

Quantum Symmetries And Marginal Deformations

Konstantinos Zoubos

Niels Bohr Institute

Nordic String Theory Meeting 2009

Jacobs University, Bremen

27/03/09

Based on arXiv: 0811.3755 with T. Månsson

$\mathcal{N} = 4$ Super–Yang–Mills

- $\mathcal{N} = 4$ SYM has played a fundamental role in the study of gauge and string theory
- It is the unique four–dimensional gauge theory with maximal global supersymmetry (16 supercharges)
- Extremely interesting properties:
 - ▶ Perturbative Finiteness [Ferrara,Zumino],[Grisaru et al. '80],[West,Sohnius '81][Stelle '81],[Brink et al. '83],[Mandelstam '83]
 - ▶ Nonperturbative Finiteness [NSVZ '86]
⇒ 4d Superconformal Field Theory
 - ▶ AdS/CFT Correspondence [Maldacena '97]
 - ▶ Planar Integrability [Minahan,Zarembo '02], many others
 - ▶ Planar Amplitudes [Anastasiou et al. '04], [Bern et al. '05], [Alday,Maldacena '07]
- But is it the unique theory with these features?
- Are there theories which share only *some* of these features?
- How does the knowledge accumulated for $\mathcal{N} = 4$ help in understanding more realistic theories?

Marginal Deformations of $\mathcal{N} = 4$ SYM

- Look for theories as close as possible to $\mathcal{N} = 4$ SYM
- Preserve conformal invariance \Rightarrow Marginal Deformations
- Focus on superpotential deformations $\Rightarrow \mathcal{N} = 1$ SUSY
- Write the $\mathcal{N} = 4$ SYM action in $\mathcal{N} = 1$ superspace

$$\mathcal{L} = \int d^4\theta \text{Tr} e^{gV} \bar{\Phi}_i e^{-gV} \Phi^i + \left(\int d^2\theta \mathcal{W} + \int d^2\bar{\theta} \bar{\mathcal{W}} \right) + \dots$$

- Chiral Superfields $\Phi^i = \phi^i + \theta^\alpha \psi_\alpha^i + \theta^2 F^i$, $i = 1, 2, 3$
- $\mathcal{N} = 4$ superpotential:

$$\mathcal{W} = g \text{Tr} \Phi^1 [\Phi^2, \Phi^3] = \frac{g}{3} \epsilon_{ijk} \text{Tr} \Phi^i \Phi^j \Phi^k$$

- Most general classically marginal deformation:

$$\delta \mathcal{W} = h_{ijk} \text{Tr} \Phi^i \Phi^j \Phi^k,$$

where h_{ijk} is a symmetric tensor

Exactly Marginal Deformations

- Not all of these deformations are *exactly marginal*
- Perturbative approaches [Parkes, West '84],[Jones,Mezincescu '84]
- Leigh and Strassler ('95) provided a non-perturbative proof using the NSVZ β function
- The Leigh-Strassler theories are defined by:

$$\mathcal{W}_{LS} = \kappa \text{Tr} \left(\Phi^1 [\Phi^2, \Phi^3]_q + \frac{h}{3} \left((\Phi^1)^3 + (\Phi^2)^3 + (\Phi^3)^3 \right) \right)$$

- q -commutator $[X, Y]_q = XY - qYX$
- Finite if $f(g, \kappa, q, h) = 0$, where f unknown in general
- 1-loop finiteness condition

$$2g^2 = \kappa \bar{\kappa} \left[\frac{2}{N^2} (1+q)(1+\bar{q}) + \left(1 - \frac{4}{N^2} \right) (1 + q\bar{q} + h\bar{h}) \right]$$

- Recover $\mathcal{N} = 4$ SYM for $q = 1, h = 0$
- An interesting case: $q = e^{i\beta}, h = 0$ ("Real β deformation")

$\mathcal{N} = 4$ SYM vs. Leigh–Strassler?

- How do the LS theories compare with $\mathcal{N} = 4$ SYM?

| | $\mathcal{N} = 4$ SYM | Leigh–Strassler |
|-----------------------|-----------------------|---|
| Conformally Invariant | ✓ | ✓ ^(*) |
| AdS/CFT dual | ✓ | real β [Lunin,Maldacena'05] |
| Planar Integrability | ✓ | basically real β ([†]) |
| Planar Amplitudes | ✓ | real β [Khoze'05] |

(*) Recent controversy over higher–loop finiteness [Elmetti et al. '06,'07], [Rossi et al. '05,'06]

([†]) Berenstein&Cherkis ('04) showed mismatch of LS deformation with integrable deformation of the spin chain for complex β . However one–loop integrability persists in a particular sector for complex β [Månsson '07]. A few other integrable choices are known.

