The PDE's of continuum physics*

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June 25, 2005

Abstract

The aim of this paper is to present to a non-physicist audience the basic principles underlying the partial differential equations encountered in physics. The audience is expected to be familiar with the mathematics of partial differential equations but to have only a limited knowledge of the physics behind. It is shown how physics provides an intuitive understanding of the mathematical equations and the limitations on their use. The emphasis is on the interplay between global and local conservation laws. The paper provides a top-down view, using a compact notation common in physics, but does neither enter into practical applications nor numeric simulations. It can in no way replace a proper course in continuum physics.

The everyday experience of the smoothness of matter is an illusion[1]. Since the beginning of the twentieth century it has been known with certainty that the material world is composed of microscopic atoms and molecules, responsible for the macroscopic properties of ordinary matter. Long before the actual discovery of molecules, chemists had inferred that something like molecules had to exist, even if they did not know how big they were. Molecules *are* small — so small that their existence may be safely disregarded in all our daily doings. Continuum physics deals with the systematic description of matter at length scales that are large compared to the molecular scale. Most macroscopic length scales occurring in practice are actually huge in molecular units, typically in the hundreds of millions. This enormous ratio of scales isolates continuum theories of macroscopic phenomena from the details of the microscopic molecular world. There might, in principle, many different microscopic models leading to the same macroscopic physics.

Whether a given number of molecules is large enough to warrant the use of a smooth continuum description depends on the precision desired. Since matter is never continuous at sufficiently high precision, continuum physics is always an approximation. But as long as the fluctuations in physical quantities caused by the discreteness of matter are smaller than the desired precision, matter may be taken to be continuous. Continuum physics is, like thermodynamics, a limit of statistical physics where all macroscopic quantities such as mass density and pressure are understood as averages over essentially infinite numbers of microscopic molecular variables. At a level intermediate between the molecular and continuum descriptions of matter, one often speaks about *material particles* as the smallest objects that may consistently be considered part of the continuum description within the required precision. Thus, for example, to suppress the random density fluctuations in air below 1%, the smallest material particle must contain at least 10,000 molecules, corresponding to the number contained in a cubic box roughly 100 nanometers on a side. Even if a material particle always contains a large number of molecules, it may nevertheless in the continuum description be thought of as infinitesimal or point-like.

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*Introductory lecture presented at the *Workshop on PDE methods in Computer Graphics*, Department of Computer Science, University of Copenhagen, Denmark, March 31–April 1, 2005.

[1] This introduction is in part composed of excerpts from the first chapter of my recent book [1].

In continuum physics a macroscopic body is seen as a huge collection of tiny material particles, each of which contains a sufficiently large number of molecules to justify the continuum description. Continuum physics does not on its own go below the level of the material particles. Although the mass density in a point may be calculated by adding together the masses of all the molecules in a material particle containing that point and dividing with the volume occupied by it, this procedure falls strictly speaking outside continuum physics. In the extreme mathematical limit, the material particles are taken to be truly infinitesimal, and all physical properties of the particles as well as the forces acting on them are described by smooth, or piecewise smooth, functions of space and time.

Continuum physics is therefore a theory of *fields*. Mathematically, a field f is simply a real-valued function $f(x, y, z, t)$ of spatial coordinates x, y, z, and time t, representing the value of a physical quantity in this point of space at the given time, for example the mass density $\rho = \rho(x, y, z, t)$. Sometimes a collection of such functions is also called a field and the individual real-valued members are called its components. Thus, the most fundamental field of fluid mechanics, the velocity field $\boldsymbol{v} = (v_x, v_y, v_z)$, has three components, one for each of the coordinate directions. Besides fields characterizing the state of the material, such as mass density and velocity, it is convenient to employ fields that characterize the forces acting on and within the material. The gravitational acceleration field $\boldsymbol{g}$ is a force field, which penetrates bodies from afar and acts on their mass. Some force fields are only meaningful for regions of space where matter is actually present, as for example the pressure field p, which acts across the imagined contact surfaces that separate neighboring volumes of a fluid at rest. Pressure is, however, not the only *contact force*. For fluids in motion, for solids and more general materials, contact forces are described by the stress field, $\{\sigma_{ij}\}$, which is a 3×3 matrix field with rows and columns labeled by the coordinates: $i, j = x, y, z$.

Mass density, velocity, gravity, pressure, and stress are the usual fields of *continuum mechanics*. The more general subject of *continuum physics* also deals with thermodynamic fields, like the temperature T, the specific internal energy density u, and the specific entropy s. There may also be fields that describe different states of matter, for example the electric charge density ρ_e and current density $\boldsymbol{j}_e$. The associated electric and magnetic field strengths, $\boldsymbol{E}$ and $\boldsymbol{B}$, are like the Newtonian field of gravity $\boldsymbol{g}$ thought to exist even in regions of space completely devoid of matter. Further fields may refer to material properties, for example the coefficient of shear elasticity μ of a solid and the coefficient of shear viscosity η of a fluid. Such fields are usually constant within homogeneous bodies, i. e. independent of space and time, and are mostly called material constants rather than true fields.

Like all physical variables, fields evolve with time according to dynamical laws, called *field equations*, taking the general form of coupled partial differential equations. In non-relativistic continuum mechanics, the central equation of motion descends directly from Newton's second law, whereas mass conservation, which is all but trivial and most often tacitly incorporated in particle mechanics, turns into an equation of motion for the mass density. Still other field equations such as Maxwell's equations for the electromagnetic fields have completely different and non-mechanical origins, although they do couple to the mechanical equations of motion.

The paper is organized in the following way. In section 1 which to some extent serves to define notation, the Poisson equation for gravity is derived directly from Newton's law of gravity. In section 2 the concept of stress is introduced and Cauchy's equilibrium equation for non-relativistic statics is derived and applied to hydrostatics and linear elastostatics. In section 3, conservation of mass is used to derive Euler's equation of continuity, and from momentum conservation Cauchy's equation of dynamics is derived. For Newtonian fluids these equations become the Navier-Stokes equations. In section 4 energy conservation is used to derive the thermodynamical heat equation, and in section 5 the relation between laws of balance and conservation is discussed, together. Finally in section 6 some concluding remarks are made about the extension of continuum mechanics beyond the traditional topics.

1 Newtonian gravity

Isaac Newton created the first dynamics of point particles and gave us the mathematics to deal with it. His Second Law states that the equation of motion for a point particle of mass m is a second order ordinary differential equation in time[2],

$$m\ddot{\boldsymbol{x}} = \boldsymbol{\mathcal{F}}(\boldsymbol{x}, \dot{\boldsymbol{x}}, t) \; , \tag{1}$$

where the vector function $\boldsymbol{\mathcal{F}}$ on the right hand side is the total force acting on the particle. If a system consists of more than one point particle, there is an equation of motion for each particle, and the force on any of the particles will in general depend on the positions and velocities of all the particles.

Newton also gave us the first theory of gravity, in which the gravitational force exerted by a point particle of mass M on a point particle of mass m a distance r away is of magnitude GmM/r^2, where G is the gravitational constant. Holding the particle M fixed at the origin of the coordinate system, and using that the force is attractive and directed along the line connecting the particles, we find

$$\boldsymbol{\mathcal{F}} = -\frac{GmM}{r^2}\boldsymbol{e}_r \; , \tag{2}$$

where $r = |\boldsymbol{x}|$ is the length of $\boldsymbol{x}$ and $\boldsymbol{e}_r = \boldsymbol{x}/r$ is the unit radius vector. Together with Newton's equation of motion for a point particle (1) this law is sufficient to calculate the leading approximation to the planetary orbits around a fixed Sun. Including the mutual gravitational interaction of the orbiting objects, Newton's theory of gravity for point particles may be extended to cover the motions of all objects in the solar system and beyond: spacecraft, meteors, comets, planets, moons, stars, and galaxies.

