Integrability in Topological String Theory and N=2 Gauge Theory

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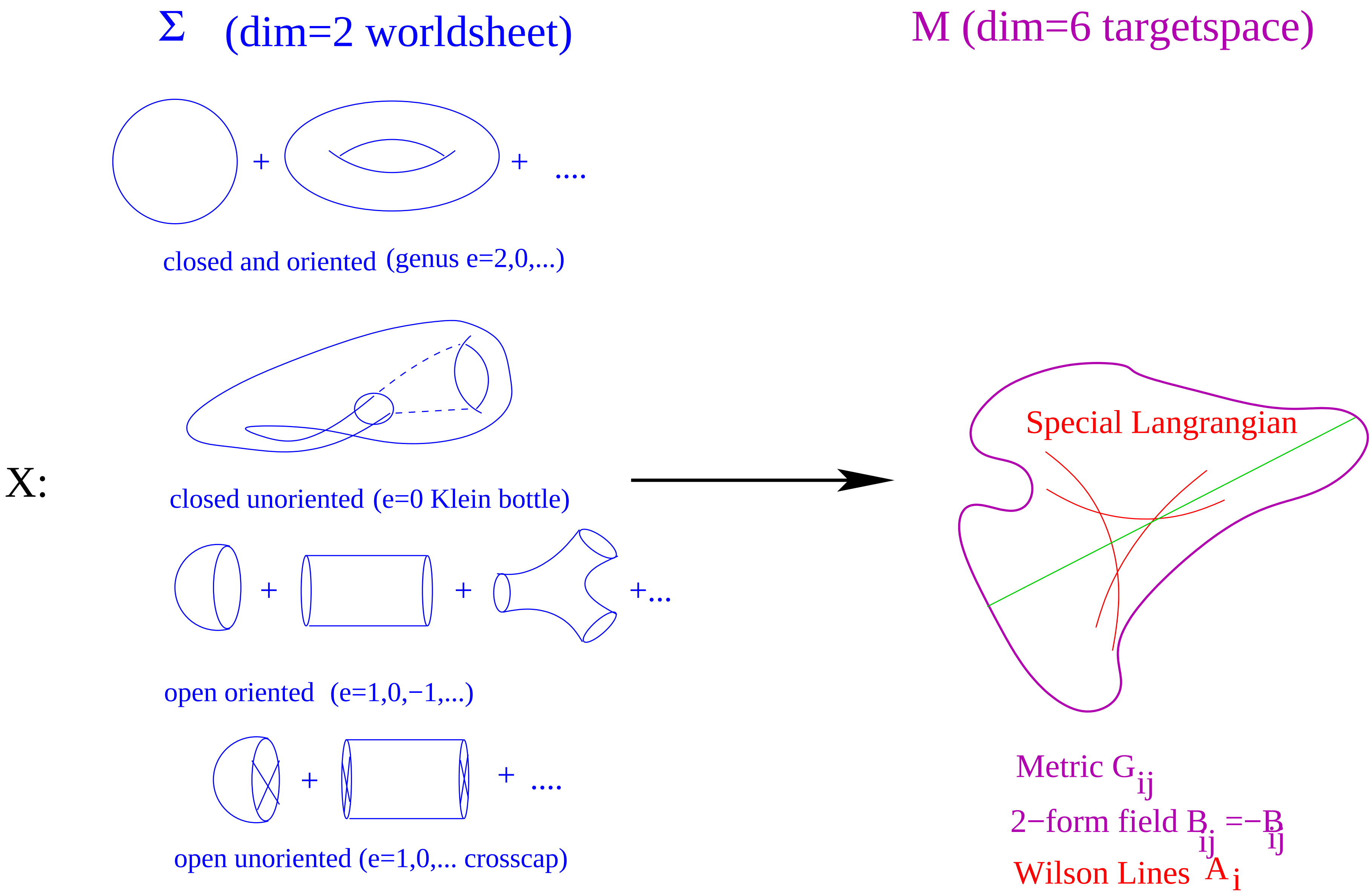