- What makes the real β deformation so special?
- Take a closer look at the symmetries
- Work at the level of the classical lagrangian

Symmetries: $\mathcal{N} = 4$ SYM

- 4d Superconformal group: PSU(2, 2|4)
- Focus on the R-symmetry subgroup SU(4) \sim SO(6)
- In $\mathcal{N} = 1$ superspace notation, the $\mathcal{N} = 4$ theory has manifest SU(3) \times U(1)_R symmetry

$$\mathcal{W} = g \text{Tr} \Phi^1 [\Phi^2, \Phi^3] = \frac{g}{3} \epsilon_{ijk} \text{Tr} \Phi^i \Phi^j \Phi^k$$

- ϵ_{ijk} is the invariant tensor of SU(3)

$$\epsilon_{ijk} U^i_l U^j_m U^k_n = (\det U) \epsilon_{lmn} = \epsilon_{lmn}$$

- Transforming $\Phi^i \rightarrow U^i_j \Phi^j$ leaves the superpotential invariant
- Note that SL(3) would be enough for the superpotential, SU(3) comes from the kinetic term $\sim \bar{\Phi}_i \Phi^i$

Symmetries: Leigh–Strassler

- The generic LS deformation breaks $SU(3)$ to a discrete subgroup

$$\mathcal{W}_{LS} = \kappa \text{Tr} \left(\Phi^1 [\Phi^2, \Phi^3]_q + \frac{\hbar}{3} ((\Phi^1)^3 + (\Phi^2)^3 + (\Phi^3)^3) \right)$$

- This superpotential has the following \mathbb{Z}_3 symmetries:

$$\mathbb{Z}_3^A : \Phi^1 \rightarrow \Phi^2 \quad , \quad \Phi^2 \rightarrow \Phi^3 \quad , \quad \Phi^3 \rightarrow \Phi^1$$

$$\mathbb{Z}_3^B : \Phi^1 \rightarrow \omega \Phi^1 \quad , \quad \Phi^2 \rightarrow \omega^2 \Phi^2 \quad , \quad \Phi^3 \rightarrow \Phi^3 \quad (\omega^3 = 1)$$

- Together with a third \mathbb{Z}_3 within $U(1)_R$ ($\Phi^j \rightarrow \omega \Phi^j$), they form a trihedral group known as Δ_{27} [Aharony et al. '02]
- For real β the symmetry group is enhanced to $U(1)^3$
- Is this all?

More symmetry?

- Let us naively rewrite \mathcal{W}_{LS} as

$$W_{LS} = \frac{1}{3} E_{ijk} \text{Tr} \phi^i \phi^j \phi^k$$

where

$$E_{123} = E_{231} = E_{312} = \kappa ,$$

$$E_{321} = E_{213} = E_{132} = -\kappa q ,$$

$$E_{111} = E_{222} = E_{333} = \kappa h$$

- Similarly $F^{ijk} = \bar{E}_{ijk}$ defines $\bar{\mathcal{W}} = \frac{1}{3} F^{ijk} \text{Tr} \bar{\phi}_i \bar{\phi}_j \bar{\phi}_k$
- Would like to find some t_j^i such that $\phi^i \rightarrow t_j^i \phi^j$ is a symmetry, i.e.

$$E_{ijk} t_j^l t_m^j t_n^k = E_{lmn}$$

- Clearly there exists no Lie group with this property...
 \Rightarrow Quantum Groups

An Example: The Manin Plane

- A simpler setting that illustrates the main ideas
- $x^1, x^2 \in V$, where V is a noncommutative space

$$x^1 x^2 = \frac{1}{q} x^2 x^1$$

- Coordinates of a 2-dimensional quantum plane
- These commutation relations can be obtained from a matrix $R : V \otimes V \rightarrow V \otimes V$

$$R = q^{-\frac{1}{2}} \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 1 & q - q^{-1} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & q \end{pmatrix},$$

via the relation $R_{k,l}^j x^k x^l = q^{\frac{1}{2}} x^j x^i$ (or $R_{12} x_1 x_2 = q^{\frac{1}{2}} x_2 x_1$)

- Think of R as acting on basis $\{|11\rangle, |12\rangle, |21\rangle, |22\rangle\}$
(e.g. $R_{21}^1 = q^{\frac{1}{2}} - q^{-\frac{3}{2}}$)

