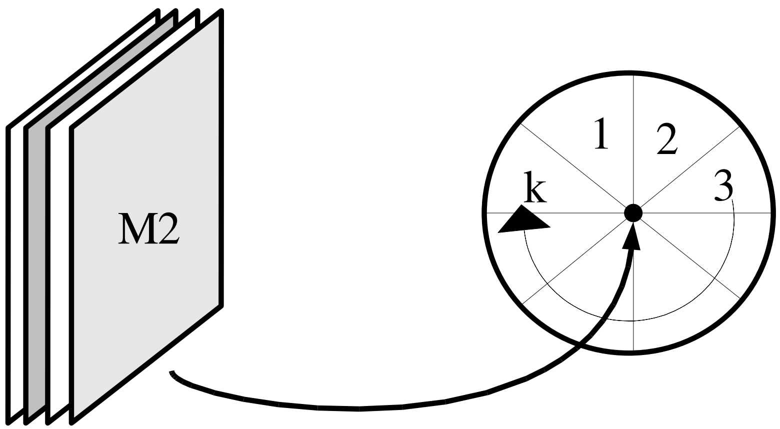
Wilson Loops in $\mathcal{N} = 6$ Supersymmetric Chern-Simons and their String Theory Duals

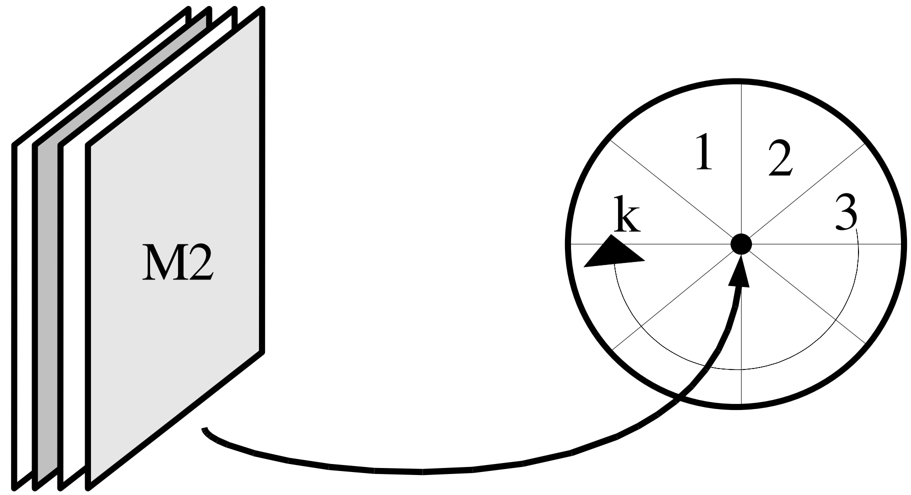


Donovan Young
Institut für Physik
Humboldt-Universität zu Berlin

Nadav Drukker, Jan Plefka, DY, arXiv:0809.2787 [hep-th]

Nordic String Theory Meeting, Jacobs University Bremen, March 26, 2009

ABJM theory: stacks of M2-branes



- The construction is to place a stack of N M2's at the orbifold point of $\mathbb{R}^8/\mathbb{Z}_k$ where $\mathbb{R}^8$ is the space transverse to the M2's.

- The M2-branes source the geometry $AdS_4 \times S^7/\mathbb{Z}_k$.

- There is also a dual 3-dimensional gauge theory which captures the low energy dynamics of the M2-stack.

IIA String theory description

We may write the metric on $S^7/\mathbb{Z}_k$ as a $U(1)$ fibration over $\mathbb{CP}^3$

$$ds^2 = \frac{1}{k^2}\left(d\zeta + A\right)^2 + ds^2_{\mathbb{CP}^3}$$

where $d\zeta + A$ is the M-theory circle. Thus the IIA limit is when $k \to \infty$. We then have $AdS_4 \times \mathbb{CP}^3$.