Σ (dim=2 worldsheet)
M (dim=6 targetspace)
+
+ ....
closed and oriented (genus e=2,0,...)
X:
closed unoriented (e=0 Klein bottle)
+ + +...
open oriented (e=1,0,−1,...)
+ + ...
open unoriented (e=1,0,... crosscap)
Special Langrangian
Metric G_{ij}
2−form field $B_{ij}=-B_{ij}$
Wilson Lines A_i

Topological String Theory

is a truncation to *cohomological string theory*, which eliminates the *oscillator modes* and calculates holomorphic amplitudes in $N = 2$ (closed version) and $N = 1$ (open version) 4-d supergravity action exactly.

This simplification makes advances questions addressible:

- Exploration of string dualities:
  - Type IIA/Type IIB duality — T-Duality
  - Heterotic/Type II duality — Str.Str.-Duality
  - String/Gauge Th. duality — large N-Duality

- Solvability and Integrability:
  - Exact solutions in $\frac{\sqrt{\alpha'}}{R}$ $\Rightarrow$ small volume expansions, e.g. orbifold points/conifold points
  - All genus recursions $\Rightarrow$ all genus expansion
  - Completion of the asymptotic genus expansions $\Rightarrow$ conjecturial non-perturpative formulation

Physical Applications: String phenomenology, Black-hole physics, maybe even the lessons about non-perturbative effects are rescrited to (highly) supersymetric situations

Perturbative string theory has a genus expansion, e.g. for partition function of string theory $X : \Sigma_g \to M$

For M Kähler the string path integral localizes in the topological A-model to a finite dimensional integral over the moduli space of the holomorphic maps. We can make a large radius expansion $\mathrm{Im}(\hat{t}) \to \infty$ and write a convergent series for the connected vacuum amplitudes

$$\boxed{\mathcal{F}_g(\hat{t}) = \sum_{\beta \in H_2(M,\mathbb{Z})} r^g_\beta e^{2\pi i \hat{t}\cdot\beta} \,,}$$

which depends only on the complexified Kähler parameter of M: $\hat{t} = \int_{C_\beta} i\omega + B$. The finite dimensional integrals are topological in the sense that they depend only on the genus of the curve and the cohomology class of the image. They are mathematically well defined

$$\boxed{r_\beta^g = \int_{\mathcal{M}(\beta,g)} c_{vir}(\beta, g) \in \mathbb{Q} \in \mathbb{Q}}$$

and known as Gromov-Witten invariants. Symplectic invariants closely related to integer invariants such as Donaldson-Thomas and Gopakumar-Vafa invariants.

Formally one can write the vacuum amplitude as an expansion

$$Z(W,\hat{t}) = \exp(F(\lambda,\hat{t})), \quad F = \sum_{g=0} \lambda^{2g-2} F_g(\hat{t})$$

in the is the string coupling λ. However this is an asymptotic expansion in λ!

The critical Case: Grothendieck-Hirzebruch-Rieman-Roch

$$\dim \overline{\mathcal{M}}_g(M,\beta) = c_1(M)\cdot\beta + (\dim(M)-3)(1-g) \geq 0$$

Special in this GHRR dimension formula are

- Calabi-Yau manifolds as $c_1(M) = 0$.
- complex 3-folds.
- the genus one amplitude.

as then $\dim \overline{\mathcal{M}}_g(M,\beta) = 0$ $\rightarrow r_g^\beta \neq 0$: a point counting problem sometimes solvable by localization with respect to torus action.

$r_g^\beta \neq 0$ Calabi Yau 4-folds relevant for M/F-theory compactifications

- GHRR $\rightarrow r_g^\beta \neq 0$ only for $g = 0, 1$. This sector is solved in arXiv:math.ag/0702189 with R. Pandharipande and new integer meeting invariants defined.

Calabi-Yau 3-folds are the critical case.

