Apparent superluminal behavior in wave propagation

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The apparent superluminal propagation of electromagnetic signals seen in recent experiments is shown to be the result of simple and robust properties of relativistic field equations. Although the wave front of a signal passing through a classically forbidden region can never move faster than light, an attenuated replica of the signal is reproduced ''instantaneously'' on the other side of the barrier. The reconstructed signal, causally connected to the forerunner rather than the bulk of the input signal, appears to move through the barrier faster than light.

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I. INTRODUCTION

Recent experimental reports of the propagation of electromagnetic signals with velocities larger than c in dispersive media [1], in waveguides [2], and in electronic circuits [3] have once again focused attention on a subject of long-standing interest. The immediate and widespread hostility that such reports seem to generate among theorists is an understandable consequence of a firm belief in the consistency of electromagnetism and relativity. In order to dispel such concerns at the outset, we distinguish between ''true'' and ''apparent'' superluminal phenomena. Consider a right-moving pulse with a wave front located at x_0 at $t=0$. A true superluminal phenomenon would permit the observation of some signal at positions $x>x_0+ct$. True superluminality is not predicted by either Maxwell's equations or the Klein-Gordon equation which we shall consider here. Indeed, most recent experimental papers are careful to emphasize their consistency with Maxwell's equations and, hence, do not claim the observation of true superluminal effects. Rather, these experiments have demonstrated the existence of ''apparent'' superluminal phenomena taking place well behind the light front. In the case of waveguide experiments, observations generically involve pulses that seem to traverse a ''classically forbidden'' region instantaneously. While these results illuminate an interesting and potentially useful effect, such transmission occurs well behind the light front and does not challenge the wisdom and authority of Maxwell and Einstein. As we shall see below, apparent superluminality is extremely general and to be expected.

Papers by Sommerfeld and Brillouin [4] represent some of the earliest and most beautiful investigations of the question of superluminality. Their concern was with unbounded, dispersive media. There were at the time abundant examples of anomalous dispersion, i.e., substances for which phase and group velocities were both larger than c. Since the group velocity was then believed to be identical to the velocity of energy propagation, Sommerfeld and Brillouin understandably found the question of superluminal propagation to be of importance. Their strategy was to write the requisite propagator using Laplace transforms and a suitable analytic form for the phase velocity $v(\omega)=\omega/k$. The fact that the singularities of $v(\omega)$ were restricted to the lower half ω plane was then sufficient to prove that the signal is necessarily zero ahead of the light front and that the light front always moves with a velocity of c. While the efforts of Sommerfeld and Brillouin would seem to have settled the issue of true superluminal propagation definitively, the situation is somewhat more subtle. Their work conclusively demonstrated that Maxwell's equations preclude superluminal propagation for media with a causal form of $v(\omega)$. It did not, however, extend to a proof that the singularities of $v(\omega)$ must lie in the lower half plane. In simple electron resonance models of dielectrics, such behavior follows from the absorptive nature of the material [5]. We are not, however, aware of any completely general proof of material causality. For this reason, present considerations will be confined to modeling dielectric-free waveguides.

The present paper addresses the current issue of apparent superluminality. In order to avoid the difficult issue of modeling $v(\omega)$, we will restrict our attention to propagation in two-dimensional wave guides with constrictions. For slow variations in the shape of the constriction, Maxwell's equations reduce to a one-dimensional Klein-Gordon equation, in which the nonuniformities can be modeled through a suitable potential. A side benefit of this replacement is that the strict impossibility of true superluminal propagation is easily demonstrated. We then consider the propagation of a wave form with a precise front initially to the left of a potential barrier located in the interval $0\leqslant x\leqslant b$. The barrier is presumed to be high relative to the dominant wave numbers contained in the pulse but is otherwise arbitrary. Our results for the incoming and transmitted waves can then be expressed simply: The incoming wave moves with a uniform velocity of $c=1$. The transmitted wave $\psi(x,t)$ (for $x>b$) is attenuated as an obvious consequence of barrier penetration, and its amplitude is proportional to the derivative of the initial pulse evaluated at the point $(x-ct-b)$. The additional displacement b suggests instantaneous transmission of the pulse through the barrier and is the source of the apparent superluminality observed empirically. The fact that the transmitted pulse is an image of the derivative of the original pulse and not the pulse itself is an elementary consequence of the fact that transmission amplitudes generally vanish in the limit $\omega\rightarrow 0$. When the signal is a low-frequency modulation of a carrier wave, as is the case in many experimental investiga-

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tions, the envelopes of the incident and transmitted waves are identical.