The IIA and M-theory regimes are valid in the following limits

$$\frac{N}{k^5} \to \begin{cases} \gg 1, & \text{M-theory} \quad AdS_4 \times S^7/\mathbb{Z}_k \\ \ll 1, & \text{IIA string theory} \quad AdS_4 \times \mathbb{CP}^3 \end{cases}$$

The gauge theory

$$S_{\mathrm{CS}} = \frac{k}{4\pi} \int d^3x \, \varepsilon^{\mu\nu\rho} \Big[\mathrm{Tr}(A_\mu \partial_\nu A_\rho + \frac{2}{3} A_\mu A_\nu A_\rho) - \mathrm{Tr}(\hat{A}_\mu \partial_\nu \hat{A}_\rho + \frac{2}{3} \hat{A}_\mu \hat{A}_\nu \hat{A}_\rho) \Big]$$

$$S_{\mathrm{gf}} = \frac{k}{4\pi} \int d^3x \Big[\frac{1}{\xi} \mathrm{Tr}(\partial_\mu A^\mu)^2 + \mathrm{Tr}(\partial_\mu \bar{c} \, D_\mu c) - \frac{1}{\xi} \mathrm{Tr}(\partial_\mu \hat{A}^\mu)^2 + \mathrm{Tr}(\partial_\mu \bar{\hat{c}} \, D_\mu \hat{c}) \Big]$$

$$S_{\mathrm{Matter}} = \int d^3x \Big[\mathrm{Tr}(D_\mu C_I \, D^\mu \bar{C}^I) + i \; \mathrm{Tr}(\bar{\psi}^I \not{D} \, \psi_I) \Big] + S_{\mathrm{int}}$$

- 3-d Chern-Simons theory with two gauge groups $U(N) \times U(N)$, each with opposite sign of level k.
- Four complex bi-fundamental scalars C_I transforming in the $(\mathbf{N}, \bar{\mathbf{N}})$.
- Four bi-fundamental Dirac fermions ψ_I transforming in the $(\mathbf{N}, \bar{\mathbf{N}})$.
- A complicated potential $S_{\mathrm{int}} \sim \int d^3x \, \big(C^6 + C^2\psi^2\big)$.
- Large-N dynamics governed by the 't Hooft coupling $\lambda = N/k$.

Constructing Wilson loops in ABJM

The simplest imaginable Wilson loop is to take

$$W = \frac{1}{N} \operatorname{Tr} \mathcal{P} \exp \left(i \int A_\mu dx^\mu \right) .$$

Experience from $\mathcal{N} = 4$ SYM instructs us that such an object is not generally supersymmetric. An exception is for light-like lines.

In $\mathcal{N} = 4$ SYM we had adjoint scalars Φ, so the coupling was obvious. Here we have bi-fundamental scalars. How do we couple them?

$$W = \frac{1}{N} \operatorname{Tr} \mathcal{P} \exp \int \left(i A_\mu \dot{x}^\mu + \frac{2\pi}{k} |\dot{x}| M_J^I : C_I \bar{C}^J : \right) ds \, .$$

In this way we have built an adjoint out of 2 bifundamentals. The matrix M_J^I is a constant matrix to be determined by SUSY.

This Wilson loop was suggested pre-ABJM by Gaiotto & Yin (2007).

Analysis in CS perturbation theory: pure CS

- Perturbation theory in pure CS has a long and rich history.
- CS was described exactly by Witten (1989). There were many subsequent attempts to recover these exact results from perturbation theory.
- The (possibly knotted) Wilson loop played a central role in this work, and important connections to knot theory were made.

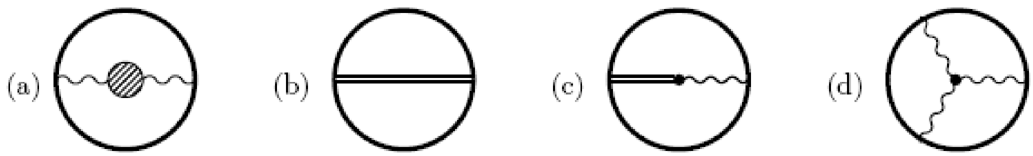
For example the unknotted Wilson loop (of arbitrary shape) may be given exactly for all k, N

$$\langle W \rangle = \frac{1}{N} \frac{q^{N/2} - q^{-N/2}}{q^{1/2} - q^{-1/2}}, \quad q = \exp\left(\frac{2\pi i}{N+k}\right)$$

$$= 1 - \frac{\lambda^2 \pi^2}{6} + \frac{\lambda^3 \pi^2}{3} + \dots, \quad \lambda = \frac{N}{k}$$

Analysis in CS perturbation theory: our case

We have matter coupled and so the situation is much less simple!