- GHRR $\rightarrow r_g^\beta \neq 0,\ \forall g$

		non–compact CY(toric)	compact CY (AS toric)
A–model	localisation Pandharipande, Graber, Zaslow, Liu, Katz	✓	g=0 Kontsevich Giventhal,Yau,Lian g>0 ?
	large N duality Vertex Aganagic,Klemm Marino, Vafa	✓	?
	Relative G–W Pandharipande	✓	in principle g small Pandharipande Okounkov, Gathman
B–model	large N duality Matrix model Aganagic,Klemm Marino, Vafa	✓	?
	DT Okounkov, Maulik,Nekrasov, Pandharipande	✓	? Pandharipande, Thomas announced
	Holomorphic anomaly this talk	✓	g=0 Candelas della Ossa, Green,Parkes g small Bershadski,Cecotti, Ooguri,Vaf Katz, Klemm, Vafa g>0 this talk
	KS–H Action BCOV, Pestun, Witten	g =0,1	
Heterotic –II duality	K3–Fiber	g=0 KLM, g=1, Harvey, Moore, all g: Gava, Narain, Taylor, Marino, Moore, Klemm, Kreuzer, Riegler, Scheidegger, Grimm Weiss 07 Maulik Pandharipande 07	

New Developments:

- Direct integration of the closed sector. Huang, Bouchard, Grimm, Haghighat, Marino, Quakenbush, Rauch, Weiss, AK
- Solution of the open sector for small radius, e.g. at Orbifold point using matrix model. Bouchard, Pasquetti, Marino, AK
- Open string sector on compact Calabi Yau. Walcher, Krefl, Alim, Hecht, Mayr, Jockers, . . .

Direct integration of the closed sector

- M.x. Huang and AK: arXiv:hep-th/0605195, M.x. Huang, S. Quakenbush and AK: arXiv:hep-th/0612125

- B. Haghighat, A.K. and M. Rauch arXiv:hep-th:0809.1674, M.x. Huang and AK: arXiv:hep-th/09024255

- The holomorphic anomaly in topological string theory
  - Duality Symmetries in Topological String
  - Special Geometry
  - The holomorphic anomaly equation

- The holomorphic anomaly as modular anomaly

- Ring of almost holomorphic functions
- Direct integration of the holomophic anomaly equation

- Integrability of the holomorphic anomaly equation
  - The gap condition
  - Applications

Modularity in Topological String Theory

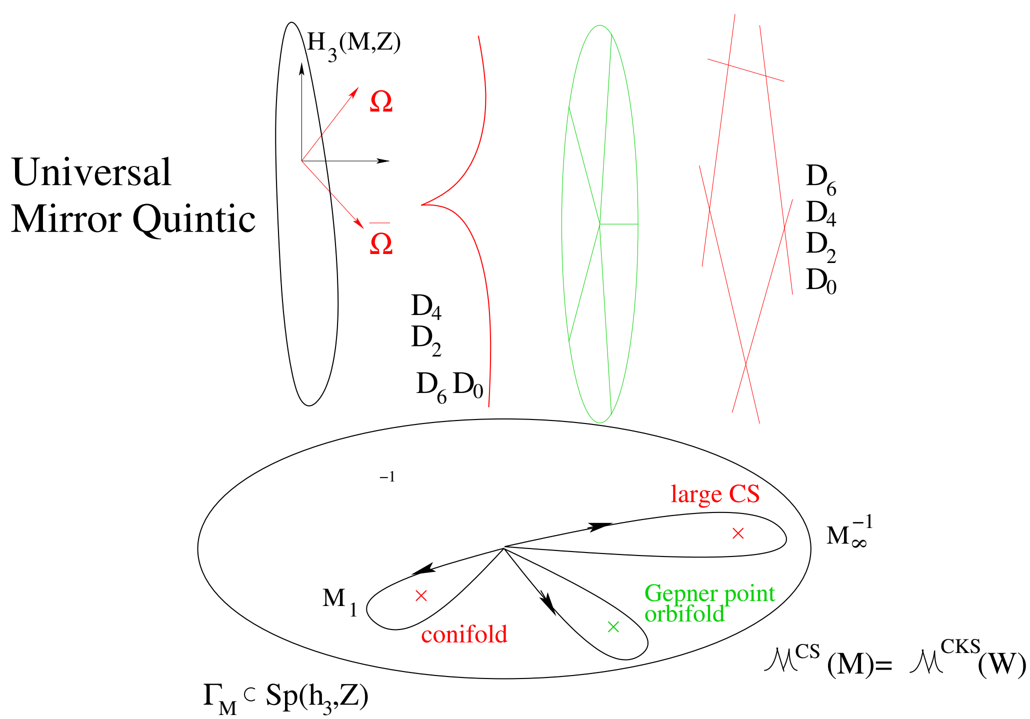
Some invariances $\hat{t} \to \hat{t} + 1$ are clear in this formulation, but full the global monodromy comes from mirror picture.

