1 Vector fields, flows and Lie derivatives

Coordinates

Consider an \( n \)-dimensional manifold \( M \). Let \( x^\mu(p) \) denote local coordinates for \( p \in M \) in a subset of \( M \) \((\mu = 1, 2, \ldots, n)\). I.e. \((x^1(p), x^2(p), \ldots, x^n(p))\) can be seen as a map from the subset of \( M \) to \( \mathbb{R}^n \).

Consider a map \( f \) from \( M \) to \( M \), i.e. \( f: M \to M \). Then we can write \( f(p) \), with \( p \in M \), using local coordinates \( x^\mu \) as \( f^\mu(p) \), i.e. \( f^\mu(p) \) are the components of \( f(p) \) in the local coordinates. In other words the translation of the map \( f: M \to M \) to a map \( \mathbb{R}^n \to \mathbb{R}^n \), given the local coordinate system \( x^\mu \), is written as \( f^\mu(x^1, x^2, \ldots, x^n) \).

Vector fields

We define a vector field \( V \) to be any smooth linear map from \( C^\infty(M) \) \( (\) the space of real-valued smooth functions on \( M \)\( ) \) to \( C^\infty(M) \). Let \( f: M \to \mathbb{R} \) be a (smooth) real-valued function on \( M \) then \( V \) on \( f \) is written as \( V[f] \). So given the function \( f \) the vector field \( V \) returns the function \( V[f] \).

For a point \( p \in M \) the vector field \( V \) for a given function \( f \) takes the value \( V[f](p) \). Seen as a function of \( f \) you have that \( V[f](p) \) is a linear map from \( C^\infty(M) \) to \( \mathbb{R} \). The collection of such linear maps for a given point \( p \in M \) defines the tangent space \( T_p(M) \).

The components \( V^\mu \) of a vector field \( V \) in the local coordinate system \( x^\mu \) are given by

\[
V[f] = V^\mu \frac{\partial f}{\partial x^\mu}
\]

In the point \( p \in M \) this is written

\[
V[f](p) = V^\mu(p) \frac{\partial f}{\partial x^\mu}(p)
\]

Abstractly, we write that

\[
V^\mu(p) \frac{\partial}{\partial x^\mu}(p) \in T_p(M)
\]

In particular,

\[
\frac{\partial}{\partial x^\mu}(p) \in T_p(M)
\]
gives a basis for the tangent space $T_p(M)$. Along these lines we can then further think of

$$V = V^\mu \frac{\partial}{\partial x^\mu}$$

as the vector field $V$ without reference to a function in $C^\infty(M)$.

**Commuting vector fields and local coordinate systems**

Given two vector field $V$ and $W$ we can define the new vector field $[V, W]$, called the commutator of $V$ and $W$, by

$$[V, W][f] = V[W[f]] - W[V[f]]$$

(1.6)

One can easily see that in terms of components we have

$$[V, W] = V^\nu \frac{\partial W^\mu}{\partial x^\nu} - W^\nu \frac{\partial V^\mu}{\partial x^\nu}$$

(1.7)

given local coordinates $x^\mu$. Using this we see that

$$\left[ \frac{\partial}{\partial x^\mu}, \frac{\partial}{\partial x^\nu} \right] = 0$$

(1.8)

Thus, the vector fields $\partial/\partial x^\mu$ associated to the coordinates $x^\mu$ all commute with each other.

One can prove the converse: If a set of $k$ linearly independent vector fields $V_i$, $i = 1, ..., k$ and $k \leq n$, all mutually commute with each other, i.e.

$$[V_i, V_j] = 0$$

(1.9)

for all $i, j = 1, 2, ..., k$, then it is possible to find a local coordinate system $x^\mu$ such that for $\mu = 1, ..., k$ we have

$$V(\mu) = \frac{\partial}{\partial x^\mu}$$

(1.10)

**The induced map**

Let $\Phi : M \to M$ be a map from $M$ into itself. For a given point $p \in M$ we can then define the induced map $D_p\Phi : T_p(M) \to T_{\Phi(p)}(M)$ as follows. Writing

$$V' = D_p\Phi(V)$$

(1.11)

we define the induced map as

$$V'[f](\Phi(p)) = V[f \circ \Phi](p)$$

(1.12)

In coordinates we have that

$$V'[f] = V^\mu \frac{\partial \Phi^\nu}{\partial x^\mu} \frac{\partial f}{\partial x^\nu}$$

(1.13)
Flows

For any vector field $V$ on $M$ there exists a smooth map $\sigma : \mathbb{R} \times M \to M$ called the flow of $V$, written here as $\sigma_t(p)$ with $t \in \mathbb{R}$ and $p \in M$, such that $\sigma_0(p) = p$, $\sigma_t(\sigma_s(p)) = \sigma_{t+s}(p)$ and

$$\frac{d}{dt}\sigma_t = V(\sigma_t(p)) \quad (1.14)$$

Since $\sigma_t$ is a map $M \to M$ we can think of the components $\sigma^\mu_t(p)$ with respect to the local coordinates $x^\mu$ (see beginning of this section). Using now (1.14) we see that

$$\sigma^\mu_t(p) = x^\mu(p) + tV^\mu(p) + \mathcal{O}(t^2) \quad (1.15)$$

when $t$ is small. This relation expresses in a straightforward way that the flow along $V$ for small $t$ goes in the direction of $V$.

