Problem 20.3

December 16, 2004

Problem 20.3 on p. 290 and its answer on p. 575 should be replaced by the following two related problems.

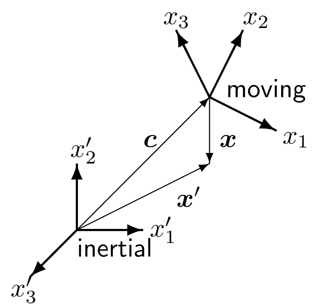
Problems

$*$ **20.3** Consider an arbitrary time-dependent orthogonal matrix $\mathbf{A}(t) = \{A_{ij}(t)\}$. Show that there exists a rotation vector $\boldsymbol{\Omega}(t) = \{\Omega_i(t)\}$ such that

$$\dot{\mathbf{A}} = -\mathbf{A} \times \boldsymbol{\Omega} \qquad \text{or} \qquad \dot{A}_{ij} = -\sum_{kl} \epsilon_{jkl} A_{ik} \Omega_l \tag{20.39}$$

and determine the form of $\boldsymbol{\Omega}$.

$*$ **20.4** In an inertial Cartesian system the coordinates of a point are denoted $\boldsymbol{x}'$ whereas in a generally non-inertial moving Cartesian system the coordinates of the same point are denoted $\boldsymbol{x}$. In the inertial system the motion of the non-inertial system is described by the time-dependent coordinates of its origin $\boldsymbol{c}(t)$ and basis vectors $\boldsymbol{a}_i(t)$.



The moving coordinate system is in general accelerated relative to the inertial frame.

(a) Show that the instantaneous relation between the two sets of coordinates is,

$$\boldsymbol{x} = \mathbf{A}(t) \cdot (\boldsymbol{x}' - \boldsymbol{c}(t)) \ . \tag{20.40}$$

where $A_{ij} = (\boldsymbol{a}_i)_j$ is the orthogonal transformation matrix.

(b) Show that the velocity $\dot{\boldsymbol{x}}$ in the moving system is

$$\dot{\boldsymbol{x}} = \mathbf{A} \cdot (\dot{\boldsymbol{x}}' - \dot{\boldsymbol{c}}) - \boldsymbol{\Omega} \times \boldsymbol{x} \tag{20.41}$$

where $\boldsymbol{\Omega} = \mathbf{A} \cdot \boldsymbol{\Omega}'$ is the rotation vector Ω' in the inertial system projected onto the axes of the moving system (Hint: use problem 20.3).

(c) Show that Newton's second law in the moving system becomes

$$m\ddot{\boldsymbol{x}} = \boldsymbol{f} - m\boldsymbol{q} - m\dot{\boldsymbol{\Omega}} \times \boldsymbol{x} - 2m\boldsymbol{\Omega} \times \dot{\boldsymbol{x}} - m\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{x}) \tag{20.42}$$

where $\boldsymbol{q}(t) = \mathbf{A}(t) \cdot \ddot{\boldsymbol{c}}(t)$ is the acceleration of the origin of the moving system projected on the axes of the moving system. This is necessary because a constant acceleration of the moving frame in the inertial system must lead to a rotating acceleration vector for the inertial system seen from the moving frame.

Answers

20.3 Orthogonality implies that $\sum_k A_{ik}A_{jk} = \sum_k A_{ki}A_{kj} = \delta_{ij}$. Differentiating the last after time we get $\sum_k \dot{A}_{ki}A_{kj} + \sum_k A_{ki}\dot{A}_{kj} = 0$. This shows that the matrix $\Omega_{ij} = \sum_k A_{ki}\dot{A}_{kj}$ is antisymmetric $\Omega_{ij} = -\Omega_{ji}$ so that we may put $\Omega_{ij} = \sum_k \epsilon_{ijk}\Omega_k$. Using again orthogonality we find $\dot{A}_{ij} = \sum_m \delta_{im}\dot{A}_{mj} = \sum_{mk} A_{ik}A_{mk}\dot{A}_{mj} = \sum_k A_{ik}\Omega_{kj} = \sum_{kl} A_{ik}\epsilon_{kjl}\Omega_l = -\sum_{kl} A_{ik}\epsilon_{jkl}\Omega_l$.

20.4 **(a)** Follows along the same lines as on p. 21 (with primed and unprimed variables interchanged).

(b) Differentiating after time the velocity becomes

$$\dot{\boldsymbol{x}} = \mathbf{A}\cdot(\dot{\boldsymbol{x}}' - \dot{\boldsymbol{c}}) + \dot{\mathbf{A}}\cdot(\boldsymbol{x}' - \boldsymbol{c})$$

The second term is now rewritten using problem 20.3 with the rotation vector $\boldsymbol{\Omega}'$ in the inertial system,

$$\begin{aligned}\dot{\mathbf{A}}\cdot(\boldsymbol{x}' - \boldsymbol{c}) &= -\mathbf{A}\times\boldsymbol{\Omega}'\cdot(\boldsymbol{x}' - \boldsymbol{c}) = -\mathbf{A}\cdot(\boldsymbol{\Omega}'\times(\boldsymbol{x}' - \boldsymbol{c}))\\ &= -(\mathbf{A}\cdot\boldsymbol{\Omega}')\times\mathbf{A}\cdot(\boldsymbol{x}' - \boldsymbol{c}) = -\boldsymbol{\Omega}\times\boldsymbol{x}\end{aligned}$$

In the next to the last step we assumed that $\det\mathbf{A} = 1$ such that the transformed cross product becomes the cross product of the transformed vectors.

(c) Differentiate once more after time and repeat the above steps to get the acceleration in the moving system

$$\begin{aligned}\ddot{\boldsymbol{x}} &= \mathbf{A}\cdot(\ddot{\boldsymbol{x}}' - \ddot{\boldsymbol{c}}) + \dot{\mathbf{A}}\cdot(\dot{\boldsymbol{x}}' - \dot{\boldsymbol{c}}) - \dot{\boldsymbol{\Omega}}\times\boldsymbol{x} - \boldsymbol{\Omega}\times\dot{\boldsymbol{x}}\\ &= \mathbf{A}\cdot(\ddot{\boldsymbol{x}}' - \ddot{\boldsymbol{c}}) - \mathbf{A}\times\boldsymbol{\Omega}'\cdot(\dot{\boldsymbol{x}}' - \dot{\boldsymbol{c}}) - \dot{\boldsymbol{\Omega}}\times\boldsymbol{x} - \boldsymbol{\Omega}\times\dot{\boldsymbol{x}}\\ &= \mathbf{A}\cdot(\ddot{\boldsymbol{x}}' - \ddot{\boldsymbol{c}}) - \boldsymbol{\Omega}\times(\dot{\boldsymbol{x}} + \boldsymbol{\Omega}\times\boldsymbol{x}) - \dot{\boldsymbol{\Omega}}\times\boldsymbol{x} - \boldsymbol{\Omega}\times\dot{\boldsymbol{x}}\\ &= \mathbf{A}\cdot(\ddot{\boldsymbol{x}}' - \ddot{\boldsymbol{c}}) - \dot{\boldsymbol{\Omega}}\times\boldsymbol{x} - 2\boldsymbol{\Omega}\times\dot{\boldsymbol{x}} - \boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\boldsymbol{x})\end{aligned}$$

Finally using Newton's second law in the inertial system, $m\ddot{\boldsymbol{x}}' = \boldsymbol{f}'$, and defining $\boldsymbol{f} = \mathbf{A}\cdot\boldsymbol{f}'$ we get (20.42).