$$\boxed{Z(W, \hat{t}) = Z(M, t)}$$

Here t is the complex structure parameter of the mirror manifold W: $H^{p,q}(M) = H^{3-p,q}(W)$ and $\hat{t} = t + O(e^{2\pi i t})$ the mirror map.

E.g. for the family of mirror quintics (over $e^{-\frac{t}{5}} \in \mathbb{P}^1$)

$$W = \sum_{i=1}^{5} x_i^5 - e^{-\frac{t}{5}} \prod_{i=1}^{5} x_i = 0 \in \mathbb{P}^4,$$



the global monodromy is generated by

$$M_0=\begin{pmatrix} 1 & 0 & 0 & 0\\ 1 & 1 & 0 & 0\\ 5 & -3 & 1 & -1\\ -8 & -5 & 0 & 1 \end{pmatrix},\ M_1=\begin{pmatrix} 1 & 0 & -1 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1 \end{pmatrix},\ M_\infty^{-1}=\begin{pmatrix} -4 & 3 & -1 & 1\\ 1 & 1 & 0 & 0\\ 5 & -3 & 1 & -1\\ 8 & -5 & 0 & 1 \end{pmatrix}.$$

as a discrete subgroup of $\Gamma_M=\mathrm{Sp}(4,\mathbb{Z})$ acting on $H^3(M,\mathbb{Z})$, i.e. on the periods

$$\Pi(t)=\begin{pmatrix} \int_{A^i}\Omega = X^i \\ \int_{B_i}\Omega = P_i = \frac{\partial F_0}{\partial X^i} \end{pmatrix}$$

$\exists\ \Omega\in H^{3,0}(M,\mathbb{Z})$ is defining property of a Calabi-Yau space. T-duality $\Rightarrow Z(M,t)$ invariant under Γ.

Special Kähler Geometry: The moduli space is Kähler with potential K, i.e. $G_{i\bar{j}} = \partial_i \partial_{\bar{j}} K$ given by,

$$\boxed{\exp(K) = i \int \Omega \wedge \bar{\Omega} = -i(P_i \bar{X}^i - \bar{P}_i X^i)\ .}$$

Further we have

$$\boxed{C_{ijk} = \Omega \partial_i \partial_j \partial_k \Omega = D_i D_j D_k \mathcal{F}_0\ .}$$

Compatibility ($P_i = \frac{\partial F_0}{\partial X^i}$, $\bar{C}_{\bar{l}}^{ij} = e^{2K} \bar{C}_{\bar{k}\bar{l}\bar{m}} G^{\bar{m}i} G^{\bar{n}j}$) implies

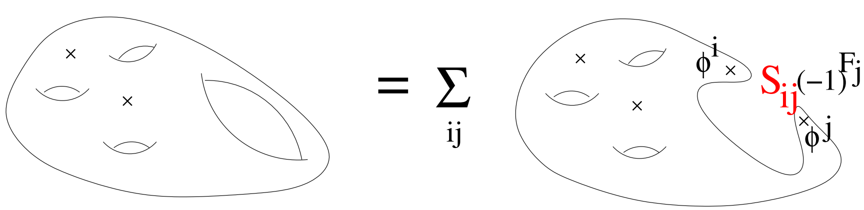
$$\partial_{\bar{l}} \Gamma^i_{km} = R^i_{k\bar{l}m} = \delta^i_k G_{\bar{l}m} + \delta^i_m G_{\bar{l}k} - C_{kmj} \bar{C}_{\bar{l}}^{ij}\ .$$