Quantum Plane Symmetries

- The function $f(x^1, x^2) = x^1 x^2 - q^{-1} x^2 x^1$ is invariant under

$$x^i \rightarrow t^i_j x^j$$

if the matrix t^i_j satisfies the FRT, or RTT relations:

$$R^i_k t^a_j t^b_l = t^k_b t^i_a R^{a b}_{j l} \quad (R_{12} t_1 t_2 = t_2 t_1 R_{12})$$

- Writing $t = \begin{pmatrix} t^1_1 & t^1_2 \\ t^2_1 & t^2_2 \end{pmatrix}$, we find

$$t^1_1 t^1_2 = q^{-1} t^1_2 t^1_1, \quad t^1_1 t^2_1 = q^{-1} t^2_1 t^1_1,$$

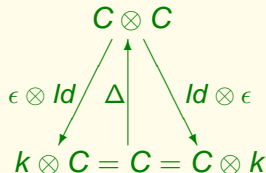
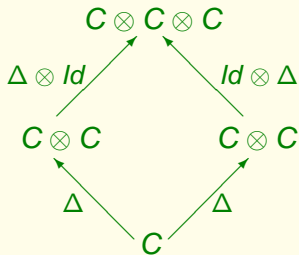
$$t^2_1 t^2_2 = q^{-1} t^2_2 t^2_1, \quad t^2_1 t^2_2 = q^{-1} t^2_2 t^2_1,$$

$$t^1_2 t^2_1 = t^2_1 t^1_2, \quad t^1_1 t^2_2 - t^2_2 t^1_1 = (q^{-1} - q) t^1_2 t^2_1,$$

- The elements of t are noncommutative!
- These commutation relations define a *quantum group*

Quantum Groups

- What are quantum groups? Some definitions
- Recall an *algebra* $(\mathcal{C}, +, \cdot, \eta; k)$ is a vector space together with a product $\cdot : \mathcal{C} \otimes \mathcal{C} \rightarrow \mathcal{C}$ and a unit map $\eta : k \rightarrow \mathcal{C}$
- A *coalgebra* $(\mathcal{C}, +, \Delta, \epsilon; k)$ is instead equipped with a coproduct $\Delta : \mathcal{C} \rightarrow \mathcal{C} \otimes \mathcal{C}$ and a counit $\epsilon : \mathcal{C} \rightarrow k$



- A *bialgebra* $(H, +, \cdot, \eta, \Delta, \epsilon; k)$ is both an algebra and a coalgebra in a compatible way

Hopf Algebras

- A Hopf Algebra is a bialgebra equipped with an antipode $S : \mathcal{C} \rightarrow \mathcal{C}$

$$\cdot (S \otimes \text{id}) \circ \Delta = \cdot (\text{id} \otimes S) \circ \Delta = \eta \circ \epsilon .$$

- We are working in the Quantum Matrix Algebra picture, where the coproduct and counit are

$$\Delta g_n^m = \sum_k g_k^m \otimes g_n^k, \quad \epsilon g_n^m = \delta_n^m$$

and the antipode s satisfies : $t_k^i s_j^k = \delta_j^i = s_k^i t_j^k$

- In this picture, it is the *product* whose noncommutativity is controlled by the matrix R through the RTT relations
- For $R = I$ we are left with a Lie Algebra
- Dual picture \Rightarrow Universal Enveloping Algebra

Quasitriangular Hopf Algebras

- In a *quasi-triangular* Hopf algebra, the matrix R controlling noncommutativity satisfies the Quantum Yang–Baxter Equation (QYBE) (but note without spectral parameter):

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$$

(in index notation: $R_{s r}^j R_{l p}^s R_{m n}^k R_{r m}^p = R_{s p}^j R_{r n}^k R_{l m}^s R_{r m}^p$)

- Among other things, this condition guarantees that the resulting algebra is not too trivial
- We call a quasitriangular Hopf algebra a *quantum group*
- The matrix R defining the Manin plane does satisfy QYBE!
- It corresponds to the quantum group $SU_q(2)$

Why $SU_q(2)$ and not $GL_q(2)$?

