RECENT DEVELOPMENTS IN THE COMPARISON BETWEEN THEORY AND EXPERIMENTS IN QUANTUM ELECTRODYNAMICS

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Abstract

This review is a survey of three main topics in quantum electrodynamics: fundamental bound systems, anomalous magnetic moments and high energy experiments. The emphasis lies particularly in recent developments concerning the electron and muon anomalous magnetic moments.
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0. Introduction

In this review we shall be concerned with recent developments in the calculation of quantum electrodynamics (QED) effects, and with recent experiments which test the validity of the theory of QED. From the onset we wish to emphasize that by theory of QED we understand the precise set of rules which stem from the fundamental works* of Feynman, Schwinger, and Tomonaga; and which allow us to predict observable effects of the interaction of light with charged leptons.

Possible breakdown of the validity of QED has been searched for in two main directions: (i) precision measurements in simple structures like the hydrogen atom, muonium and positronium, and measurements of the electron and muon anomalous magnetic moments; (ii) scattering experiments at high momentum transfer. Experiments of the first type attain accuracies which in some instances** are of a few parts per million (ppm), and they are sensitive to higher order terms predicted by the perturbation expansion in the fine structure constant \( \alpha (\alpha \sim 1/137) \). The second type of experiments involves large momentum transfers, typically of a few GeV/c, and they specifically probe the validity of QED at small distances (of the order of a tenth to a hundredth of a fermi, \( 1 \text{ fm} = 10^{-13} \text{ cm} \)).

Historically, it was the measurement of the \( 2S_{1/2} \) and \( 2P_{1/2} \) level displacement in the hydrogen atom*** which led to the development of quantum electrodynamics. Since then, parallel progress has taken place in higher order calculations and in the accuracy of experiments. In their first experiment, Lamb and Retherford [001] showed that the \( 2S \) level of hydrogen, which in the theory of Dirac coincides with the \( 2P_{1/2} \) level, was actually higher than the latter "by about" 1000 MHz. At present, the experimental accuracy of this Lamb shift is \( \sim 50 \text{ ppm} \) and is sensitive to corrections† of relative order \( \alpha^2 \). Precision experiments of energy level splittings have now been made in hydrogen, in deuterium, in helium, in lithium, and also in other hydrogen-like systems, muonium, positronium, and \( \mu \)-mesic atoms.

The experiment of Kusch and Foley [002], which first gave direct indication of an anomalous magnetic moment for the electron, was accurate enough (\( \alpha_e = 0.01115 \pm 0.0004 \)) to confirm Schwinger's calculation‡† (\( \alpha_e = \alpha/2\pi \)). At present the accuracy of the measurement of \( \alpha_e \) is \( 1 \text{ ppm} \) and an adequate comparison with the theory requires the inclusion of the sixth order contribution to \( \alpha_e \)‡‡.

Does the muon have an anomalous magnetic moment as predicted by QED? A positive answer to this important question, which bears upon one of the greatest puzzles in physics, i.e., the origin of the muon-electron mass difference, has been given by beautiful experiments performed at CERN. The accuracy of the first \( g-2 \) muon experiment [004] was \( 4 \times 10^{-4} \). The latest \( g-2 \) experiment [007, 008] has attained an accuracy of 28 parts in \( 10^8 \) and has clearly confirmed the vacuum polarization terms with virtual electron pairs predicted by QED+. The accuracy will very likely improve by a factor of 10 to 30 in the next generation of \( g-2 \) experiments++.

---

*There exists a very useful compilation of the early parts on quantum electrodynamics edited by Schwinger, see J. Schwinger in the list of review articles at the end, ref [R 15].
**For the hyperfine-splitting of the hydrogen ground state the present experimental accuracy is actually 2 parts to \( 10^7 \).
***See Lamb and Retherford [001], reprinted in the compilation quoted in ref [R 15], paper No 11.
†See the discussion in section 11.
‡‡See Schwinger [003], reprinted in the compilation quoted in ref [R 15], paper No 13.
‡‡‡See the discussion in section 4.
+The calculation of this contribution was first done by Peterman [005], and by Suura and Wichman [006]. For further details see the discussion of section 5.
++Private communication from John Barley, Francis Barley and Milko Picasso.
The understanding of the $g_\mu - 2$ experimental result has been a challenge to theorists which has led to new developments in computational techniques. An early discrepancy between theory and experiment motivated the calculation of all the terms predicted by QED which at sixth order make the anomaly of the muon different from that of the electron. The result of these calculations has brought the theoretical prediction again within the errors of the experiment.*

An important question about the precision tests of QED is the accuracy they can attain before they are also sensitive to the electromagnetic interactions of hadrons. The electron anomaly and systems like positronium and muonium are still far from being influenced by the electromagnetic interactions of hadrons. This makes them the more excellent candidates for future precision tests of QED. As we shall see the situation is different for the Lamb shift, the hyperfine structure in hydrogen, and for the muon anomaly. Here, an adequate comparison between theory and experiment requires the knowledge of contributions due to the electromagnetic interactions of hadrons. It is remarkable that in some cases, like the muon anomaly, the hadronic contributions can be related to empirical information already available from the high energy electron experiments. A very interesting link between high energy experiments and precision low energy experiments is thereby developing.

The high energy experiments are also useful from another point of view. They can test the validity of the QED rules for the lepton and photon propagators in processes where the Born approximation is adequate yet the momentum transfers involved are large (of a few GeV/c with the present machines).