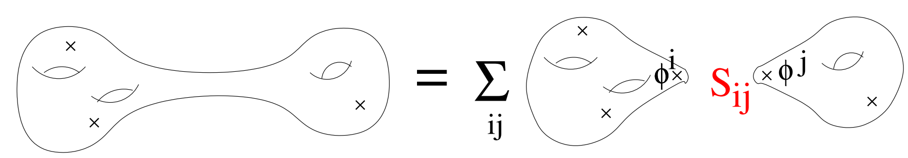
The holomorphic anomaly equations:

World-sheet analysis of Bershadski, Cecotti, Ooguri and Vafa

$$\begin{aligned}\bar\partial_{\bar t_{\bar k}} F_g &= \int_{\overline{\mathcal{M}}(g)} \partial\bar\partial\lambda \\ &= \tfrac{1}{2}\bar C_{\bar k}^{ij}\left(D_i D_j F_{g-1} + \textstyle\sum_{r=1}^{g-1} D_i F_r D_j F_{g-r}\right) .\end{aligned}$$

B-model Parameters are complex structur def. in $\mathcal{M}_{CS}(W)$ of mirror W





Equations come from factorization of *higher genus world-sheets*.

Note that the covariant derivatives are determined from the special Kähler metric, which follows from the genus zero prepotential $\mathcal{F}_0$.

Recursive equations in the genus but leave

- an *holomorphic ambiguity* (functions)
- *s-t modularity* $\longrightarrow$ *modular ambiguity* (discrete data)
- eventually fixed by *gap conditions*.

Implementation of interplay between world-sheet and space-time arguments requires

- an understanding of modular group Γ_M,
- control over the metaplectic transformation property of $Z(W, t, \bar{t})$ under Γ_M.

These ideas apply and are in fact easier explained in the $N = 2$ gauge theory limit of type II string compactifications:

Geometrically this is a decompatification limit of

(M, W), where the compact part of W reduces to a Riemann surface $\mathcal{C}$ and the holomophic $(3,0)$-form Ω reduces to a meromorphic one form λ on $\mathcal{C}$

Local non-compact geometry limit of W

$$v \cdot w = H(x, y, t) \ ,$$

where $v, w \in \mathbb{C}$ and $x, y \in \mathbb{C}^*$. The information about the complex structure is encoded in the periods

$$\begin{pmatrix} \int_{a^i} \lambda = x^i \\ \int_{b_i} \lambda = p_i = \frac{\partial F_0}{\partial x^i} \end{pmatrix}$$

of the Riemann surface

$$H(x, y, t) = 0$$

with $(a^i, b_i) \in H_1(\mathcal{C}, \mathbb{Z})$ a symplectic basis.

$F_0(x^i)$ is the Seiberg-Witten prepotential, which gives the exact $N = 0$ gauge coupling and all dyon masses on the Coulomb branch of the gauge theory parametrized locally by x^i.

Coupling Seiberg-Witten gauge theory to gravity HK1 & 2

Geometric engineering realizes e.g. pure N=2 SU(2) as double scaling limit of TST on $0(-2,-2) \longrightarrow \mathbb{P}^1 \times \mathbb{P}^1$. $H(x, yt) = 0$ is then simply related to the Seiberg-Witten curve for pure N=2 SU(2) gauge theory.