**Lie derivative of a function**

We now want to define the Lie derivative acting on a given function $f : M \to \mathbb{R}$. The Lie derivative is defined with respect to a vector field $V$. When acting on a function $f$ it quantifies how much $f$ changes along the flow of $V$. We should therefore define it using the difference $f(\sigma_t(p)) - f(p)$, i.e. the difference between $f$ in the point $p$ and the translated point $\sigma_t(p)$. In that way we can get a measure of the change of $f$ in the direction of the flow of $V$.

We define therefore the Lie derivative $\mathcal{L}_V f$ of the function $f : M \to \mathbb{R}$ along the vector field $V$ as

$$\mathcal{L}_V f(p) = \lim_{t \to 0} \frac{1}{t} \left[ f(\sigma_t(p)) - f(p) \right] \quad (1.16)$$

for any point $p \in M$ where $\sigma_t$ is the flow of $V$.

Using (1.15) we see that for small $t$

$$f(\sigma_t(p)) = f(p) + tV^\mu(\sigma_t(p)) \frac{\partial f}{\partial x^\mu}(p) + \mathcal{O}(t^2) \quad (1.17)$$

In the local coordinate system $x^\mu$ we have therefore

$$\mathcal{L}_V f = V^\mu \frac{\partial f}{\partial x^\mu} \quad (1.18)$$

**Lie derivative of a vector field**

We now want to define the Lie derivative of a vector field $W$. The Lie derivative is defined with respect to a vector field $V$. When acting on a vector field $W$ it quantifies how much $W$ changes along the flow of $V$. Naively one could imagine taking the difference $W(\sigma_t(p)) - W(p)$, i.e. the difference between $W$ at $p$ and at $\sigma_t(p)$, and then look at the limit $t \to 0$. However, this is not well defined since $W(\sigma_t(p)) \in T_{\sigma_t(p)}(M)$ and $W(p) \in T_p(M)$ i.e. they are vectors in two different tangent spaces, hence one cannot subtract them from each
other. Thus, we should find a way to take \( W(\sigma_t(p)) \) and map it into a vector in the tangent space \( T_p(M) \). This can be done with the induced map \( D_{\sigma_t(p)}\sigma^{-t} : T_{\sigma_t(p)}(M) \rightarrow T_p(M) \).

We define therefore the Lie derivative \( \mathcal{L}_V W \) of the vector field \( W \) along the vector field \( V \) as

\[
\mathcal{L}_V W(p) = \lim_{t \to 0} \frac{1}{t} \left[ D_{\sigma_t(p)}\sigma^{-t}(W) - W(p) \right]
\](1.19)

for any point \( p \in M \) where \( \sigma_t \) is the flow of \( V \).

In coordinates we have

\[
\frac{\partial}{\partial x^\mu}(\sigma_t(p)) = \delta^\mu_\nu - t \frac{\partial V^\mu}{\partial x^\nu}(p) + O(t^2)
\](1.22)

We compute from this that the components of the Lie derivative \( \mathcal{L}_V W \) are

\[
(L_V W)^\mu = V^\nu \frac{\partial W^\mu}{\partial x^\nu} - W^\nu \frac{\partial V^\mu}{\partial x^\nu}
\](1.23)

We see that this is nothing but the components of the commutator \([V, W]\) of \( V \) and \( W \).

Thus we have shown that

\[
\mathcal{L}_V W = [V, W]
\](1.24)

## 2 One-forms, tensors and Killing vector fields

### The cotangent space and one-form fields

Given a point \( p \in M \) we define the cotangent space \( T^*_p(M) \) as the space of linear maps from \( T_p(M) \) to \( \mathbb{R} \). The elements of \( T^*_p(M) \) are called one-forms.

A one-form field \( A \) on \( M \) is a one-form \( A(p) \in T^*_p(M) \) for each value of \( p \in M \). I.e. for each point \( p \in M \) \( A(p) \) is a linear map from \( T_p(M) \) to \( \mathbb{R} \).

Given a local coordinate system \( x^\mu \) we now define (locally) the one-form field \( dx^\mu \) so that for each point \( p \) (in the subspace of \( M \) where the coordinate system is defined) we have

\[
dx^\mu(p)(V) = V^\mu
\](2.1)

for all \( V \in T_p(M) \).