Some of the topics treated here have been discussed elsewhere both analytically and numerically [6,3]. Our intention is to emphasize the generality and extreme simplicity of this phenomenon.

II. THE MODEL

We consider a scalar wave ψ moving in a one-dimensional potential according to the Klein-Gordon equation

$$-\frac{\partial^2\psi}{\partial x^2}+V(x)\psi=-\frac{\partial^2\psi}{\partial t^2}. \tag{1}$$

This problem is closely related to that of propagation in a two-dimensional waveguide according to

$$\nabla^2\Psi=\frac{\partial^2\Psi}{\partial t^2}. \tag{2}$$

The waveguide is assumed to be infinite in the x direction and to extend from $0\leqslant z\leqslant h(x)$ in the z direction. If Ψ vanishes at the transverse bounding surfaces and if h is a slowly varying function of x, we can approximate Ψ as the product $\psi(x,t)\sin(\pi z/h)$. Neglecting derivatives of h, Eq. (2) reduces to Eq. (1). The potential is determined by the width of the waveguide so that $V(x)=\pi^2/h(x)^2$ for the lowest transverse mode. For simplicity, however, we will consider choices of $V(x)$ that are nonzero only in the region $0\leqslant x\leqslant b$.

We seek solutions to Eq. (1) that describe the motion of an initial pulse $\psi(x,0)=f(x)$, which has arbitrary shape but satisfies the following two conditions. First, the pulse has a well-defined wave front x_0 initially to the left of the barrier $V(x)$, i.e., $\psi(x,0)\neq 0$ only when $x\leqslant x_0<0$. Second, at $t=0$ the pulse moves uniformly to the right with a velocity of 1, so that $\partial\psi/\partial t=-\partial\psi/\partial x$ at $t=0$. For any given potential, this problem can be solved with the aid of the corresponding Green's function.

This model was considered in detail in Ref. [7], where it was shown in generality that the transmitted wave is given by

$$\psi(x,t)=\int_{-\infty}^{x_0}\tilde{T}(x-t-x')f(x')dx', \tag{3}$$

where $\tilde{T}(x)$ is the retarded transmission kernel, which may be expressed in terms of its Fourier transform $T(\omega)$

$$\tilde{T}(u)=\int_{-\infty}^{\infty}T(\omega)e^{i\omega u}\frac{d\omega}{2\pi}. \tag{4}$$

The physical interpretation of $T(\omega)$ as a transmission amplitude is elementary: An incoming plane wave $\exp(i\omega x)$, incident on the potential barrier from the left, leads to a transmitted wave $T(\omega)\exp(i\omega x)$. Since $|T(\omega)|\to 1$ for $|\omega|\to\infty$, the integration contour in Eq. (4) can be closed in the upper half ω plane for $x>0$. If $T(\omega)$ is free of singularities in the upper half plane, it follows that $\tilde{T}(x-t-x')=0$ for $x>x'+t$ and thus that $\psi(x,t)$ is strictly zero for $x>x_0+t$. Nothing precedes the light front.

A. A special case

The authors of Ref. [7] considered the special case where $V(x)=m^2$ is a positive constant inside the barrier region $0\leqslant x\leqslant b$ and found