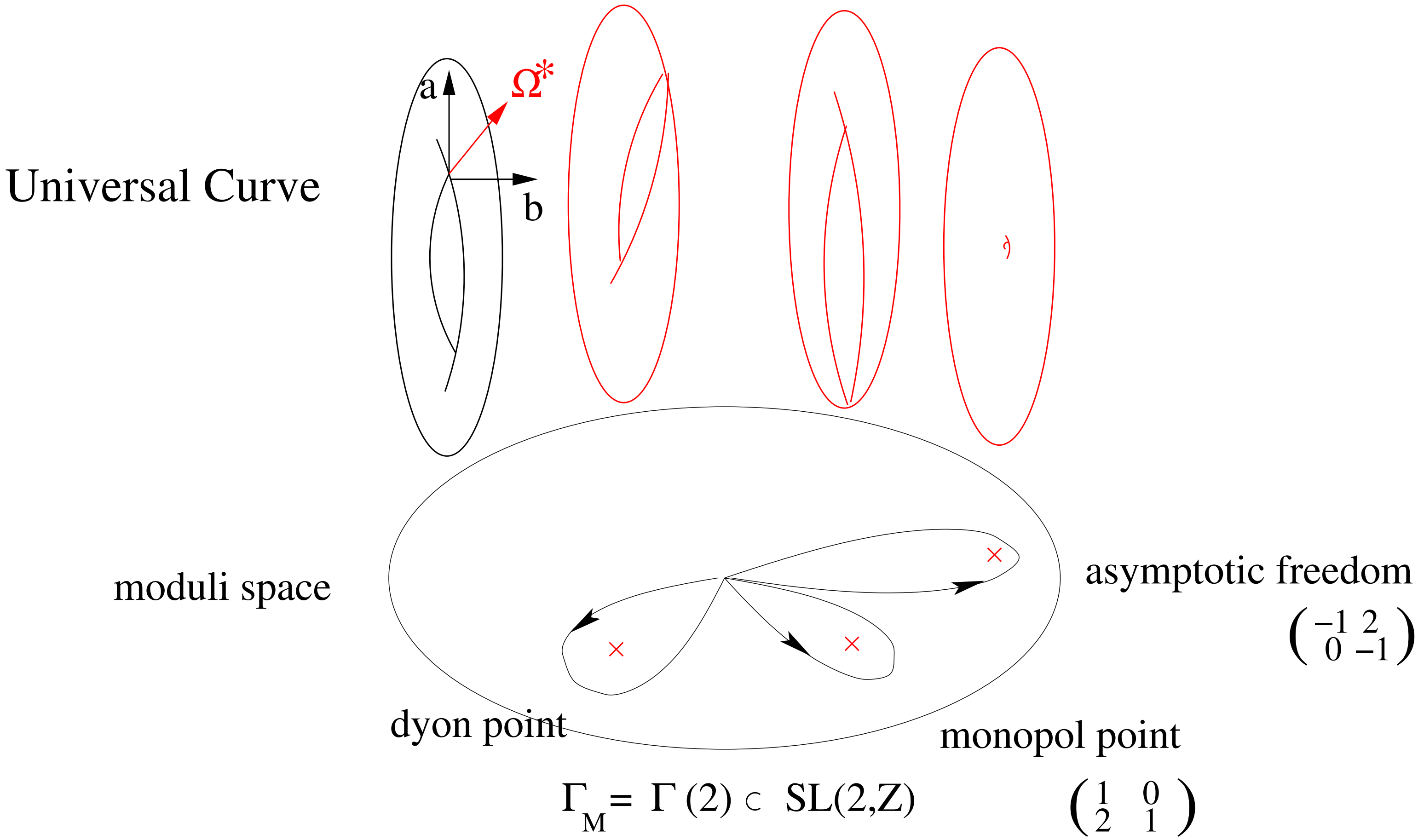
In general with matter one has the curves

$$y^2 = C(x)^2 - G(x) \ ,$$

where $C(x)$ and $G(x)$ are defined as

$$\begin{aligned}
N_f = 0: \quad C(x) &= x^2 - u, & G(x) &= \Lambda^4, \\
N_f = 1: \quad C(x) &= x^2 - u, & G(x) &= \Lambda^3(x+m_1), \\
N_f = 2: \quad C(x) &= x^2 - u + \tfrac{\Lambda^2}{8}, & G(x) &= \Lambda^2(x+m_1)(x+m_2), \\
N_f = 3: \quad C(x) &= x^2 - u + \tfrac{\Lambda}{4}(x + \tfrac{m_1+m_2+m_3}{2}), & & \\
G(x) &= \Lambda(x+m_1)(x+m_2)(x+m_3) & & .
\end{aligned}$$

The pure N=2 SU(2) curve is an elliptic curve with $\Gamma(2) \in \mathrm{SL}(2,\mathbb{Z})$ monodromy.