- We have constructed the quantum group $SU_q(2)$ as the invariance group of the Manin plane
- The quantum determinant $\mathbb{D} := t_1^1 t_2^2 - q^{-1} t_2^1 t_1^2$ is *central* and we can set it equal to one $\implies SL_q(2)$
- There is an analogous construction for the coplane $u_1 u_2 = q u_2 u_1$ ($u_i \in V^*$) defined by

$$q^{\frac{1}{2}} u_a u_b = u_j u_i R_{ba}^{ji}$$

- Compatibility of the plane and coplane imposes a reality condition on the matrix R :

$$\overline{R_{kl}^{ij}} = R_{ji}^{lk} \quad (\hat{R} = PR \text{ hermitian})$$

- This lets us define $t_j^{i*} = s_j^i \implies SU_q(2)$

Three–Dimensional Quantum Planes

- Apply these ideas to the Leigh–Strassler theories!
- Note the F –term conditions: [Berenstein et al. '00]

$$\phi^1 \phi^2 = q \phi^2 \phi^1 - h(\phi^3)^2$$

$$\phi^2 \phi^3 = q \phi^3 \phi^2 - h(\phi^1)^2$$

$$\phi^3 \phi^1 = q \phi^1 \phi^3 - h(\phi^2)^2$$

(non–commutative moduli space)

- The three scalars will play the role of the quantum plane coordinates: $\phi^i \rightarrow x^i$
- Can think of the \mathbb{C}^3 transverse to the $D3$ –branes becoming noncommutative
- We need to examine the symmetries of three–dimensional quantum planes

Quantum deformations of $GL(3)$

- Ewen & Ogievetsky ('94) classified quantum deformations of $GL(3)$, and the corresponding quantum planes
- Starting point were the q -epsilon tensors E_{ijk} and their duals F^{ijk}
- Given the following conditions

$$\delta_j^i = \frac{1}{2} E_{jmn} F^{mni} \quad \text{and} \quad E_{ajm} F^{mib} E_{ebk} F^{kcj} = \delta_a^c \delta_e^i + \delta_a^i \delta_e^c$$

they show that

$$\hat{R}_{kl}^{ij} = \delta_k^i \delta_l^j - E_{kln} F^{nij}$$

satisfies QYBE and defines a quantum plane through

$$\hat{R}_{12} x_1 x_2 = x_1 x_2$$

LS as a quantum symmetry deformation

- Can the E&O approach be applied to LS?
- The desired q -epsilon tensors are:
 $E_{123} = \kappa$, $E_{132} = -\kappa q$, $E_{111} = \kappa h$ (+cyclic), $F^{ijk} = \bar{E}_{ijk}$
- Imposing the first E&O condition ($\delta_j^i = \frac{1}{2} E_{jmn} F^{mni}$) we find
 $\kappa \bar{\kappa} = 1/(1 + q\bar{q} + h\bar{h}) \Rightarrow$ planar 1-loop finiteness condition!
- $H_{kl}^{ij} = E_{kln} F^{nij}$ is the 1-loop spin chain Hamiltonian [Roiban '03]

$$H_{l,l+1} = \frac{1}{1 + q\bar{q} + h\bar{h}} \begin{pmatrix} h\bar{h} & 0 & 0 & 0 & 0 & \bar{h} & 0 & -\bar{h}q & 0 \\ 0 & 1 & 0 & -q & 0 & 0 & 0 & 0 & h \\ 0 & 0 & q\bar{q} & 0 & -h\bar{q} & 0 & -\bar{q} & 0 & 0 \\ 0 & -\bar{q} & 0 & q\bar{q} & 0 & 0 & 0 & 0 & -h\bar{q} \\ 0 & 0 & -\bar{h}q & 0 & h\bar{h} & 0 & \bar{h} & 0 & 0 \\ h & 0 & 0 & 0 & 0 & 1 & 0 & -q & 0 \\ 0 & 0 & -q & 0 & h & 0 & 1 & 0 & 0 \\ -h\bar{q} & 0 & 0 & 0 & 0 & -\bar{q} & 0 & q\bar{q} & 0 \\ 0 & \bar{h} & 0 & -\bar{h}q & 0 & 0 & 0 & 0 & h\bar{h} \end{pmatrix}$$

- Hermitian, cyclic: $H_{kl}^{ij} = H_{(k+1)(l+1)}^{(i+1)(j+1)}$ etc.
- Define $\hat{R}_{kl}^{ij} = \delta_k^i \delta_l^j - H_{kl}^{ij} \implies R_{kl}^{ij} = \hat{R}_{kl}^{ji}$

Does it work?