$$T(\omega)=\frac{4\omega\kappa}{D}e^{i(\kappa-\omega)b} \tag{5}$$

with $\kappa=i\sqrt{m^2-(\omega+i\epsilon)^2}$ and

$$D=(\omega+\kappa)^2-(\omega-\kappa)^2e^{2i\kappa b}. \tag{6}$$

[The discontinuity in $V(x)$ implies that Eq. (1) is not a numerically reliable approximation to the corresponding two-dimensional problem of a waveguide with a sudden constriction. Nevertheless, the apparent superluminal behavior displayed below is strictly analogous to that seen in real waveguides [6].] The singularities of $T(\omega)$ are due to the zeros of D in the ω plane. Given the form of κ, the zeros of D are confined to the lower half plane, and $T(\omega)$ is indeed analytic in the upper half plane. As expected, this model precludes genuine superluminal propagation. The analytic properties of $T(\omega)$ are, of course, dictated by those of $V(\omega)$. The general proof that any given potential will lead to such analyticity is more challenging.[1] All real, local, and bounded potentials that vanish sufficiently rapidly as $x\to\infty$ are expected to respect these analyticity conditions, and the absence of true superluminality is thus to be expected for all physically sensible choices of V.

B. Apparent superluminal behavior

Confident that our Klein-Gordon model is free of genuine acausal propagation, we turn to apparent superluminal phenomena. Consider a strong barrier (with $mb\gg 1$) and imagine that the initial wave form $\psi(x,0)$ is dominated by low-frequency components for which $|\omega|\ll m$. In this case, the form of $\psi(x,t)$ is both simple and intuitive. Specifically, we need only consider $\omega\approx 0$ for which $\kappa\approx im$. In this domain, the transmission amplitude can be approximated as

$$T(\omega)\approx-\omega\frac{4i}{m}e^{-i\omega b}e^{-mb}. \tag{7}$$

We shall see shortly that this form of the transmission amplitude is quite general. Using Eq. (7), we see that Eq. (4) reduces to

[1]A general proof can be constructed along the following lines. Write the transmission amplitude as a linear integral equation of the form $\psi=\varphi+\int G_0 V\,\psi$, where G_0 is a suitable free propagator. Singularities in ψ arise when singularities of the integrand pinch the integration contour. Analyticity of $V(\omega)$ in the upper half plane then ensures the desired analyticity properties of $T(\omega)$.

$$\tilde{T}(x-t-x')=-\frac{4}{m}e^{-mb}\frac{\partial}{\partial u}\delta(u-b)\Big|_{u=x-t-x'}. \quad (8)$$

Thus, we find that

$$\psi(x,t)\approx-\frac{4}{m}e^{-bm}f'(x-t-b). \quad (9)$$

When a pulse dominated by low-frequency components impinges on a strongly repulsive barrier, the transmitted wave is a strongly attenuated replica of the derivative of the original pulse. The transmitted pulse appears to traverse the region of the potential barrier in zero time. This is the apparent superluminal phenomenon observed empirically. It occurs well behind the light front of the original signal and is not an indication of true superluminal propagation. Rather, it is an interference phenomenon which is in no sense acausal.

There is an evident inconsistency between the present assumption that $\psi(x,0)$ is dominated by low-frequency components and our initial assumption that the signal has a well-defined light front (which necessarily implies the presence of high-frequency components). The consideration of signals that are the product of, e.g., a Gaussian pulse (with clear low-frequency dominance) and a step function to impose the light front makes it clear that the effects of this inconsistency can be made arbitrarily small [7].

C. Carrier waves

Experiments frequently involve a modulated carrier wave

$$f(x)=e^{i\omega_0 x}F(x), \quad (10)$$

where $F(x)$ provides a slowly varying modulation of the carrier. Inserting this into Eq. (9), the transmitted signal becomes

$$\psi(x,t)=-\frac{4}{m}e^{-bm}[i\omega_0F(u)+F'(u)]e^{i\omega_0 u}|_{u=x-t-b}. \quad (11)$$

Since the Fourier transform of $F(x)$ is presumed to have support only for frequencies $|\omega|\ll\omega_0$, the second term may be ignored. We conclude that the envelope of the pulse, $|\psi(x,t)|$, is unaltered by the transmission. Again, the argument of the right side of Eq. (11) suggests that transmission of the envelope through the barrier is instantaneous.