Treating all these bodies as point particles is an approximation justified only by the enormous distances in space in comparison with the sizes of the objects. Closer to home, the fall of an apple is also governed by gravity, but in this case it would seem like madness to treat the Earth as a point particle[3]. For an extended static mass distribution field $\rho(\boldsymbol{x})$, the force on a small test particle of mass m may be calculated by adding the contributions from each and every material particle in the body[4]. The mass of a material particle situated at $\boldsymbol{x}$ is $dM = \rho(\boldsymbol{x})\, dV$ where $dV = dxdydz$ is the "infinitesimal‘" volume of the particle, and replacing $\boldsymbol{x}$ by $\boldsymbol{x} - \boldsymbol{x}'$, we find,

$$\boldsymbol{\mathcal{F}} = m\boldsymbol{g} \; , \qquad\qquad \boldsymbol{g}(\boldsymbol{x}) = -G\int \frac{\boldsymbol{x} - \boldsymbol{x}'}{|\boldsymbol{x} - \boldsymbol{x}'|^3}\,\rho(\boldsymbol{x}')dV' \; . \tag{3}$$

Here we have also for convenience factored the force into the product of the test particle mass m and the static *gravitational acceleration* field $\boldsymbol{g}(\boldsymbol{x})$. This is the force you, for example, would use to calculate the small corrections to the orbit of a near-earth satellite, taking into account the Earth's deviation from perfect spherical symmetry, and the uneven distribution of land, sea, and mountains, as well as its complex material composition.

It is now a simple mathematical exercise to show that the vector gravitational field $\boldsymbol{g}$ may be written as the gradient[5] of a single (scalar) field, the *gravitational potential* Φ,

$$\boldsymbol{g} = -\boldsymbol{\nabla}\Phi \; , \qquad\qquad \Phi(\boldsymbol{x}) = -G\int \frac{\rho(\boldsymbol{x}')}{|\boldsymbol{x} - \boldsymbol{x}'|}\,dV' \; . \tag{4}$$

[2]Cartesian coordinates (and vectors) in ordinary space are denoted by boldface symbols $\boldsymbol{x} = (x, y, z)$. In veneration of Newton we use here a dot to indicate differentiation with respect to time.

[3]It was Newton's great luck that for a spherically symmetric body the field outside the body will in fact be that of a point particle, permitting him with remarkable precision to connect the Moon's orbital motion with the anecdotal fall of an apple.

[4]That the gravitational forces obey this *superposition principle* should be viewed as an empirical law of nature.

[5]The vector gradient operator, called *nabla* or *del*, is defined as $\boldsymbol{\nabla} = (\partial/\partial x, \partial/\partial y, \partial/\partial z)$.

In principle, this allows us to calculate the gravitational field from any prescribed mass distribution.

It is, however, easy to set up a general physical situation where the mass density is not prescribed, but depends on the actual field of gravity, which in turn depends on the mass density, and so forth. To handle such circularity, it is much better to convert the above relation (4) between the mass density and the potential into a partial differential equation. There are numerous ways of doing so, but to make a long story short we shall use the well-known expression for the Laplace operator $\nabla^2 = \boldsymbol{\nabla} \cdot \boldsymbol{\nabla} = \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$ applied to the archetypal Coulomb potential $1/|\boldsymbol{x}|$,

$$\nabla^2 \frac{1}{|\boldsymbol{x}|} = -4\pi\delta(\boldsymbol{x}) \ , \tag{5}$$

where $\delta(\boldsymbol{x}) = \delta(x)\delta(y)\delta(z)$ is the three-dimensional Dirac delta-function. Replacing $\boldsymbol{x}$ by $\boldsymbol{x} - \boldsymbol{x}'$, and applying the Laplace operator to (4), we finally obtain *Poisson's equation* (1812),

$$\boxed{\nabla^2 \Phi = 4\pi G \rho \ .} \tag{6}$$

Even if this is not the first partial differential equation in the history of physics, it is certainly one of the most fundamental ones. Mathematically it is the prototype of elliptic differential equations, arising in hydrostatics, electrostatics, magnetostatics, and many other areas of physics.

Notice that the relation (4) between the potential and the mass density is *non-local*, meaning that the potential in a given point depends on the mass density in all points of space[6]. For a time-independent static mass density this does not matter, because in infinite time any signal has time to arrive from even the farthest corners of space. But should the mass density vary with time, as it for example does in the solar system because of the motions of the planets, the Newtonian field of gravity must according to (4) respond instantaneously everywhere to reflect this variation[7].

2 Non-relativistic statics

The world is not static, but dynamic. Living on the surface of earth, we are nevertheless — and luckily so — surrounded by objects that do not move, or at least do not move much. The study of the static configurations of matter under the influence of external and internal forces is in a sense the baseline for the physics of continuous matter. Mathematically, the physics of static matter leads to elliptic partial differential equations, that are much more amenable to analytic or numeric treatment than the parabolic and hyperbolic partial differential equations of dynamics to be presented in the following section.

2.1 Cauchy's equation of static equilibrium

The principle behind continuum statics is the *vanishing of the total force* acting on any volume of matter. In Newtonian particle mechanics the concept of a *body* usually covers an arbitrary collection of fixed mass point particles, whereas in continuum physics *any volume of matter* may serve as a body.

[6]The inverse relation (6) is on the other hand local, meaning that the mass density in a point only depends on the potential in the immediate vicinity of this point. Since we think of the potential (and thus the gravitational force) as *caused* by the mass density, and not the other way around, this is of little interest. We shall later return to the question of locality in dynamics.

[7]Although viewed with unease by Newton himself, this action-at-a-distance did not conflict with any known principles of physics at that time. After Einstein created the special theory of relativity in 1905, instantaneous action-at-a-distance became an acute problem because relativity predicts that no signal can travel faster than light. The resolution of the problem was also given by Einstein in 1916 in his general theory of relativity in which he set up the dynamics of gravity (in the framework of curved spacetime). Though not yet directly confirmed by experiment, gravity is today firmly believed to spread through space at the speed of light.

Two kind of forces act on the material of a body. First, there are long range forces penetrating the whole volume V of the body. These forces are described by a force density $\boldsymbol{f} = d\boldsymbol{\mathcal{F}}/dV$, the prime example[8] being gravity with $\boldsymbol{f} = \rho\boldsymbol{g}$. Besides these, there are short-range contact forces acting on the surface S of the volume V. Even if they act only on surfaces, contact forces are nevertheless described by fields thought to exist throughout the body. In the simplest case, hydrostatics of isotropic fluids, there is only the pressure field $p(\boldsymbol{x})$ which acts along the normal to any real or imagined surface with a force proportional to the area. Most materials are not as simple as isotropic fluids. Even isotropic elastic solids have a more complicated structure, and the plethora of modern materials with intermediate properties between the fluid and solid state adds further dimensions to the description.

The fundamental concept for describing contact forces is a generalization of the pressure field, called the *stress* field. It is a tensor[9] (i.e. matrix) field, $\sigma_{ij}(\boldsymbol{x})$ with indices running over the coordinate labels, here $i, j = x, y, z$. The stress field is defined such that the component σ_{ij} equals the local force per unit of area acting in the coordinate direction i on a surface with normal along the coordinate direction j. To calculate the local contact force on any surface element $d\boldsymbol{S} = \{dS_i\}$ one invokes *Cauchy's stress hypothesis*[10], and simply adds the contributions from each coordinate direction $d\mathcal{F}_i = \sum_j \sigma_{ij} dS_j$. The total force on a body of volume V with surface S then becomes the sum of the volume and surface contributions,

$$\mathcal{F}_i = \int_V f_i \, dV + \oint_S \sum_j \sigma_{ij} \, dS_j \ . \tag{7}$$

Converting the closed surface integral into a volume integral by means of Gauss' theorem, we may express it as,

$$\mathcal{F}_i = \int_V \left(f_i + \sum_j \nabla_j \sigma_{ij} \right) dV \ . \tag{8}$$

In compact notation the integrand one may also be written $\boldsymbol{\mathcal{F}} = \int_V (\boldsymbol{f} + \boldsymbol{\nabla} \cdot \boldsymbol{\sigma}^\mathsf{T}) \, dV$, where $\boldsymbol{\sigma}^\mathsf{T}$ is the transposed stress tensor.

