in A_n^*$ = the set of words of length n.

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- shuffle product of words $x = x_1 \cdots x_n$ and $y = y_1 \cdots y_m \in A^*$

$$x \# y := x_1(x_2 \cdots x_n \# y) + y_1(x \# y_2 \cdots y_m) \in A^*_{n+m}$$

linear extension $\Rightarrow$ $(\bigoplus W_n, \#)$ shuffle algebra $Sh(A)$.

Shuffle product of words $x = x_1 \cdots x_n$ and $y = x_{n+1} \cdots x_{n+m} \in A^*$

$$\mu(x \otimes y) := x \# y = x_1(x_2 \cdots x_n \# y) + x_{n+1}(x \# x_{n+2} \cdots x_{n+m}) = \sum_{\pi \in S_{n,m}} x_{\pi(1)} \cdots x_{\pi(n+m)}$$

$S_{n,m} = S_{n+m}/S_n \times S_m$ = set of shuffle permutations (mixing two decks of cards).

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Deconcatenation coproduct $\Delta : Sh(A) \to Sh(A) \otimes Sh(A)$

$$\Delta(x) = \sum_{I \sqcup J = \underline{n}} x_I \otimes x_J \qquad \text{for } x \in W_n$$

where $\underline{n} = \{1, \dots, n\}$, $I \sqcup J = \underline{n}$ as ordered sets (including $I, J = \emptyset$), and for $I = \{1, \dots, k\}$ we set $x_I = x_1 \cdots x_k$, and $x_\emptyset = 1$. Then $(Sh(A), \mu, \Delta, S)$ is a commutative, non-cocommutative Hopf algebra with Antipode $S(x_1 \cdots x_n) = (-1)^n x_n x_{n-1} \cdots x_1$.

Eulerian idempotents

Associative convolution in a Hopf Algebra (H, μ, Δ, S):
$f \star g = \mu \circ (f \otimes g) \circ \Delta$ for any two linear maps $f, g : H \to H$.

For $H = Sh(A)$ consider Euler idempotent $e : H \to H$, $e^2 = e$,

$$e(x) = \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k} (\mathrm{id}_H - \epsilon)^{\star n} \qquad \text{for } x \in W_n$$

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Rem Closed formula for e in terms of permutations exists.

Prop $e(x \# y) = 0$ for x and y nontrivial.

Prop Any word can be decomposed into a unique linear combination of shuffle products of elements from ime.

Eulerian idempotents

Decomposition of $x = x_1 \cdots x_n$

$$x = \sum_{k=1}^{n} x^{(k)}$$

where

$$x^{(k)} = \frac{1}{k!} \sum_{J_1 \sqcup \cdots \sqcup J_k = \underline{n}} e(x_{J_1}) \# \cdots \# e(x_{J_k})$$

with $\underline{n} = \{1, \ldots, n\}$, $J_i \neq \emptyset$, and where $J \sqcup I$ denotes the ordered union (i.e. append the second set to the first) and where for $J = \{r, r+1, \ldots, s\}$, we set $x_J = x_r x_{r+1} \cdots x_s$.

Grading from this decomposition: $\deg(x^{(k)}) = n - k - 1 \geq -1$ for x a word of length n. Observe that for any two (nonempty) words x, y we have $\deg(x^{(k_1)} \# y^{(k_2)}) = \deg(x^{(k_1)}) + \deg(y^{(k_2)}) + 1$.

Poisson structure

Prop The image $\mathrm{ime} \subset Sh(A)$ is a graded Lie algebra (w.r.t. deg) with bracket

$$[e(x), e(y)] = \sum_{\rho,\nu \in A} 2\,\eta^{\rho\nu}\ e(\partial^R_\rho \star S(x) \cdot S \star \partial^L_\nu(y))$$

where $\partial^L_\rho(x) = \delta_{\rho,x_1} x_2 \cdots x_n$ and $\partial^R_\nu(y) = \delta_{\rho,y_m} y_1 \cdots y_{m-1}$

Proof: Antisymmetry easy, Jacobi idenity tedious.
Grading: $[,] : e(W_n) \otimes e(W_m) \to e(W_{n+m-2})$
$\Rightarrow \deg([e(x), e(y)]) = \deg(e(x)) + \deg(e(y))$.

Let x, y be words, decompose them into linear combinations of shuffle products of elements from ime, extend the Lie bracket from ime via the Leibniz rule $\to$ Poisson bracket on $Sh(A)$.