We begin with diagrams (a) and (b):

$$\langle A_\mu(x)_{ij} A_\nu(y)_{kl}\rangle = \delta_{ik}\delta_{jl}\frac{1}{k}\left[-i\frac{\varepsilon_{\mu\nu\rho}(x-y)^\rho}{2|x-y|^3} + \frac{N}{k}\left(\frac{\delta_{\mu\nu}}{|x-y|^2} - \partial_\mu\partial_\nu \ln|x-y|\right)\right]$$

$$\langle (C_I)_{i\hat{i}}(x)\,(\bar{C}^J)_{\hat{j}j}(y)\rangle = \delta_I^J\,\delta_{ij}\,\delta_{\hat{i}\hat{j}}\,\frac{1}{4\pi|x-y|}$$

The **4-d propagator** emerges from (a) + (b):

$$\mathcal{D}[x_1(\tau_1), x_2(\tau_2)] \equiv -\frac{N^3}{k^2}\left[\frac{\dot{x}_1\cdot\dot{x}_2 - |\dot{x}_1||\dot{x}_2|}{(x_1-x_2)^2} - \partial_{\tau_1}\partial_{\tau_2}\ln|x_1-x_2|\right].$$

The logarithm can be removed by a gauge transformation. Finally, the contribution to $\langle W\rangle$ is $\frac{1}{N}\frac{1}{2!}\oint d\tau_1\oint d\tau_2\,\frac{N^3}{2k^2} = \frac{\pi^2 N^2}{k^2}$.

Analysis in CS perturbation theory cont'd

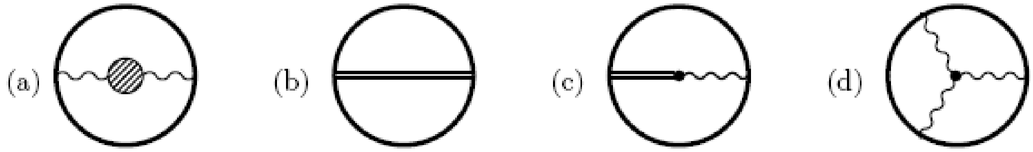


Diagram (c) vansihes by the antisymmetric nature of the CS propagator

Diagram (d) is well-known in pure CS theory, it gives $-\pi^2\lambda^2/6$. However should we take a linear combination of WL in the two gauge groups

$$\frac{1}{2}\Bigg[\frac{1}{N}\,\mathrm{Tr}\,\mathcal{P}\exp\int\left(iA_\mu\dot{x}^\mu+\frac{2\pi}{k}|\dot{x}|M_J^I:C_I\bar{C}^J:\right)ds$$
$$\pm\frac{1}{N}\,\mathrm{Tr}\,\mathcal{P}\exp\int\left(-i\hat{A}_\mu\dot{x}^\mu-\frac{2\pi}{k}|\dot{x}|M_J^I:\bar{C}^JC_I:\right)ds\Bigg]$$

we can either destroy this pure CS contribution, or keep it only, throwing away the matter contribution. The string duals we find should correspond to the "+" choice.

Matrix model?

Our perturbative result is

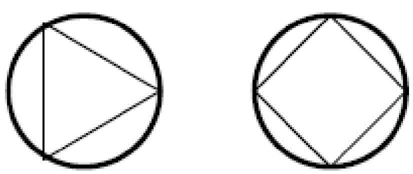
$$\langle W \rangle = 1 + \frac{\pi^2 N^2}{k^2} - \frac{\pi^2 N^2}{6k^2} + \mathcal{O}(k^{-3})\,.$$

As we saw, in $\mathcal{N} = 4$ SYM, the circular Wilson loop also has a constant propagator, and there non-ladder graphs vanish leaving a matrix model:

$$\langle W_{\mathcal{N}=4} \rangle_{\text{planar}} \sim \frac{2}{\sqrt{\lambda}} I_1(\sqrt{\lambda}) = \begin{cases} 1 + \frac{1}{8}\lambda + \ldots & \text{for} \quad \lambda \ll 1 \\ e^{\sqrt{\lambda}} & \text{for} \quad \lambda \gg 1 \end{cases}$$

Could the same work here for the matter contribution?

A study of higher loops (λ^4) would be necessary to answer this question. Unlike $\mathcal{N} = 4$ SYM, here we will encounter graphs like



A higher loop analysis would be very welcome indeed.

Giant Magnon dispersion relation

- The analysis of the Giant Magnon dispersion relation in ABJM has revealed that, compared to $\mathcal{N} = 4$ SYM, one replaces

$$\sqrt{1 + \frac{\lambda}{\pi^2} \sin^2 \frac{p}{2}} \qquad \rightarrow \qquad \frac{1}{2}\sqrt{1 + 16h^2(\lambda) \sin^2 \frac{p}{2}}$$

where

$$h(\lambda) \sim \begin{cases} \lambda & \lambda \ll 1 \\ \sqrt{\lambda/2} & \lambda \gg 1 \end{cases}$$

- Natural to propose $\sqrt{\lambda} \rightarrow \text{const.} \cdot h(\lambda)$ in the $\mathcal{N} = 4$ SYM matrix model for Wilson loops.