In static equilibrium the total force must vanish for any volume, and it follows that the integrand must vanish everywhere, or

$$\boxed{f_i + \sum_j \nabla_j \sigma_{ij} = 0 \ .} \tag{9}$$

This is *Cauchy's equilibrium equation* from 1827. Although derived here from the global consideration, it follows from (8) that Cauchy's equilibrium equation may be viewed equivalently as expressing the vanishing of the effective force $d\boldsymbol{\mathcal{F}} = (\boldsymbol{f} + \boldsymbol{\nabla} \cdot \boldsymbol{\sigma}^\mathsf{T}) \, dV$, acting on each and every material particle in the body. It is thus in accordance with Newton's Second Law (1) applied to a particle at rest.

Cauchy's equilibrium equation (9) consists of three differential conditions. In conventional

[8] The only other example of a long range force is in fact the electromagnetic Lorentz force $\boldsymbol{f} = \rho_c \boldsymbol{E} + \boldsymbol{j}_c \times \boldsymbol{B}$.

[9] The simple boldface vector notation used up to this point is not sufficient to handle more complicated expressions involving tensors. Instead we shall use a component notation with indices $i, j, k, \ldots$ running implicitly over the coordinate labels. This notation coexists peacefully with the ordinary vector notation. We shall, however, refrain from using the Einstein convention of implicit summation over all repeated indices, but write each sum explicitly.

[10] Cauchy's stress hypothesis is really a theorem which can be derived from physical arguments.

mathematical notation they are,

$$f_x + \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} = 0$$
$$f_y + \frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} = 0$$
$$f_z + \frac{\partial \sigma_{zx}}{\partial x} + \frac{\partial \sigma_{zy}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} = 0$$

To transform these conditions into a closed set of partial differential equations, it is necessary to add suitable *constitutive equations*, expressing the local relations between the stresses and the local state of matter, described by suitable fields. Different kinds of continuous matter — gases, liquids, solids, or intermediate — only differ by their constitutive equations. We shall see below how this is done for isotropic Newtonian fluids and isotropic linear elastic solids.

The stress tensor has a priori nine different components. In classical continuum theory the stress tensor is assumed to be symmetric,

$$\sigma_{ij} = \sigma_{ji} \ , \tag{10}$$

and thus has only six independent components. This relation, called *Cauchy's second law* (1827), is however not a law of nature [2, 1] but should rather be viewed as belonging to the constitutive equations. There exist in fact non-classical extensions of continuum theory with manifestly asymmetric stress tensors (see for example [3]).

So far we have not discussed boundary conditions. If the material properties vary smoothly across a body, the stress tensor may also be assumed to vary continuously. At an interface between different materials[11] where material properties jump discontinuously, boundary conditions are, however, necessary to bridge the discontinuity. For the stress tensor, these are provided by Newton's Third Law which states that action and reaction must be equal and opposite. Since the normals are opposite on the two sides of the interface, this law implies the continuity of the *stress vector*[12], defined as the vector force $(\boldsymbol{\sigma} \cdot \boldsymbol{n})_i = \sum_j \sigma_{ij} n_j$ per unit of area acting on a surface with normal $\boldsymbol{n}$. The continuity condition may then be written as, $\Delta\boldsymbol{\sigma} \cdot \boldsymbol{n} = \boldsymbol{0}$ where $\Delta\boldsymbol{\sigma}$ is the difference of the stress tensors on the two sides of the interface. Newton's Third Law only demands continuity of three linear combinations of the stress tensor components, whereas the remaining three linear combinations of the symmetric stress tensor are in general allowed to jump across an interface.

2.2 Hydrostatics

An isotropic fluid that is everywhere at rest can, as mentioned, only support pressure forces acting along the normal to any real or imagined surface. The *pressure* is defined as the force per unit of area and acts on any vector surface element $d\boldsymbol{S}$ with a force $d\boldsymbol{\mathcal{F}} = -p\, d\boldsymbol{S}$ (with a conventional minus-sign). Evidently this corresponds to the stress tensor,

$$\sigma_{ij} = -p\, \delta_{ij} \ , \tag{11}$$

where δ_{ij} is the Kronecker symbol. Upon insertion into Cauchy's equilibrium equation (9) we obtain for the case of gravity the basic differential equation of hydrostatics[13],

$$\rho \boldsymbol{g} - \boldsymbol{\nabla} p = \boldsymbol{0} \ . \tag{12}$$

[11] Materials typically interface across a few molecular diameters. Interfaces therefore fall outside the continuum description and are replaced by mathematical discontinuities. Interfaces may nevertheless possess physical properties, for example surface tension.

[12] The stress vector is not a vector field in the usual sense of the word, depending only on the location $\boldsymbol{x}$, because it also depends on the local normal $\boldsymbol{n}$ to the surface.

[13] This equation may be viewed as expressing Archimedes Law, *"buoyancy balances weight"*, for material particles.

Due to the non-local character of the gravitational field (3), this is in general an integro-differential equation. If the field of gravity is prescribed, as it for example is to a good approximation on the surface of earth, one obtains in combination with an equation of state (see below) a partial differential equation which can be applied to determine the density and pressure of the sea and the atmosphere.

In the general case where the gravitational field is not prescribed, we introduce the potential (4) and write (12) in the form $\boldsymbol{\nabla} p = -\rho \boldsymbol{\nabla} \Phi$. Dividing by the mass density and calculating the divergence of both sides, we obtain by means of the Poisson equation (6), the following fundamental equation of hydrostatics[14],

$$\boxed{\boldsymbol{\nabla} \cdot \left(\frac{1}{\rho} \boldsymbol{\nabla} p \right) = -4\pi G \rho \ .} \tag{13}$$

This is the equation you could use to calculate the interior properties of ordinary stars and planets. Notice that it is in general non-linear (except for $p \sim \rho^2$).

There are still two unknown fields in the above equation. To arrive at an equation for a single field, we need a relation between pressure and density. It is a well-known result of thermodynamics that for simple materials in thermodynamic equilibrium there will always exist an *equation of state* relating the values of pressure p, density ρ, and temperature T. For example, for an ideal gas the equation of state takes the simple form $p \sim \rho T$ whereas in liquids it is more complicated. Assuming that every material particle is in local thermodynamic equilibrium, the equation of state becomes a relation between the local values of the fields,

$$p(\boldsymbol{x}) = f(\rho(\boldsymbol{x}), T(\boldsymbol{x})) \ . \tag{14}$$

If the temperature field is specified, for example constant temperature, this relation may be used to eliminate the pressure, and eq. (13) becomes indeed a nonlinear elliptic partial differential equation for the density field. If on the other hand the temperature field is not specified, an extra partial differential equation for that field must be added (see section 4)[15].

2.3 Linear elastostatics

Any deformation of a material body may be described by an exhaustive account of how each material particle in the body is displaced from its initial position. The *displacement* of a material particle is naturally defined as the vector $\boldsymbol{u} = \boldsymbol{x} - \boldsymbol{x}_0$, where $\boldsymbol{x}$ is the current and $\boldsymbol{x}_0$ the initial position of the particle. In keeping with our general definition of a field as indicating the actual state of matter in a given point, we view the displacement as a function of the current position, $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x})$. This is the Eulerian representation of the displacement field. Alternatively, and equivalently, one may use the Lagrangian (or material) displacement field, $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x}_0)$, defined as a function of the *initial* position of a material particle in the undeformed body. The distinction between the Eulerian and Lagrangian formalisms is of no consequence to leading order when the displacement field has small gradients satisfying $|\nabla_i u_j| \ll 1$ for all i, j. In the remainder of this subsection we shall assume this to be the case.