- Given R , the RTT relations will produce a bialgebra $\mathcal{A}(R)$
- But our R is *not* part of the E&O classification
- In particular, R does not satisfy QYBE!
(apart from special cases, e.g. real β)
- Differences from E&O:
 - ▶ We have not imposed the second E&O condition
 - ▶ We are interested in a cyclic quantum plane structure (while E&O look at ordered planes, e.g. $x^i x^j = q x^j x^i$, $i < j$)
- We cannot have a quasitriangular Hopf Algebra, but is it still a Hopf Algebra?
- Need careful analysis of the RTT relations
- Two main new features:
 - Possibility of no (nontrivial) solutions
 - Associativity will imply higher relations

Solving the RTT relations

- When R satisfies QYBE, we are guaranteed that

$$R_{a b}^{i k} t_j^a t_l^b = t_b^k t_a^i R_{j l}^{a b}$$

has nontrivial solutions for t_j^i

- In our case we need to explicitly show that out of the 81 equations, only 36 are independent
- This turns out to be the case!
- Quadratic commutation relations of $\mathcal{A}(R)$:

$$(a) \quad t_c^a t_c^{a+1} - q t_c^{a+1} t_c^a + h t_c^{a-1} t_c^{a-1} = h (t_{c+1}^a t_{c-1}^{a+1} - \bar{q} t_{c-1}^a t_{c+1}^{a+1} + \bar{h} t_c^a t_c^{a+1})$$

$$(b) \quad q[t_{c+1}^{a+1}, t_c^a] = -q^2 t_{c+1}^{a+1} t_c^a + h q t_{c+1}^{a-1} t_c^{a-1} + h t_{c+1}^{a-1} t_c^{a-1} + t_{c+1}^a t_c^{a+1}$$

$$(c) \quad -q t_c^{a+1} t_{c+1}^a + \bar{q} t_{c+1}^a t_c^{a+1} = \bar{h} t_{c-1}^a t_{c-1}^{a+1} - h t_c^{a-1} t_{c+1}^{a-1}$$

$$(d) \quad h(t_{c+1}^a t_{c-1}^a - \bar{q} t_{c-1}^a t_{c+1}^a) = \bar{h}(t_c^{a+1} t_c^{a-1} - q t_c^{a-1} t_c^{a+1})$$

Associativity

- The QYBE also guarantees no new relations arise at higher levels
- In our case, associativity leads to new cubic relations

$$a) R_{12}R_{13}R_{23}t_1t_2t_3 = R_{12}R_{13}t_1t_3t_2R_{23} = R_{12}t_3t_1t_2R_{13}R_{23} = t_3t_2t_1R_{12}R_{13}R_{23}$$

$$b) R_{23}R_{13}R_{12}t_1t_2t_3 = R_{23}R_{13}t_2t_1t_3R_{12} = R_{23}t_2t_3t_1R_{13}R_{12} = t_3t_2t_1R_{23}R_{13}R_{12}$$

- These relations would be the same if QYBE were satisfied, but now they have to be imposed to guarantee associativity
- Danger is that they will trivialise the quantum determinant

$$\begin{aligned} \mathbb{D} &= \frac{1}{6} E_{ijk} t_i^j t_m^k t_m^l F^{lmn} \\ &= t_1^1 t_2^2 t_3^3 - q t_1^2 t_2^1 t_3^3 + h t_1^3 t_2^3 t_3^3 + t_1^3 t_2^1 t_3^2 - q t_1^1 t_2^3 t_3^2 + h t_1^2 t_2^2 t_3^2 \\ &\quad + t_1^2 t_2^3 t_3^1 - q t_1^3 t_2^2 t_3^1 + h t_1^1 t_2^1 t_3^1 \end{aligned}$$

- We have checked that this is not the case
- \mathbb{D} is nontrivial and central \Rightarrow Can set $\mathbb{D} = 1$

The Quantum Symmetry Algebra

- We have also shown that there exists an antipode

$$s_{1+k}^{1+i} = t_{2+i}^{2+k} t_{3+i}^{3+k} - \bar{q} t_{3+i}^{2+k} t_{2+i}^{3+k} + \bar{h} t_{1+i}^{2+k} t_{1+i}^{3+k} = t_{2+i}^{2+k} t_{3+i}^{3+k} - q t_{2+i}^{3+k} t_{3+i}^{2+k} + h t_{2+i}^{1+k} t_{3+i}^{1+k}.$$