Given a one-form field \( A \) we can now define the components \( A_\mu \) of \( A \) with respect to the coordinates \( x^\mu \) as

\[
A = A_\mu dx^\mu
\](2.2)

Given a one-form field \( A \) and a vector field \( V \) we see using the above that

\[
A(V) = A_\mu V^\mu
\](2.3)
Moreover, we find that
\[ dx^\mu \left( \frac{\partial}{\partial x^\nu} \right) = \delta^\mu_\nu \] (2.4)
which for a given one-form field \( A \) means that
\[ A \left( \frac{\partial}{\partial x^\mu} \right) = A_\mu \] (2.5)

**Tensor fields**

One can furthermore generalize the vector fields and one-form fields to tensor fields. We will not define tensors in a precise way but merely note that an \((r, s)\)-tensor field \( T \) can be written in terms of components as
\[ T = T^{\mu_1 \mu_2 \ldots \mu_r}_{\nu_1 \nu_2 \ldots \nu_s} \frac{\partial}{\partial x^{\mu_1}} \otimes \frac{\partial}{\partial x^{\mu_2}} \otimes \cdots \otimes \frac{\partial}{\partial x^{\mu_r}} \otimes dx^{\nu_1} \otimes dx^{\nu_2} \otimes \cdots \otimes dx^{\nu_s} \] (2.6)
A \((1, 0)\)-tensor field is a vector field while a \((0, 1)\)-tensor field is a one-form field. A \((0, 2)\)-tensor field is
\[ T = T_{\mu\nu} dx^\mu \otimes dx^\nu \] (2.7)
At a point \( p \in M \) this can be thought of as a linear map from \( T_p(M) \times T_p(M) \) into \( \mathbb{R} \). I.e.
\[ T(V, W) = T_{\mu\nu} V^\mu W^\nu \] (2.8)
for two vector fields \( V \) and \( W \).

In particular a metric \( g \) is a \((0, 2)\)-tensor field which is symmetric, i.e.
\[ g(V, W) = g(W, V) \] (2.9)
This is equivalent to the component identity
\[ g_{\mu\nu} = g_{\nu\mu} \] (2.10)

In the following we will write the metric as
\[ ds^2 = g_{\mu\nu} dx^\mu \otimes dx^\nu = g_{\mu\nu} dx^\mu dx^\nu \] (2.11)
where the last equality is an abbreviation.

Note that a general \((r, s)\)-tensor field at a point \( p \in M \) can be thought of as a linear map from \((T_p^* (M))^r \times (T_p (M))^s \) into \( \mathbb{R} \). Thus for instance given a \((2, 3)\)-tensor field \( T \) we can write
\[ T(A, B; V, W, U) = T^\mu_{\rho\alpha} A_\mu B_\nu V^\rho W^\alpha U^\beta \] (2.12)
Lie derivatives of tensor fields

The Lie derivative of an \((r, s)\)-tensor field \(T\) along the vector field \(V\) is an \((r, s)\)-tensor with components

\[
(L_V T)_{\mu_1\mu_2...\mu_r}^{\nu_1\nu_2...\nu_s} = V^\rho \frac{\partial T^{\mu_1\mu_2...\mu_r}}{\partial x^\rho} _{\nu_1\nu_2...\nu_s}
\]  

(2.13)

\[
- \frac{\partial V^{\mu_1}}{\partial x^\rho} T^{\rho\mu_2...\mu_r}_{\nu_1\nu_2...\nu_s} - \frac{\partial V^{\mu_2}}{\partial x^\rho} T^{\mu_1\mu_3...\mu_r}_{\nu_1\nu_2...\nu_s} - \frac{\partial V^{\mu_r}}{\partial x^\rho} T^{\mu_1\mu_2...\rho}_{\nu_1\nu_2...\nu_s} 
\]  

(2.14)

\[
+ \frac{\partial V^\rho}{\partial x^{\mu_1}} T^{\mu_1\mu_2...\mu_r}_{\nu_1\nu_2...\nu_s} + \frac{\partial V^\rho}{\partial x^{\mu_2}} T^{\mu_1\mu_2...\mu_r}_{\nu_1\nu_2...\nu_s} + \ldots + \frac{\partial V^\rho}{\partial x^{\mu_s}} T^{\mu_1\mu_2...\mu_r}_{\nu_1\nu_2...\nu_s} 
\]  

(2.15)

The first term in the Lie derivative of the \((r, s)\)-tensor is the change of the value of \(T^{\mu_1\mu_2...\mu_r}_{\nu_1\nu_2...\nu_s}\) in the direction of the vector field \(V\), just as for the Lie derivative of a function (1.18). The next \(r\) terms come in the same way as for the Lie derivative of vector fields, see the second term in (1.23). The last \(s\) terms instead generalize what one would have for a one-form field (which we haven’t derived here).