Displacement includes bodily translations and rotations that should not be classified as deformations. A true deformation must involve changes in the local geometric relationships in the body.

[14] One may wonder how the vector equation (12) can be replaced by the scalar equation (13). But since $\boldsymbol{\nabla} p = -\rho \boldsymbol{\nabla} \Phi$ tells us that the gradient of the pressure must everywhere be parallel with the gradient of the gravitational potential, it follows that the isobaric surfaces (of constant pressure) must coincide with the equipotential surfaces of gravity everywhere in a fluid at rest. Hydrostatics is thus completely described by a single family of surfaces, i.e. by a single scalar field.

[15] The coupling between temperature and density caused by the nearly universal heat expansion of matter may, however, in a gravitational field give rise to convective instabilities, invalidating the assumption of a static state.

To expose this we consider an infinitesimal material "needle" initially connecting the nearby points $\boldsymbol{x}_0$ and $\boldsymbol{x}_0 + \boldsymbol{a}_0$ in the body. After the displacement it connects the points $\boldsymbol{x} = \boldsymbol{x}_0 + \boldsymbol{u}(\boldsymbol{x})$ and $\boldsymbol{x}+\boldsymbol{a} = \boldsymbol{x}_0 + \boldsymbol{a}_0 + \boldsymbol{u}(\boldsymbol{x}+\boldsymbol{a})$. Expanding to first order in $\boldsymbol{a}$, we find the change in the needle vector $\delta\boldsymbol{a} = \boldsymbol{a} - \boldsymbol{a}_0 = \boldsymbol{u}(\boldsymbol{x}+\boldsymbol{a}) - \boldsymbol{u}(\boldsymbol{x}) \approx (\boldsymbol{a}\cdot\boldsymbol{\nabla})\boldsymbol{u}(\boldsymbol{x})$. In view of the assumption of small displacement gradients, $|\nabla_i u_j| \ll 1$, this is just a tiny correction to the needle vector. Since the scalar product $\boldsymbol{a}\cdot\boldsymbol{b}$ of two needle vectors is a purely geometric quantity, unaffected by translations and rotations, we may isolate proper deformations by calculating the change in the scalar product,

$$\delta(\boldsymbol{a}\cdot\boldsymbol{b}) = \delta\boldsymbol{a}\cdot\boldsymbol{b} + \boldsymbol{a}\cdot\delta\boldsymbol{b} = \boldsymbol{a}\cdot(\boldsymbol{\nabla}\boldsymbol{u})\cdot\boldsymbol{b} + \boldsymbol{b}\cdot(\boldsymbol{\nabla}\boldsymbol{u})\cdot\boldsymbol{a} \ .$$

Evidently, the scalar product is controlled by the symmetrized displacement gradient tensor, $\nabla_i u_j + \nabla_j u_i$. Cauchy's *strain tensor* is conventionally defined to be half of that,

$$u_{ij} = \frac{1}{2}\left(\nabla_i u_j + \nabla_j u_i\right) \ . \tag{15}$$

such that we have $\delta(\boldsymbol{a}\cdot\boldsymbol{b}) = 2\sum_{ij} u_{ij} a_i b_j$. A displacement is classified as a deformation when the strain tensor does not vanish everywhere. Geometrically, the diagonal component u_{ii} represents the relative length increase along the i-th axis, whereas the non-diagonal component u_{ij} is proportional to the change in angle between the initially orthogonal i-th and j-th coordinate axes.

Bodily displacements, translations and rotations, should not create stresses, implying that in an elastic material the local stresses can only depend on the local strains. When the strains are small, the relation between the stress tensor and the strain tensor for an elastic material will be approximatively linear (Hooke's law). In full generality it takes up to 18 different parameters, called *elastic moduli*, to characterize the most complex linear elastic material (a triclinic crystal), whereas for an isotropic material only two parameters are needed. Assuming that all the stresses in the body vanish before the deformation[16], the isotropic relation can for symmetry reasons only take the tensor form

$$\sigma_{ij} = 2\mu\, u_{ij} + \lambda\, \delta_{ij} \sum_k u_{kk} \ . \tag{16}$$

The elastic moduli, λ and μ (also called the Lamé coefficients), may in principle vary across a material, but we shall for simplicity assume that they are constant. They are usually huge in macroscopic units, for example about 100 gigapascals for iron. Everyday stresses are usually smaller than one bar (100 kilopascals), and thus typically generate strains of the order of parts per million.

We are now in position to write down the equilibrium equation for the displacement field. Inserting the constitutive equation (16) into the equilibrium equation (9) and afterwards inserting the Cauchy strain tensor (15), we arrive after a bit of index manipulation at the following relatively simple vector partial differential equation for the displacement field, called the *Navier-Cauchy equilibrium equation*,

$$\boxed{\boldsymbol{f} + \mu\nabla^2\boldsymbol{u} + (\lambda+\mu)\boldsymbol{\nabla}\boldsymbol{\nabla}\cdot\boldsymbol{u} = \boldsymbol{0} \ .} \tag{17}$$

If the force field is prescribed, which it normally is for small deformations, this elliptic equation may be solved for for the displacement field (with suitable boundary conditions on the displacement and/or stress fields).

If we relax the assumption of small displacement gradients, the Cauchy strain tensor (15) acquires a term quadratic in the displacement gradients, and elastostatics becomes nonlinear even if Hooke's linear law (16) is maintained. In the 20'th century, large efforts have been devoted to the study of nonlinear elasticity because of the technological interest in large deformations, but although the mathematics becomes forbidding [5, 6, 7], no new physical principles are involved.

[16]There are many situations where this condition is not fulfilled, for example prestressed armored concrete or glass with frozen-in stresses.

3 Non-relativistic dynamics

Newton's Second Law (1) is *the* fundamental classical equation of motion for particles. Viewing continuous matter as a collection of material particles, this law must a fortiori apply to each and every material particle, and it is perfectly possible to carry through such an argument and derive the fundamental equations of continuum mechanics. It is, however, more instructive to begin with a global argument of the same kind as we did for statics in the preceding section, and only afterwards derive the local laws.

3.1 Mass conservation

In non-relativistic physics, mass cannot be created or destroyed. This fundamental law does not appear explicitly among Newton's laws, but is implicitly contained in the assumptions that the mass of a particle is a constant, and that a body consists of a fixed number of constant mass particles. In continuum physics a body is simply any volume of matter, and material may in the course of time flow in and out through the surface of this volume, thereby changing its mass. Denoting the velocity field by $\boldsymbol{v}(\boldsymbol{x}, t)$, this immediately leads to the following expression[17] for the global law of *mass conservation* in a fixed volume V with surface S,

$$\frac{d}{dt}\int_V \rho \, dV + \oint_S \rho \boldsymbol{v} \cdot d\boldsymbol{S} = 0 \ . \tag{18}$$

The first term on the left hand side represents the (signed) rate of change of mass in the volume. Since in a small time interval dt, a volume of matter $dV = \boldsymbol{v}dt \cdot d\boldsymbol{S}$ flows *out* through the surface element $d\boldsymbol{S}$, the second term denotes the (signed) rate of mass *loss* from the volume through its surface. Mass conservation expressed through the vanishing of the right hand side implies that if mass is lost by flow through the surface there must be a corresponding decrease in the mass in the enclosed volume, and conversely[18].