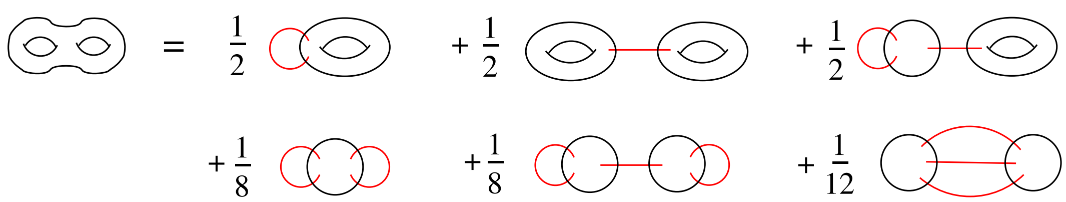

Universal Curve
a
Ω*
b
moduli space
asymptotic freedom
$\begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix}$
dyon point
monopol point
$\Gamma_M = \Gamma(2) \subset SL(2,Z)$
$\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$

Modularity and WS degenerations:

- $F_g(\tau, \bar{\tau})$ invariant under $\Gamma_M = \Gamma(2)$, e.g.

$$F_1 = -\log(\sqrt{\text{Im}(\tau)}\eta\bar{\eta})$$

- degenerations cap. by Feynmann rules:



- 'Propagator' transforms as form of weight 2 (derivative)

$$= S = \frac{\partial}{\partial \tau} 2F_1 = \frac{1}{12}\left(E_2 - \frac{3}{\pi \mathrm{Im}\tau}\right) =: \hat{E}_2$$

- $F_g(\tau, \bar{\tau}) = \xi^{2g-2} \sum_{k=0}^{3(g-1)} \hat{E}_2^k(\tau, \bar{\tau}) c_k^{(g)}(\tau) =: \xi^{2g-2} f_g \;, x$

  where $\xi = \frac{\theta_2^2}{1728\theta_3^4\theta_4^4} = \frac{1}{F_{aaa}^{(0)}}$ is of weight -3.

- Invariance means mathematically

$$\boxed{f_g \in \hat{\mathcal{M}}_{6(g-1)}(\hat{E}_2, \Delta, h)}$$

the *ring* of *almost holomorphic functions* of $\Gamma(2)$ of weight $6(g-1)$ *finitely generated* by

$$(\hat{E}_2, h = \theta_2^4 + 2\theta_4^4, \Delta = \theta_3^4\theta_4^4) \ .$$

Modular origin of the homolomorphic anomaly

Example $\Gamma = PSL(2,\mathbb{Z})$. Ring of modular forms $\mathcal{M}[E_4, E_6]$ generated by E_4 and E_6.

$$\tau \to \tau_\gamma = \frac{a\tau + b}{c\tau + d}$$

$$E_k = \frac{1}{2} \sum_{\substack{m,n\in\mathbb{Z} \\ (m,n)\neq 0}} \frac{1}{(m\tau + n)^k}$$

$$E_k(\tau_\Gamma) = (c\tau + d)^k E_k(\tau)$$

Converges for $k > 2$. However we need a ring on which we can differentiate. It is easy to see that the differential operator $\frac{d}{d\tau}$ is of weight 2.