- The bialgebra $\mathcal{A}(R)$ is thus a Hopf algebra
- We have found a Hopf algebra underlying the general Leigh–Strassler deformation
 - Transform $\Phi^i \rightarrow t_j^i \Phi^j$, $t \in \mathcal{A}(R)$
 - $\mathbb{D} = 1$ guarantees invariance of the superpotential:

$$\mathcal{W} = \frac{1}{3} E_{ijk} \text{Tr} \Phi^i \Phi^j \Phi^k \implies E_{ijk} t_l^i t_m^j t_n^k = \mathbb{D} E_{lmn}.$$

- The antipode guarantees invariance of the kinetic terms

$$\bar{\Phi}_i \Phi^j \rightarrow \bar{\Phi}_j t_i^j {}^* t_k^i \Phi^k = \bar{\Phi}_j \delta_k^j \Phi^k$$

- The full Leigh–Strassler lagrangian is invariant under $\mathcal{A}(R)$
- The \mathbb{Z}_3 's appear as automorphisms of $\mathcal{A}(R)$

Integrable Cases

- The Hopf algebra $\mathcal{A}(R)$ becomes quasitriangular for special choices of (q, h)
- All known integrable deformations of $\mathcal{N} = 4$ can be obtained in this way
- Can show that they arise as Hopf algebra twists of the real β case

$$R_\beta = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & q^{-1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & q & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & q & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & q^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & q^{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & q & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

(this is also a twist of the $\mathcal{N} = 4$ case [Beisert, Roiban '05])

- E.g. $q = 0, \bar{h} = 1/h$ can be obtained by

$$R' = \mathcal{F}_{21} R_\beta \mathcal{F}^{-1}, \quad \mathcal{F} = U \otimes U^2$$

where

$$U = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

Relation to Noncommutativity

- It is known that the real β deformation can be described by a star product [Berenstein et al. '00], [Kulaxizi, Zoubos '04], [Lunin, Maldacena '05]
- Non(–anti)commutativity on open string side leads to NS (and RR) fields on closed string side [Schomerus '99], [Seiberg, Witten '99]
- Can try to apply these ideas to construct the dual gravity background to the Leigh–Strassler theories [Kulaxizi '06]
- Works fine for real β , but problems with associativity for general case. Still, a solution was found to third order
- The resulting noncommutativity relations

$$[z^i, z^j]_* = i\beta\Theta_{kl}^{ij}z^kz^l, \quad [z^i, \bar{z}^j]_* = i\beta\Theta_{k\bar{l}}^{i\bar{j}}z^k\bar{z}^l, \quad [\bar{z}^i, \bar{z}^j]_* = i\beta\Theta_{\bar{k}\bar{l}}^{i\bar{j}}\bar{z}^k\bar{z}^l$$

can be mapped to our (extended) quantum plane relations

$$R^i{}_k R^j{}_l x^k x^l = x^j x^i, \quad u_k u_l R^k{}_i R^l{}_j = u_j u_i, \quad u_l R^j{}_k R^l{}_i x^k = x^j u_i, \quad x^k \tilde{R}^i{}_k R^l{}_j u_l = u_j x^i$$

by expanding $R = I + \rho r + O(\rho^2)$, ($\rho = \beta, \hbar$)

- r is the classical r -matrix

Summary

- We have exhibited a Hopf algebra structure underlying the general Leigh–Strassler deformation
- The $SU(3) \times U(1)$ R–symmetry of $\mathcal{N} = 4$ is not broken, it is q –deformed to $\mathcal{A}(R) \times U(1)$
- This algebra appears to be a new deformation of $SU(3)$
- This quantum symmetry appears at the level of the classical Lagrangian
- It is also a symmetry of the 1–loop spin chain Hamiltonian

$$R_{12}t_1t_2 = t_2t_1R_{12} \Rightarrow \hat{R}_{12}t_1t_2 = t_1t_2\hat{R}_{12} \Rightarrow (t_2)^{-1}(t_1)^{-1}H_{12}t_1t_2 = H_{12}$$

- It reduces to known structures: Quasitriangular Hopf for integrable cases, star products at first order

Still To Do

- Mathematical side

- ▶ Better understanding of the algebra $\mathcal{A}(R)$ (e.g. higher order relations)
- ▶ Classification of such algebras?
- ▶ Could $\mathcal{A}(R)$ be reformulated as a (non-associative) quasi-Hopf algebra? [Drinfel'd '89], [Mack, Schomerus '92]
- ▶ Add spectral parameter dependence?
- ▶ Are there other integrable deformations?

- Physics side

- ▶ What happens at quantum level?
- ▶ Regularisation at higher loops
- ▶ Construction of dual backgrounds
- ▶ Is there a relation between perturbative finiteness and quantum symmetry?