Applying Gauss' theorem the surface integral in eq. (18), and using that the volume is arbitrary, we arrive at the *equation of continuity* due to Euler (1753),

$$\boxed{\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \boldsymbol{v}) = 0 \ .} \tag{19}$$

The zero on the right hand side explicitly states that there are no local sources of mass in classical mechanics. Pulling the second term on the left over to the right, the equation of continuity may of course also be viewed as an *equation of motion* for the mass density field, expressing the local time derivative of the density, $\partial\rho/\partial t$, in terms of the instantaneous values of the density and velocity fields.

It is instructive to look at the continuity equation from the point of view of a particle following the flow. The path $\boldsymbol{x}(t)$ of such a particle is a solution to the ordinary differential equation

$$\frac{d\boldsymbol{x}(t)}{dt} = \boldsymbol{v}(\boldsymbol{x}(t), t) \ . \tag{20}$$

Differentiating $\rho(\boldsymbol{x}(t), t)$ with respect to t, the rate of change of the mass density along the path

[17] Here we shall only be concerned with fixed (static) volumes, although in full generality one must also consider time-dependent volumes, generically called control volumes, that move around in any way one pleases [1].

[18] This formulation of mass conservation may initially appear a bit awkward. For some readers, eq. (18) is easier to understand if the second term on the left is moved over to the right, such that mass conservation now states that the rate of change of the mass in a volume equals the (signed) rate of mass flow *into* the volume through its surface.

of the particle, also called the comoving time derivative, becomes[19],

$$\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + (\boldsymbol{v} \cdot \boldsymbol{\nabla})\rho \ . \tag{21}$$

Using the mathematical relation, $\boldsymbol{\nabla} \cdot (\rho \boldsymbol{v}) = (\boldsymbol{v} \cdot \boldsymbol{\nabla})\rho + \rho \boldsymbol{\nabla} \cdot \boldsymbol{v}$, the equation of continuity (19) may be reformulated as,

$$\frac{d\rho}{dt} = -\rho \, \boldsymbol{\nabla} \cdot \boldsymbol{v} \ . \tag{22}$$

This form of the continuity equation shows that the comoving rate of change of the density is proportional to the divergence of the velocity field[20]. For the volume per unit mass, $V = 1/\rho$, we obtain from (22),

$$\frac{dV}{dt} = V \boldsymbol{\nabla} \cdot \boldsymbol{v} \ , \tag{23}$$

which shows the divergence must vanish everywhere, $\boldsymbol{\nabla} \cdot \boldsymbol{v} = 0$, in an incompressible material[21].

3.2 Momentum balance

The momentum of a particle is defined as the product of its mass and velocity, $\boldsymbol{p} = m\dot{\boldsymbol{x}}$, and Newton's Second Law (1) states that the rate of change of momentum equals force, $\dot{\boldsymbol{p}} = \boldsymbol{\mathcal{F}}$. Due to its linearity this equation retains its form when summed over a collection of particles. In Newtonian particle mechanics, *momentum balance* simply expresses that the rate of change of the total momentum of any fixed collection of particles always equals the sum of all the forces acting on the collection. Equivalently, one may read this statement as saying that the (signed) momentum produced by the total force on a body of fixed mass is always accumulated in the body.

In continuum mechanics the momentum of a material particle of mass $dM = \rho \, dV$ is $\boldsymbol{v} \, dM = \rho \boldsymbol{v} \, dV$. Taking into account that momentum may be carried by the mass flow through the surface S of a volume V, *global momentum balance* takes the form,

$$\frac{d}{dt} \int_V \rho \boldsymbol{v} \, dV + \oint_S \rho \boldsymbol{v} \, \boldsymbol{v} \cdot d\boldsymbol{S} = \boldsymbol{\mathcal{F}} \ , \tag{24}$$

where $\boldsymbol{\mathcal{F}}$ is the total force (7). On the left hand side we find the rate of change of the momentum contained in the volume plus the (signed) rate of momentum flow *out* of the volume. If these two terms canceled each other, momentum would like mass be strictly conserved, but since the total force on the right hand side in general does not vanish, the momentum of a system is in general not conserved[22]. Reading from right to left, this equation expresses that the momentum produced by all the forces acting on the material must either be accumulated in the volume (the first term on the left hand side) or leave through the surface (the second term).

[19] There is a bit of ambiguity in this notation when the time-dependent path is not explicitly kept in the argument. Often the mixed time and space derivative operator $\frac{D}{Dt} = \frac{\partial}{\partial t} + \boldsymbol{v}(\boldsymbol{x}, t) \cdot \boldsymbol{\nabla}$, called the *material time derivative*, is defined. This operator can be applied to any field and produces a field which when evaluated on the path of the particle equals the comoving time derivative of the field.

[20] Eq. (22) also brings contact with the Lagrangian representation, because the general particle path is a function of the initial position $\boldsymbol{x}_0$ of the particle at $t = t_0$, i. e. $\boldsymbol{x}(t) = \boldsymbol{f}(t, \boldsymbol{x}_0, t_0)$.

[21] The vanishing of the divergence does not imply that the density takes the same value everywhere. A boulder is by most counts incompressible, but may have a spatially varying density due to local variations in the material composition. Eq. (22) with $\boldsymbol{\nabla} \cdot \boldsymbol{v} = 0$ then tells us the rather self-evident fact that near any particular grain, the density is always the same, even when the boulder rolls ands skips down a mountain side.

[22] Momentum is, however, conserved (i. e. constant) for an isolated system in a comoving volume, subject to no environmental forces. Since a system plus its total environment is isolated, it follows that any change in momentum of a non-isolated system can be accounted for through the exchange of momentum with its environment, described above by the external forces. Momentum balance is for this reason often called momentum conservation, although here we shall reserve the term "conservation" to apply only to strictly conserved quantities, such as mass.

Applying again Gauss' theorem to the surface integral, and using the expression (8) for the total force, we arrive at the *local equation of momentum balance*,

$$\frac{\partial(\rho \boldsymbol{v})}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \boldsymbol{v}\boldsymbol{v}) = \boldsymbol{f} + \boldsymbol{\nabla} \cdot \boldsymbol{\sigma}^{\mathsf{T}} \ . \tag{25}$$

On the left hand side it has the same general form as the equation of continuity (19), with the mass density ρ replaced by the (vector) momentum density $\rho\boldsymbol{v}$. The right hand side represents the effective density of force acting on a material particle, also called the *local source of momentum.*

Using the product rules for differentiation, the left hand side of the above equation may be simplified to,

$$\frac{\partial(\rho \boldsymbol{v})}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \boldsymbol{v}\boldsymbol{v}) = \boldsymbol{v}\frac{\partial \rho}{\partial t} + \rho\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v}\boldsymbol{\nabla} \cdot (\rho \boldsymbol{v}) + \rho(\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{v} = \rho\left(\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{v}\right) \ ,$$

where in the last step the continuity equation (19) has been used to eliminate the first and third terms. Finally this brings us to the conventional form of *Cauchy's equation (1827)*,

$$\boxed{\rho\left(\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{v}\right) = \boldsymbol{f} + \boldsymbol{\nabla} \cdot \boldsymbol{\sigma}^{\mathsf{T}} \ .} \tag{26}$$

Solved for $\partial \boldsymbol{v}/\partial t$, it becomes an equation of motion for the velocity field. Together with the equation of motion for the mass density obtained from the continuity equation and a specification of the volume force $\boldsymbol{f}$ and the stress tensor $\boldsymbol{\sigma}$ through suitable constitutive equations, we have established the general foundation of non-relativistic continuum dynamics[23].

Since the parenthesis on left hand side is of the same form as the comoving time derivative of the density (21), it represents the comoving time derivative of the velocity field, also called the *material acceleration*,

$$\boldsymbol{w} = \frac{d\boldsymbol{v}}{dt} = \frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{v} \ . \tag{27}$$

Multiplying with the volume element dV, we may interpret Cauchy's equation (26) as Newton's second law (1) applied to comoving material particles (which by definition have constant mass).