- This doesn't work as asymptotics are off by a factor of two at weak coupling.

Description of our Wilson loops at strong coupling

- We have now type IIA string theory on $AdS_4 \times \mathbb{CP}^3$.
- A fundamental string ending along the loop at the boundary of AdS_5 describes the $\mathcal{N} = 4$ Wilson loop.
- But that Wilson loop sits in an $AdS_2 \subset AdS_5$, and at a point on the S^5. Thus we can borrow that solution and place it in AdS_4 times a point in $\mathbb{CP}^3$.

We know that in traditional AdS/CFT there is a natural association between the scalars of $\mathcal{N} = 4$ SYM and the S_5 of the string theory background.

There, coupling a single scalar $\theta^I \Phi_I$ in the WL corresponds to sitting at a point on the S^5, i.e. $SO(5)$ symmetry.

Here we are coupling to **two** scalars: C and $\bar{C}$, through the matrix M_J^I. This breaks the $SU(4)$ symmetry which rotates the C_I (and $\bar{C}^J$) into themselves to $SU(2) \times SU(2)$.

How do we impose $SU(2) \times SU(2)$ symmetry in the string solution?

Strong coupling: fundamental string

Our result for $\langle W \rangle$ at strong coupling is then simply borrowed from traditional AdS/CFT. In an AdS_4 coordinate system given by

$$ds^2_{AdS_4} = du^2 + \cosh^2 u\, ds^2_{AdS_2} + \sinh^2 u\, d\phi^2\,,$$

$$ds^2_{AdS_2} = \begin{cases} d\rho^2 - \cosh^2\rho\, dt^2\,, & \text{appropriate for a time-like line,} \\ d\rho^2 + \sinh^2\rho\, d\psi^2\,, & \text{appropriate for a space-like circular loop,} \end{cases}$$

our circular Wilson loop is described by the ρ, ψ AdS_2 at $u = 0$. This ends in a circle parametrized by ψ at the boundary $\rho = \infty$.

$$\mathcal{S}_{\text{string, cl.}} = \frac{R^3}{8\pi k}\int_0^{2\pi} d\psi \int_0^{\rho_0} d\rho\, \sinh\rho = \pi\sqrt{2\lambda}\,(\cosh\rho_0 - 1)\,,$$

$$\langle W \rangle_{\text{string}} \sim e^{\pi\sqrt{2\lambda}}\,.$$

- We then smear over an $S^2 \subset CP^3$ to restore $SU(2)\times SU(2)$ symmetry.

The brane solutions

- When the representation of the trace is taken large ($\sim N$) a "Myers" effect kicks-in and the fundamental string "blows-up" into a higher dimensional object: a D-brane.

- In traditional AdS/CFT a large symmetric representation Wilson loop becomes a D3-brane $\subset AdS_5$, while a large antisymmetric representation Wilson loop becomes a D5-brane $\subset S^5$.

- We have found candidates for the analog of both these objects in the ABJM dual, i.e. $AdS_4 \times \mathbb{CP}^3$. They are D2-branes and D6-branes respectively.

- They preserve the correct supersymmetries, charges, and symmetries.

The D2-brane solution

$$ds^2_{AdS_4} = R^2\left[d\rho^2 + \sinh^2\rho\, d\psi^2 + \sinh^2\rho\big(d\theta^2 + \sin^2\theta\, d\phi^2\big)\right] .$$

$$ds^2_{\mathbb{CP}^3} = \frac{R^2}{4}\Bigg[d\alpha^2 + \cos^2\frac{\alpha}{2}(d\vartheta_1^2 + \sin^2\vartheta_1^2\, d\varphi_1^2) + \sin^2\frac{\alpha}{2}(d\vartheta_2^2 + \sin^2\vartheta_2^2\, d\varphi_2^2)$$
$$+ \sin^2\frac{\alpha}{2}\cos^2\frac{\alpha}{2}(d\chi + \cos\vartheta_1\, d\varphi_1 - \cos\vartheta_2\, d\varphi_2)^2\Bigg].$$

We take worldvolume coordinates ρ, ψ and χ from $\mathbb{CP}^3$, leaving α arbitrary.