Prop On cyclic sums of words this reproduces Poisson bracket of Pohlmeyer's Z.

Quantization: History

Poisson algebra of Z's not freely generated $\rightarrow$ identify generators and relations. Important tool: grading (work stratum per stratum).

Quantization: deforme to a filtered associative s.t. number of relations remains unchanged (admit for quantum corrections)

Never-ending programme (infinitely many strata), but a lot has been learned about the algebra's structure by explicit calculations. Important conjectures were derived.

Quantization: Breakthrough [Rehren + Meusburger]

again reformulated in $Sh(A)$-framework

Let V be d-dim vectorspace with basis labelled by A, i.e. $v_0, v_1, \ldots, v_{d-1}$. Consider the linear map $\tilde{\partial} : \bigoplus W_n \to \bigoplus W_n \oplus V$ (as vectorspaces) given by

$$\tilde{\partial}(x) = (x_2 \cdots x_n, v_{x_1}) - (x_1 \cdots x_{n-1}, v_{x_n}) \qquad \text{for } x \in W_n \ , \ \ n \geq 2$$

and $\tilde{\partial}(x) = 0$ otherwise.

Prop $\tilde{\partial} : \mathrm{ime} \to \mathrm{ime} \oplus V$ (compatibility with e)

Prop $\ker \tilde{\partial} = \{$cyclically symmetric elements of $\oplus W_n\}$,
i.e. $\ker \tilde{\partial}$ = set of physically meaningful objects.
Characterization as kernel of a **derivation**.

Let $\mathfrak{g} = \mathrm{ime}$ as a Lie algebra.

Prop V becomes a $\mathfrak{g}$-module via the action

$$\begin{aligned} e(x).v &= 0 \text{ for } x \in W_n,\ \ n \neq 2 \text{ for all } v \in V \\ e(\mu\rho).v_\nu &= const\,(\eta^{\nu\rho} v_\mu - \eta^{\nu\mu} v_\rho) \text{ for } \mu, \rho, \nu \in A \end{aligned}$$

With this action, $\mathfrak{g} \oplus V$ becomes a trivial extension $\mathfrak{g}_{ext}$ of $\mathfrak{g}$.

Quantization: universal enveloping algebras $U(\mathfrak{g})$ and $U(\mathfrak{g}_{ext})$.

Prop The linear map $\partial : \mathfrak{g} \to U(\mathfrak{g}_{ext})$

$$\partial(e(x)) = \frac{1}{2}[v_{x_1}, e(x_2 \cdots x_n)]_+ \; - \; \frac{1}{2}[v_{x_n}, e(x_1 \cdots x_{n-1})]_+$$

is a derivation of Lie algebras and can be extended canonically to a derivation $\partial : U(\mathfrak{g}) \to U(\mathfrak{g}_{ext})$.

Prop $\ker \partial \subset U(\mathfrak{g})$ is an associative subalgebra of $U(\mathfrak{g})$.

Let z be a cyclically symmetric element in $\oplus W_n$, i.e. $z \in \ker \tilde{\partial}$. consider $z^{(2)} = \frac{1}{2} \sum_{J \sqcup I = \underline{n}} e(x_J) \# e(x_I)$.

Correspondence principle:
$z^{(2)} \in \ker \partial$ has a uniquely determined *quantum counterpart* $\hat{z}^{(2)} \in \ker \partial$.

Not in general true for all words in $\ker \tilde{\partial}$, but:

Conjecture: Elements of the form $z^{(2)}$ generate $\ker \tilde{\partial}$.

If true, then $\ker \partial \subset U(\mathfrak{g})$ is a quantum algebra of the Nambu-Goto string!. If false, then construction yields quantization of a (large, relevant) subalgebra.

So far, all calculations within the old Pohlmeyer approach reproduced!

Outlook

Prove conjecture that the set of $z^{(2)}$ generates $\ker \tilde{\partial}$.

Even if disproved: prescription yields a quantization of a subalgebra of $\ker \tilde{\partial}$.

Outlook

Interesting algebraic questions: classical Poisson algebra is not finitely generated (proved in the 80's), but long standing conjecture that quantum algebra is finitely generated. Use the mathematically sound $\ker \partial \subset U(\mathfrak{g})$ formulation of quantization to check this.

Look for Hilbert space representation of $\ker \partial$. Role of Casimir operators (which replace the constraints) in the formal closure of the classical algebra of invariants remains to be understood.