$k = 2$ is a borderline case as far as convergence is concerned, which can be regularized

$$E_2 = \frac{1}{2} \sum_{n\neq 0} \frac{1}{2} + \frac{1}{2} \sum_{m\neq 0} \sum_{n\in\mathbb{Z}} \frac{1}{(m\tau + n)^2}$$

Breaks the symmetry

$$E_2(\tau_\Gamma) = (c\tau + d)^2 E_2(\tau) - \pi i c(c\tau + d) \ .$$

But it can be restored by defining

$$\hat{E}_2(\tau) = E_2(\tau) - \frac{3}{\pi \mathrm{Im}(\tau)} \ .$$

Now $\mathcal{M}[\hat{E}_2, E_4, E_6]$ is a ring of almost holomorphic forms on which we can differentiate!

Direct integration:

The only antiholomorphic dependence is in the $S \propto \hat{E}_2$:

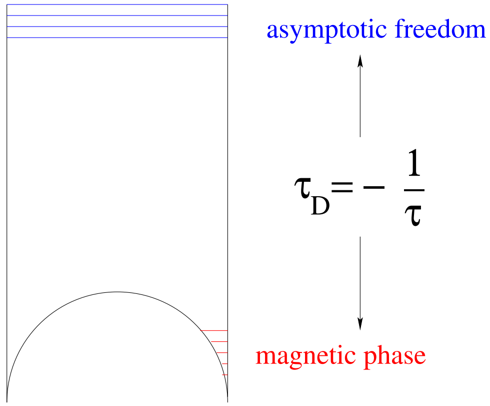
$\frac{\partial}{\partial\bar{\tau}} \longrightarrow \frac{\partial}{\hat{E}_2}$:

$$\frac{1}{24^2}\frac{\mathrm{d}}{\mathrm{d}\hat{\mathrm{E}}_2}f_g = d_\xi^2 f_{g-1} + \frac{1}{3}\frac{(\partial_\tau\xi)}{\xi}d_\xi f_{g-1} + \sum_{r=1}^{g-1} d_\xi f_r d_\xi f_{g-r},$$
$$\text{with} \quad d_\xi f_k = \partial_\tau f_k + \frac{k}{3}\frac{(\partial_\tau\xi)}{\xi}f_k \quad \text{Serre operator}$$

- Only the degree 0 part in $\hat{E}_2$ remains undetermined. Ambiguity is a holomorphic modular form $c_0^{(g)}(\tau)\in\mathcal{M}_{6(g-1)}(\Delta, h)$.

- $\dim\left(\mathcal{M}_{6(g-1)}(h,\Delta)\right) = \left[\frac{3g}{2}\right]$ number of required boundary conditions

Global properties:



$$\mathbb{F}(\Gamma(2))$$

$$F_g^D(\tau_D,\bar\tau_D)=F_g(-\frac{1}{\tau_D},-\frac{1}{\bar\tau_D})$$

- ST-instanton expansion

$$\mathcal{F}_g(\tau(a))=lim_{\bar\tau\to\infty}F_g(\tau,\bar\tau)$$

- Strong-coupling expansion

$$\mathcal{F}_g^D(\tau_D(a_D))=lim_{\bar\tau_D\to\infty}F_g^D(\tau_D,\bar\tau_D)$$

Can be seen as metaplectic transformation on $\Psi=Z$

The strong coupling gap :

$$\mathcal{F}_g^D = \frac{B_{2g}}{2g(2g-2)a_D^{2g-2}} + \ldots + k_1^{(g)} a_D + \mathcal{O}(a_D^2)$$

$$\uparrow$$

$2g-2$ independent vanishing conditions

$$\boxed{2g-2 > \left[\frac{3g}{2}\right]}$$

- theory completly solved

Conclusions:

- The gap conditions provides enough boundary to solve
  - non-conformal Seiberg-Witten theories with matter HK.
  - topological string theory on local toric Calabi-Yau BRK
- The all genus solution is constructed all over the moduli space
  - For local toric Calabi-Yau this allows to calculate orbifold GW-invariants. Recently checked using

equivariant GW theory by Ruan and Chiodo

– For Seiberg-Witten theory it allows to solve the theory at the special conformal points (Argyres, Douglas, Plesser, Seiberg, Witten)