$$\mathcal{S}_{\mathrm{D2}} = T_{\mathrm{D2}}\int e^{-\Phi}\sqrt{\det(g + 2\pi\alpha' F)} + T_{\mathrm{D2}}\int\Big[P[C_3] + 2\pi i\alpha' P[C_1]\wedge F\Big].$$

$$F_{\psi\rho} \sim i\beta E\sinh\rho, \quad C_1 = \frac{k}{4}(\cos\alpha \mp 1)\, d\chi\,,$$

$$\mathcal{S}_{\mathrm{D2}} = \frac{T_{\mathrm{D2}}R^3}{8}\int d\rho\, d\psi\, d\tau\ \sinh\rho\left[\sin\alpha\sqrt{1+\beta^2E^2} - i\beta E(\cos\alpha - 1)\right] ,$$

The equation of motion for α allows it to be an arbitrary constant but gives the relation

$$i\beta E = -\cos\alpha\,.$$

The D2-brane solution cont'd

The gauge field is a cyclic variable and the string charge p is proportional to its conjugate momentum

$$p = -4\pi i \frac{\delta \mathcal{L}}{\delta F} = \pm \frac{k}{2}\,.$$

$$\mathcal{S}_{\text{L.T, classical}} = \mathcal{S}_{\text{clasical}} - pE = \frac{R^3}{8}\int d\rho \sinh\rho = \frac{k}{2}\pi\sqrt{2\lambda}(\cosh\rho_0 - 1).$$

In order to match the $SU(2)\times SU(2)$ symmetry here, we must smear over **both** $\mathbb{CP}^1$'s since $\alpha \neq 0$.

When we do this we find that the SUSY's are exactly those of the smeared fundamental string.

Action is p times that of a fundamental string. This is **not** the case for the analogous D3-brane solution in $AdS_5 \times S^5$. But we must have $N/k \gg 1$, which precludes p approaching N.

$$S_{D3}^{\text{reg.}} = -2N\left(\kappa\sqrt{1+\kappa^2} + \sinh^{-1}\kappa\right), \qquad \kappa = \frac{p\sqrt{\lambda}}{4N}$$

The D6-brane solution

We take $\rho, \psi, \vartheta_1, \vartheta_2, \varphi_1, \varphi_2, \chi$ as worldvolume coordinates.

$$\mathcal{S}_{\rm D6} = T_{\rm D6} \int \left[e^{-\Phi} \sqrt{\det(g + 2\pi\alpha' F)} + 2\pi i P[C_5] \wedge F \right] .$$

$$\mathcal{S}_{\rm D6} = \frac{R^9 T_{\rm D6}}{2^{10} k^2} \int \sinh\rho \sin\vartheta_1 \sin\vartheta_2 \left[\sin^3\alpha \sqrt{1+\beta^2 E^2} - i\beta E \Big((\sin^2\alpha + 2)\cos\alpha - 2 \Big) \right] .$$

$$i\beta E = -\cos\alpha \, .$$

The string charge carried by the D6-brane is the conjugate to the gauge field

$$p = -i \frac{\delta S}{\delta E} = \frac{\pi^3 R^9 T_{\rm D6} \beta}{8k^2} (1 - \cos\alpha) = \frac{N}{2} (1 - \cos\alpha) \, ,$$

The value of p ranges between 0 and N, where the appearance of N is a manifestation of the “stringy exclusion principle,” and is an indication that this D-brane represents Wilson loops in anti-symmetric representations.

$$\mathcal{S}_{\rm L.T.} = \mathcal{S} - ipE = -\frac{\pi^4 R^9 T_{\rm D6}}{8k^2} \sin^2\alpha = -\pi\sqrt{2\lambda} \, \frac{p\,(N-p)}{N} \, .$$

This is indeed symmetric under $p \leftrightarrow N - p$, as should be the case of the antisymmetric representation. Also for small p it agrees with the result of p fundamental strings.

Summary and open questions

- We have constructed 1/6 BPS Wilson loops in the shapes of infinite straight lines and circles in $\mathcal{N} = 6$ supersymmetric Chern-Simons theory.
- We have constructed string theory duals for fundamental, large symmetric, and large anti-symmetric reprresentations in the form of fundamental strings, D2-branes and D6-branes.
- We have shown that these duals have the expected action, symmetries, and supersymmetries.

Open questions

- Is there a Matrix model?
- Can we construct an unsmeared (i.e. 1/2 BPS) Wilson loop in the gauge theory?
- Can we construct other BPS Wilson loops; maybe with motion on the $\mathbb{CP}^3$?