Materials differ, as mentioned before, only by the form of their stress tensors. Fluids are generically characterized by stresses that only depend on the velocity field, and Cauchy's equation for fluids typically turns into a parabolic partial differential equation (see below). If the stresses only depend on the displacement field gradients, the material is generically said to be *elastic.* For such materials it is best to eliminate the velocity field, using that it (in the Euler representation) may be written as the comoving time derivative of the displacement field,

$$\boldsymbol{v} = \frac{d\boldsymbol{x}}{dt} = \frac{d\boldsymbol{u}}{dt} = \frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{u} \ . \tag{28}$$

Combined with Cauchy's equation, it follows that elastic materials are typically governed by hyperbolic partial differential equations. Although the above equation may be solved for the velocity field in terms of the displacement field[24], it is often better to view it as an independent (local) equation of motion for the displacement field. Finally, materials that depend on both displacement and velocity are generically called *viscoelastic.*

[23] Mass conservation and momentum balance are also called *Euler's laws of motion* [7].

[24] Eq. (28) is an implicit equation for $\boldsymbol{v}$ with solution

$$\boldsymbol{v} = \frac{\partial \boldsymbol{u}}{\partial t} \cdot (1 - \boldsymbol{\nabla}\boldsymbol{u})^{-1} \ ,$$

where the last factor is the inverse of the Jacobian matrix $\partial x_{0,i}/\partial x_i = \delta_{ij} - \nabla_i u_j = (1 - \boldsymbol{\nabla}\boldsymbol{u})_{ij}$. For small displacement gradients $|\nabla_i u_j| \ll 1$ this factor can be ignored in the leading approximation, such that $\boldsymbol{v} \approx \partial \boldsymbol{u}/\partial t$. In the same approximation one also finds $\boldsymbol{w} \approx \partial \boldsymbol{v}/\partial t \approx \partial^2 \boldsymbol{u}/\partial t^2$.

3.3 Incompressible Newtonian fluid

The simplest full-fledged example of a continuum dynamics is offered by an incompressible fluid with constant density ρ and viscosity η, for which the constitutive equations take the Newtonian form[25],

$$\sigma_{ij} = -p\,\delta_{ij} + \eta\,(\nabla_i v_j + \nabla_j v_i) \;, \tag{29}$$

where p is the pressure field. Inserting this into Cauchy's equation (26) and including the incompressibility condition derived from the continuity equation (19) for constant density, we obtain the simplest form of the *Navier-Stokes equations* in a field of gravity[26],

$$\boxed{\rho\left(\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v}\right) = \rho\boldsymbol{g} - \boldsymbol{\nabla}p + \eta\nabla^2\boldsymbol{v} \;, \qquad \boldsymbol{\nabla}\cdot\boldsymbol{v} = 0 \;.} \tag{30}$$

If gravity is specified, for example constant, these equations close among themselves. Although superficially simple, the nonlinearity of the material acceleration on the left hand side makes the space of solutions extremely complex, as witnessed by the richness of form displayed by a waterfall. It is not even known whether these equations have smooth non-singular solutions[27]. Physically, the nonlinearity opens for chaotic time evolution which may lead to turbulence, so well-known from everyday dealings with fluids. After more than hundred years of intense studies, there is still no complete and accepted theory of turbulence.

In an incompressible fluid, the pressure is not directly related to the density by an equation of state (14), but rather determined by the divergence condition. This can be explicitly seen by calculating the divergence of the Navier-Stokes equation (30), leading to

$$\nabla^2 p = \rho\boldsymbol{\nabla}\cdot(\boldsymbol{g} - (\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v}) \;. \tag{31}$$

Evidently, the pressure is determined by Poisson's equation, and thus a non-local function of the velocity field. Although the first Navier-Stokes equation is local in the sense that the rate of change of the velocity field in a given point, $\partial\boldsymbol{v}/\partial t$, only depends on the fields in the immediate vicinity of this point, the inherent non-locality of the pressure implies that any local change in the velocity field is instantly communicated to all other parts of the fluid.

That is of course unphysical. Truly incompressible fluids do not exist, and in real compressible fluids the speed of sound sets an upper limit to the propagation of small-amplitude disturbances[28]. In simulations of incompressible fluids, the instantaneous Poisson equation for pressure also creates trouble, because it must be solved separately for each step in time to secure the continued vanishing of the divergence. Incompressibility is nevertheless always an important approximation. The Navier-Stokes equations for compressible fluids eliminate of course the problem of infinite sound speed, but the price to pay is that the system of differential equations becomes stiff, and more so for nearly incompressible fluids where flow velocities are everywhere much smaller than the sound velocity.

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[25] Viscosity represents internal friction in the fluid caused by neighboring layers of fluid "rubbing" against each other. Since a constant velocity field should not give rise to friction, the stress tensor can in the linear approximation only depend on the velocity gradients $\nabla_i v_j$. Requiring the stress tensor to be symmetric, one arrives at the above expression.

[26] Credited to Navier (1822) and Stokes (1845). For $\eta = 0$ the Navier-Stokes equation degenerate into the Euler equation (1755). Since all classical fluids are viscous, we shall not discuss the Euler equation here.

[27] Among the seven Millenium Prizes, each of one million dollars, offered by the Clay Mathematics Institute of Cambridge, Massachusetts, one concerns precisely the existence of smooth, non-singular solutions to (30) [8].

[28] Large amplitude disturbances can propagate much faster than the speed of sound in the form of shock waves. It should also be mentioned that in spite of the locality of the equations of motion, any solution to a diffusion equation (such as the Navier-Stokes equation with non-vanishing viscosity) has a Gaussian tail that even for arbitrarily small time intervals stretches all the way to infinite distances. But even if disturbances in principle can move with infinite velocity, the Gaussian damping implies that the diffusion front effectively only runs ahead of the front of a sound wave for a tiny time interval, satisfying $t < \eta/\rho c_S^2$ where c_S is speed of sound in the fluid.

3.4 Compressible Newtonian fluids

All fluids are in fact compressible, but behave as effectively incompressible when flow speeds are much smaller than the speed of sound, c_S. Provided the measurement precision is everywhere larger than the corrections due to the finite sound speed, the fluid may be taken to be incompressible. Compressibility always becomes important at sonic speeds and above, and at very high frequencies (ultrasound).

The stress tensor for isotropic compressible Newtonian fluids takes the more general form, analogous to the form of the stress tensor for linear elastic materials (16),

$$\sigma_{ij} = -p\,\delta_{ij} + \eta(\nabla_i v_j + \nabla_j v_i) + \left(\zeta - \frac{2}{3}\eta\right)\delta_{ij}\boldsymbol{\nabla}\cdot\boldsymbol{v}\ , \tag{32}$$

where $p = p(\rho, T)$ is the thermodynamic pressure given by the equation of state, and the parameter ζ is the so-called *expansion viscosity.* The Navier-Stokes equation now becomes,

$$\boxed{\rho\left(\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v}\right) = \rho\boldsymbol{g} - \boldsymbol{\nabla}p + \eta\nabla^2\boldsymbol{v} + \left(\zeta + \frac{1}{3}\eta\right)\boldsymbol{\nabla}\boldsymbol{\nabla}\cdot\boldsymbol{v}\ .} \tag{33}$$

If the temperature field is specified, for example a constant, this equation together with the equation of continuity (19) and the equation of state(14) form a complete set of five coupled partial differential equations for the density ρ, the pressure p, and the three velocity components (v_x, v_y, v_z).