III. GENERALITY OF THE RESULTS

The various factors contributing to the approximate form Eq. (7) for $T(\omega)$ are all of general origin. The factor $\exp(-i\omega b)$ represents the phase difference between the free plane wave $\exp(i\omega x)$ at the boundaries of the region of nonzero potential. It will always appear. Similarly, the linear vanishing of the transmission amplitude as $\omega\to 0$ is a feature common to all potentials that do not have a zero-energy bound state.[2]

The final barrier penetration factor is also familiar and is expected whenever there is strong attenuation. Consider the transmission amplitude for a strongly repulsive but otherwise arbitrary potential using the WKB approximation.[3] The resulting transmission amplitude is readily calculated, and shows that the factor $\exp(-mb)$ is replaced by

$$\exp\left[-\int_0^b\sqrt{V(x)}dx\right]. \quad (12)$$

A localized repulsive barrier of sufficient strength will transmit an instantaneous image of the derivative of the incoming signal according to Eq. (9) independent of the details of both the potential barrier and the pulse. Apparent superluminal behavior is a robust phenomenon.

A. The time delay

The above results can also be expressed as a time delay of the pulse τ defined as the difference between the time actually required for transmission across the barrier and the time required for a free wave to travel the same distance. In the case of a square barrier and a low-frequency pulse, $\tau=-b$. Negative values of τ correspond to apparent superluminal propagation. For the modulated carrier wave Eq. (10), one may expand $T(\omega)$ about the carrier frequency ω_0 and obtain

$$T(\omega)\approx|T(\omega_0)|e^{i\Phi(\omega_0)}e^{i(\omega-\omega_0)\Phi'(\omega_0)}, \quad (13)$$

where $\Phi(\omega)$ is the phase of $T(\omega)$. The second exponential factor gives rise to the time delay $\tau(\omega_0)=\Phi'(\omega_0)$ through the Fourier exponential in Eq. (4). Familiar results from quantum mechanics for purely repulsive potentials remind us that $\Phi(\omega)$ is less than 0 for all ω and that $\Phi(0)=\Phi(\infty)=0$.[4] The time delay is necessarily negative for sufficiently small ω_0 and apparent superluminal effects can be observed for all repulsive potentials. The time delay changes sign for

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[2]Consider scattering from an arbitrary potential that is zero except in the interval $0\leqslant x\leqslant b$ with an interior solution $\varphi(x)$. Join this interior solution to the right-moving plane wave $\exp i\omega(x-b)$ at $x=b$ with the usual requirement of continuity of the wave function and its first derivative. In the limit $\omega\to 0$, $\varphi(b)=1$ and $\varphi'(b)=0$. Similarly, join the interior solution to the linear combination $A\exp(i\omega x)+B\exp(-i\omega x)$ at $x=0$. If $\varphi'(0)\neq 0$, the coefficients A and B will diverge like $1/\omega$. The transmission amplitude $T(\omega)=e^{-i\omega b}/A$ will vanish linearly with ω unless $\varphi'(0)$ is zero. The condition that $\varphi'(0)=\varphi'(b)=0$ is precisely the condition that the potential should support a zero-energy bound state.

[3]We consider a potential that is strongly repulsive for all $0\leqslant x\leqslant b$ and zero elsewhere. Hence, it is appropriate to match the plane wave solutions directly to the WKB wave function.

[4]This second result is a consequence of Levinson's theorem which relates the asymptotic behavior of the phase shift to the number of bound states supported by the potential n through $\Phi(0)-\Phi(\infty)=n\pi$. There are no bound states for purely repulsive potentials.

some value of ω_0 comparable to the height of the potential barrier, and it approaches zero from above as ω_0 tends to infinity. Apparent superluminality is a very general phenomenon.

IV. CONCLUSIONS

Using the model of a Klein-Gordon equation with a potential, we have presented a simple description of the apparent superluminal phenomena seen in waveguides: Low-frequency waves seem to traverse such barriers in zero time. Strongly repulsive barriers always transmit an attenuated image of the spatial derivative of the incident signal. When the signal consists of a modulated carrier wave, the envelope of this wave is transmitted unaltered. We have attempted to demonstrate that the phenomena described here are both extremely general and noncontroversial. Their experimental observation does not challenge received wisdom and in no sense compromises our confidence in general notions of causality. It will be interesting to see whether this interesting and general consequence of wave theory will have practical applications.

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