Killing vector fields

The Lie derivative of a metric \(g\) along the vector field \(V\) is

\[
(L_V g)_{\mu\nu} = V^\rho \frac{\partial g_{\mu\nu}}{\partial x^\rho} + \frac{\partial V^\rho}{\partial x^\mu} g_{\rho\nu} + \frac{\partial V^\rho}{\partial x^\nu} g_{\mu\rho}
\]  

(2.16)

For a given metric \(g\) a Killing vector field \(V\) is now defined to be a vector field for which \(L_V g = 0\).

That \(L_V g = 0\) expresses mathematically that the metric \(g\) is invariant under the flow of \(V\). Thus, if we travel along the flow of a Killing vector field, the metric, and thereby also our space-time, is completely unchanged. It is therefore a mathematical way of expressing translation invariance of the metric \(g\), in a coordinate invariant fashion. In other words, if \(L_V g = 0\) in a particular coordinate system it is true in all coordinate systems.

Using the above statement on commuting vector fields (in this case just for a single vector field) we see that given a vector field \(V\) we can choose a coordinate system \(x^\mu\) such that

\[
V = \frac{\partial}{\partial x^1}
\]  

(2.17)

In this particular coordinate system we have therefore that

\[
(L_V g)_{\mu\nu} = \frac{\partial g_{\mu\nu}}{\partial x^1}
\]  

(2.18)

Thus, in this particular coordinate system \(V\) is a Killing vector field provided the metric \(g_{\mu\nu}\) is independent of \(x^1\). This shows that if a metric admits a Killing vector field then we can always find a coordinate system in which this Killing vector field correspond to an explicit independence of a particular coordinate, \(i.e.\) a translational invariance for that coordinate. The converse statement is also true in the sense that if in a particular coordinate system a metric is independent of one of the coordinates, the vector field for that coordinate is a Killing vector field.
3 Killing vector fields for particular metrics

The two-sphere

The metric of the two-sphere is
\[ ds^2 = d\theta^2 + \sin^2 \theta d\phi^2 \] (3.1)

A Killing vector field for the two-sphere
\[ V = u \frac{\partial}{\partial \theta} + v \frac{\partial}{\partial \phi} \] (3.2)
should obey
\[ \frac{\partial u}{\partial \theta} = 0, \quad \frac{\partial v}{\partial \theta} \sin^2 \theta + \frac{\partial u}{\partial \phi} = 0, \quad u + \frac{\partial v}{\partial \phi} \tan \theta = 0 \] (3.3)

This is solved provided
\[ u = u(\phi), \quad u''(\phi) = -u(\phi), \quad v = \cot \theta u'(\phi) + C \] (3.4)

where \( C \) is a constant. We find the three linearly independent Killing vector fields
\[ X = \frac{\partial}{\partial \phi}, \quad Y = \sin \phi \frac{\partial}{\partial \theta} + \cos \phi \cot \theta \frac{\partial}{\partial \phi}, \quad Z = \cos \phi \frac{\partial}{\partial \theta} - \sin \phi \cot \theta \frac{\partial}{\partial \phi} \] (3.5)

One can show the following commutator relations holds
\[ [X, Y] = Z, \quad [Z, X] = Y, \quad [Y, Z] = X \] (3.6)

Notice that due to the commutator relations (3.6) we can have at most one of the three Killing vector fields being associated to a coordinate since we cannot find two linearly independent Killing vector fields that commute with each other. In the above coordinate system this single Killing vector field associated to a coordinate is \( X = \partial/\partial \phi \).

The Schwarzschild black hole

Consider the Schwarzschild black hole metric
\[ ds^2 = -\left(1 - \frac{2GM}{r}\right)dt^2 + \frac{1}{1 - \frac{2GM}{r}}dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2) \] (3.7)

One can show that this has the following Killing vector fields
\[ T = \frac{\partial}{\partial t}, \quad X = \frac{\partial}{\partial \phi}, \quad Y = \sin \phi \frac{\partial}{\partial \theta} + \cos \phi \cot \theta \frac{\partial}{\partial \phi}, \quad Z = \cos \phi \frac{\partial}{\partial \theta} - \sin \phi \cot \theta \frac{\partial}{\partial \phi} \] (3.8)

We see that the Killing vector \( T \) corresponds to time-translation invariance of the Schwarzschild metric, while \( X, Y \) and \( Z \) corresponds to the spherical symmetry, \( i.e. \) these are the Killing vector fields of the two-sphere.

We see furthermore that the above metric has two Killing vector fields associated with a coordinate: \( T \) and \( X \). This is the maximal number of linearly independent commuting Killing vector fields that we have for the Schwarzschild black hole metric.