The most important boundary condition is the *no-slip condition*, which requires the velocity field to vanish at any solid wall. The velocity component normal to the wall must vanish, because the fluid cannot penetrate into the wall. The tangential component must also vanish, because of viscous friction which would otherwise generate infinite restoring stresses if the velocity jumped. At an open surface between two different fluids, the normal velocity component must be continuous for the fluids to stay in contact, and the tangential component must as before be continuous because of viscosity[29]. Furthermore, as discussed before, the stress vector must also be continuous, whereas the pressure in general will jump at an interface because of the different viscosities. Pressure loses in fact much of its intuitive appeal, known from hydrostatics, when it comes to materials with non-trivial constitutive equations.

4 The heat equation

The mechanical equations of continuum physics tell, however, only half the story. Except for idealized circumstances, for example enforced constant temperature, heat also plays a role in almost all physical and chemical systems. Not only do all materials conduct heat, but the constitutive equations depend in general on temperature, either explicitly as in the equation of state, or implicitly through the temperature dependence of the material parameters. From the First Law of thermodynamics generalized to continuous matter and Cauchy's equation, a local equation of *energy balance* may be derived. Using thermodynamic relations this equation can be converted into a parabolic partial differential equation for the temperature field, also called the *heat equation*, which in simple matter takes the form of a diffusion equation.

[29] It should be mentioned that in supersonic flows, shock fronts may arise that display very rapid transitions in velocity as well as in density, temperature, and pressure. Shocks do not correspond to interfaces between different fluids but take place inside the volume of a homogeneous fluid. In nearly inviscid fluids, the shock fronts are so narrow that they may (in fact must) be represented by true discontinuities.

4.1 Energy balance

The First Law of Thermodynamics states that a change in the total energy of a system is either caused by heat added to the system or by work performed on it[30]. In continuum physics it turns out to be most convenient to factor out the mass density and write the energy density as $\rho\,\epsilon$ where ϵ is called the *specific energy*, i. e. the energy per unit of mass[31]. Taking into account that energy can be transported through the surface of a body, we obtain the equation of *global energy balance*

$$\frac{d}{dt}\int_V \rho\,\epsilon\,dV + \oint_S \rho\,\epsilon\,\boldsymbol{v}\cdot d\boldsymbol{S} = \int_V h\,dV - \oint_S \boldsymbol{q}\cdot d\boldsymbol{S} + \oint_S \sum_{ij} \boldsymbol{v}\cdot\boldsymbol{\sigma}\cdot d\boldsymbol{S}\;. \tag{34}$$

The left hand side takes the now familiar form of the rate of change of energy in the body's volume plus the rate of energy flow out of the volume. The first term on the right hand side represents the production of heat by chemical and nuclear processes at the local rate $h(\boldsymbol{x}, t)$, whereas the second represents the conduction of heat into the volume through its surface with current density $\boldsymbol{q}(\boldsymbol{x}, t)$. The last term is the rate of work of the contact forces, calculated as the scalar product of the local velocity $\boldsymbol{v}$ and the surface force $d\boldsymbol{\mathcal{F}} = \boldsymbol{\sigma}\cdot d\boldsymbol{S}$. Converting as before the surface integrals to volume integrals by means of Gauss' theorem, we obtain the equation of *local energy balance*,

$$\frac{\partial(\rho\,\epsilon)}{\partial t} + \boldsymbol{\nabla}\cdot(\rho\,\epsilon\boldsymbol{v}) = h - \boldsymbol{\nabla}\cdot\boldsymbol{q} + \sum_{ij}\nabla_j(v_i\sigma_{ij})\;. \tag{35}$$

The local source terms on the right hand side reflect the double origin of energy change, heat or work, as stated by the First Law. Using the continuity equation (19) the left hand side may (as we did for momentum balance) be rewritten as $\rho d\epsilon/dt$, where $d\epsilon/dt = \partial\epsilon/\partial t + (\boldsymbol{v}\cdot\boldsymbol{\nabla})\epsilon$ is the comoving time derivative.

From ordinary Newtonian particle mechanics, we expect that the energy of a material particle is composed primarily of its kinetic energy $\frac{1}{2}\boldsymbol{v}^2 dM$, and its potential energy ΦdM in an external gravitational potential $\Phi(\boldsymbol{x})$, assumed here to be time independent. This leads us to decompose the specific energy into a sum of three contributions,

$$\epsilon = \frac{1}{2}\boldsymbol{v}^2 + \Phi + u\;, \tag{36}$$

where the last term u represents the *specific internal energy* of the material, due to its state of compression, temperature, and chemical composition. From this expression we obtain the comoving time derivative,

$$\rho\frac{d\epsilon}{dt} = \rho\boldsymbol{v}\cdot\frac{d\boldsymbol{v}}{dt} + \rho(\boldsymbol{v}\cdot\boldsymbol{\nabla})\Phi + \rho\frac{du}{dt} = \boldsymbol{v}\cdot(\boldsymbol{f} + \boldsymbol{\nabla}\cdot\boldsymbol{\sigma}^{\mathsf{T}}) - \boldsymbol{v}\cdot\rho\boldsymbol{g} + \rho\frac{du}{dt}\;, \tag{37}$$

where we in the last step have used Cauchy's equation (26) in the form $\rho d\boldsymbol{v}/dt = \boldsymbol{f} + \boldsymbol{\nabla}\cdot\boldsymbol{\sigma}^{\mathsf{T}}$, and the definition of the gravitational field (4). Inserting this into the local energy balance (35) and substituting the effective force density (8) (with $\boldsymbol{f} = \rho\boldsymbol{g}$), we arrive at the equation of *internal energy balance*,

$$\boxed{\rho\left(\frac{\partial u}{\partial t} + (\boldsymbol{v}\cdot\boldsymbol{\nabla})u\right) = h - \boldsymbol{\nabla}\cdot\boldsymbol{q} + \sum_{ij}\sigma_{ij}\nabla_j v_i\;.} \tag{38}$$

[30] The First Law is strictly speaking a law of energy balance. It is, however, often called energy conservation, because energy is conserved for isolated systems, implying that for a non-isolated system any change in a system's energy may be expressed through its exchange of energy with the environment (see also footnote 22).

[31] Mass conservation is the deeper reason for the importance of specific energy (and other specific quantities).

The first source terms represent as before the rate of local heat production and conduction, whereas the last represents the rate of local mechanical work of the stresses in the material, i.e. work due to local compression as well as to local internal friction, also called *dissipation.* It is the last term which provides the heat for melting the surface of a meteorite entering the atmosphere, and even for its complete evaporation.

4.2 Heat equation for incompressible fluid

The conversion of the equation of internal energy balance into a dynamic equation for the temperature field requires in general the full apparatus of thermodynamics [9, 10]. To elucidate the steps we consider the simplest of all materials, an incompressible Newtonian fluid with constant density and stress tensor given in (29). We shall, somewhat unphysically[32], assume that the density and viscosity do not depend on the temperature. The specific energy can in that case only depend on the local temperature, and for simplicity we shall assume that the relation is linear, $u = c_v T$ where c_v is the specific heat at constant volume (heat capacity per unit of mass). Finally we adopt Fourier's law

$$\boldsymbol{q} = -k\boldsymbol{\nabla}T \ , \tag{39}$$

where the positive coefficient k is called the *thermal conductivity.* Fourier's law expresses that heat always flows against the temperature gradient, i. e. from hot to cold[33]. Putting it all together we arrive at the heat equation

$$\boxed{\rho c_v \left(\frac{\partial T}{\partial t} + (\boldsymbol{v}\cdot\boldsymbol{\nabla})T\right) = k\nabla^2 T + h + \frac{1}{2}\eta\sum_{ij}\left(\nabla_i v_j + \nabla_j v_i\right)^2 \ .} \tag{40}$$

The last term has here been rewritten in a symmetric form which clearly exposes that it is always positive, showing that the internal viscous friction always adds heat to the fluid[34]. The coupling between the velocity field and the temperature field on the left hand side expresses the well-known bathroom knowledge that heat can be transported (advected) by moving fluid[35].

The derivation of the heat equation for a general compressible fluid is as mentioned considerably more involved[9, 10]. If the flow speed is much smaller than the velocity of sound, the resulting equation becomes nearly identical to eq. (40) except that the on the left hand side c_v is replaced by the isobaric specific heat capacity c_p, and that on the right hand side is added an extra positive dissipation term, $\zeta(\boldsymbol{\nabla}\cdot\boldsymbol{v})^2$, representing the contribution from the expansion viscosity.

The boundary conditions for the temperature field depend on the kind of walls that enclose the fluid. At a wall that conducts heat, the temperature field must be continuous, whereas at an insulating wall that does not conduct heat it follows from Fourier's law that the gradient of the temperature must vanish.

[32] Most materials expand in fact slightly when they are heated, so that the density decreases. In combination with gravity, even a tiny decrease in the density of heated fluid may cause it to rise buoyantly and create convective currents in a fluid otherwise at rest. Heat driven flow is of great importance (and sometimes a nuisance) in the kitchen as well as in industry.

[33] The positivity of the thermal conductivity is a consequence of the Second Law of Thermodynamics.

[34] In most practical settings this dissipative heat production is tiny and can be disregarded. There are important cases where viscous friction heats the fluid significantly, for example in a journal bearing with failing circulation of the lubricant.

[35] For a fluid at rest, $\boldsymbol{v} = \boldsymbol{0}$, with no heat production, $h = 0$, the heat equation simplifies to Fourier's famous diffusion equation for the temperature field,

$$\frac{\partial T}{\partial t} = \kappa\nabla^2 T \ ,$$

where $\kappa = k/\rho c_v$ is the thermal diffusivity.

5 Laws of balance and conservation

In the preceding discussion we have distinguished sharply between mass *conservation* and momentum and energy *balance*. As mentioned already in footnotes 22 and 30, the distinction is perhaps more a matter of taste than of substance, because momentum and energy are in fact conserved for isolated bodies, and consequently, for a non-isolated body a change in momentum or energy can be accounted for through exchange of these quantities with the environment

The local laws of conservation and balance that we have so far derived may all be written in the form,

$$\frac{\partial D}{\partial t} + \boldsymbol{\nabla} \cdot \boldsymbol{J} = S \tag{41}$$

where D is a density, $\boldsymbol{J}$ is a current density, and S is the source. For mass, momentum and energy we have[36],

$$\begin{aligned}
&D \to \rho \ , && \boldsymbol{J} \to \rho \boldsymbol{v} \ , && S \to 0 \ , \\
&D \to \rho \boldsymbol{v} \ , && \boldsymbol{J} \to \rho \boldsymbol{v}\boldsymbol{v} \ , && S \to \boldsymbol{f} + \boldsymbol{\nabla} \cdot \boldsymbol{\sigma}^{\mathsf{T}} \ , \\
&D \to \rho \epsilon \ , && \boldsymbol{J} \to \rho \epsilon \boldsymbol{v} \ , && S \to h - \boldsymbol{\nabla} \cdot \boldsymbol{q} + \boldsymbol{\nabla} \cdot (\boldsymbol{v} \cdot \boldsymbol{\sigma}) \ .
\end{aligned}$$

Notice that the "current density" for momentum is actually a tensor ($J_i \to T_{ij} = \rho v_i v_j$). To facilitate readability, the source of energy has been written in a compact matrix form.

Moving the source terms that are divergenses over to the current density we obtain,

$$\begin{aligned}
&D \to \rho \ , && \boldsymbol{J} \to \rho \boldsymbol{v} \ , && S \to 0 \ , \\
&D \to \rho \boldsymbol{v} \ , && \boldsymbol{J} \to \rho \boldsymbol{v}\boldsymbol{v} - \boldsymbol{\sigma}^{\mathsf{T}} \ , && S \to \boldsymbol{f} \ , \\
&D \to \rho \epsilon \ , && \boldsymbol{J} \to \rho \epsilon \boldsymbol{v} + \boldsymbol{q} - \boldsymbol{v} \cdot \boldsymbol{\sigma} \ , && S \to h \ .
\end{aligned}$$

In the absence of volume forces such as gravity (and electromagnetism), $\boldsymbol{f} = \boldsymbol{0}$, and heat production, $h = 0$, the equations of balance may formally be written as true conservation laws. Mathematically, and especially for numerical simulations, it may be convenient to cast the laws of balance in this common form, whereas physically it is conceptually better to retain the distinction between advection of, for example, momentum and the contact forces acting on the surface[37].

To obtain the above conservation laws we had to leave out all true volume forces like gravity and electromagnetism. This can be traced back to our omission of the dynamical laws for gravitational and electromagnetic fields. Were these included, the equations of motion of momentum and energy for the coupled systems could in fact have been written as conservation laws. Whereas it is reasonably straightforward to include the dynamics of electromagnetic fields (i. e. the Maxwell equations), gravity requires the full apparatus of general relativity.

6 Beyond traditional continuum physics

The laws of mass conservation, momentum balance and energy balance, constitute the general framework for establishing the equations of motion in continuum physics. Extensions to nonlinear

[36] One may wonder what happened to angular momentum. For classical continua angular momentum balance is, however, like kinetic energy balance, a simple consequence of local momentum balance, as expressed through Cauchy's equation (26). Practically, it means that any exact solution to the local equations of continuum dynamics automatically satisfies all the global laws of conservation and balance, including angular momentum balance. For approximate solutions, for example numerical simulations, the global laws impose useful constraints that may be used to judge the precision of the approximation.

[37] Although formally correct, it does seem strange to say about a person standing still on the floor that the momentum constantly transferred to all his parts by the field of gravity is leaving through the contact area of his feet, rather than saying that gravity is balanced by the contact forces acting on his feet.

elastic, viscoelastic or any other types of materials are in this respect straightforward[12], although the details can be highly complex and involve further equations of motion for other quantities.

Leaving the realm of classical (Newtonian) physics and entering the subject of special or general relativistic continuum physics, it is necessary to modify the mechanical equations of motion, even if the fundamental conservation laws can be maintained [9]. The principle of mass conservation, which is so important in non-relativistic physics, is generally lost in relativity because of the equivalence of mass and energy. Instead there are strict laws of baryon (and lepton) number conservation which under suitable circumstances may mimic mass conservation[38], and lead to a continuity equation for baryonic (or leptonic) mass. Relativistic flows are essentially only known from the extreme objects encountered in astrophysics and subnuclear physics. For the most compact astrophysical objects, for example neutron stars and black holes, gravitational fields are strong and general relativity has to be invoked. In other cases gravity plays little or no role, as for example in high-energy heavy-ion collisions, and the much simpler special theory of relativity can be brought into play[39].

Although traditional continuum physics is always an approximation to the underlying discrete atomic level, this is not the end of the story. At a deeper level it turns out that matter is best described by another continuum formalism, relativistic quantum field theory, in which the discrete particles — electrons, protons, neutrons, nuclei, atoms, and everything else — arise as quantum excitations in the fields. Relativistic quantum field theory without gravitation emerged in the twentieth century as *the* basic description of the subatomic world, but in spite of its enormous success it is still not clear how to include gravity. Just as the continuity of macroscopic matter is an illusion, the quantum field continuum may itself one day become replaced by even more fundamental discrete or continuous descriptions of space, time, and matter. It is by no means evident that there could not be a fundamental length in nature setting an ultimate lower limit to distance and time, and theories of this kind have in fact been proposed. It appears that we do not know, and perhaps will never know, whether matter at its deepest level is truly continuous or truly discrete.

[38] Provided inelastic collisions, decay, and matter/antimatter annihilation can be disregarded.
[39] See for example the recent review [11] of numerical methods in this strongly evolving field.

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