The main purpose in writing this review was to discuss the recent developments concerning the anomalous magnetic moment of the muon. This is done in sections 5, 6, and 7. Clearly this requires a parallel discussion of the electron anomaly as well, which is made in sections 4 and 7. We felt, however, that in order to keep some balance, there should also be a brief review of other fundamental QED topics where there has been some recent developments. We have therefore included a first part on low energy tests in fundamental systems, sections 1, 2, and 3, and a third part on electron and photon high energy experiments, section 8.

There is an important topic with implications for QED which is not discussed in this review, the determination of $\alpha$ from the ac Josephson effect. The reason for this is that there already exists an extremely detailed exposition of the subject** to which we have nothing to add. Of course, in all the numerical estimates in the text we shall indicate the appropriate origin for the value of $\alpha$ which is used.

We should finally like to point out that there are various excellent review articles on different aspects of QED in the literature***. In writing this review we have attempted more to complement the already existing literature rather than to write an encyclopaedic review of QED.

One comment about notations. We use the same metric and Dirac matrices as in J D Bjorken and S D Drell's textbooks (McGraw Hill, New-York).

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*This is discussed in detail in section 5

**See Taylor, Langenberg and Parker [R 16]

***See the list of review articles at the end
In the first part, we shall consider three fundamental bound systems: the hydrogen atom, muonium, and positronium.* We shall review the present status in the comparison between the theory and experiments concerning: the Lamb-shift in atomic hydrogen, the fine structure in hydrogen, the hyperfine structure of the hydrogen ground state, the fine-structure of the positronium ground state, the annihilation rates of orthopositronium and parapositronium, and the hyperfine-splitting of the muonium ground state.

1. The hydrogen atom

The basic features of the lower energy-level structure in the hydrogen atom are summarized in fig. 1.1. In the Dirac theory, the degeneracy between the \( n = 2 \) \( P_{3/2} \) and \( P_{1/2} \) levels is removed by the spin-orbit interaction. This leads to the fine structure \( \Delta E(2P_{3/2} - 2P_{1/2}) \), proportional** to \((Z\alpha)^4 m\), which corresponds to a level splitting of 10 969.1 MHz. The interaction of the electron with the quantized electromagnetic field removes the degeneracy between the levels \( n = 2 S_{1/2} \) and \( P_{1/2} \). The corresponding level structure: \( \Delta E(2S_{1/2} - 2P_{1/2}) = 1057.9 \) MHz is the Lamb-shift. It is proportional to \( a m(Z\alpha)^4 \log(Z\alpha) \). Another cause of level splitting in the hydrogen-atom is the interaction of the magnetic moment of the orbital electron with the magnetic moment of the proton. In the ground state \( n = 1 \), this leads to a hyperfine structure between the triplet \( F = 1 \) and singlet \( F = 0 \) levels of 1420.4 MHz. The effect is proportional to \( (m/M)(Z\alpha)^4 m \).

1.1. The Lamb-shift

The two dominant contributions to the Lamb-shift can be qualitatively understood in the following way.

On the one hand the electron, in the presence of the electromagnetic field of the proton, can emit and reabsorb a photon (see fig. 1.2). This leads to a physical spreading of the electron charge over a mean squared radius \( \langle r^2 \rangle \) which for a free electron is

\[
\langle r^2 \rangle = \frac{6}{m^2} \frac{\alpha}{\pi} \left( \frac{1}{3} \log \frac{m}{\lambda} - \frac{1}{8} \right),
\]

where \( \lambda \) is an arbitrary small mass assigned to the photon. In the hydrogen atom, however, there is an effective lower limit (of the order of the hydrogen binding energy) to the energy of the photons which the bound electron can emit and reabsorb. Qualitatively one expects that a correct treatment of the binding effects will replace \( \lambda \) by the Rydberg which is the ionization energy of the ground state of the hydrogen atom.

*For a review of other hydrogen-like systems we recommend the reader the excellent review articles of Brodsky and Drell [R. 4] and of Wu and Widel [R. 17].

**Z is the atomic number, which for hydrogen is one. We shall, however, keep \( Z \) in our expressions as an indicative of the binding effects in contrast to the purely radiative effects.
The potential corresponding to this charge density distribution diminishes the Coulomb binding \(-Za/r\) and as a consequence the S levels are pushed higher by an amount

\[ \rho \sim \alpha(Za)^4 m \log(Za)^{-1} \sim 1000 \text{ MHz} \]

The detailed calculation [101-106] which also includes the contribution from the anomalous magnetic moment of the electron, leads to the result (see the first two entries in table 1.1 where the reduced mass correction has also been taken into account)

\[ \rho \text{ (self-energy)} = 1077.64 \text{ MHz} \]

On the other hand, the effective potential seen by the electron is modified by the vacuum polarization due to virtual electron-positron pairs [107, 108] (see fig. 1.3)
Comparison between theory and experiments in quantum electrodynamics

Fig. 1.2. Lowest order electron vertex contribution to the Lamb-shift

\[
\frac{Z\alpha}{q^2} \rightarrow \frac{Z\alpha}{q^2} + \frac{Z\alpha}{q^2} \frac{1}{\pi} \int_{4m^2}^\infty \frac{dt}{t} \text{Im} \Pi(t) \frac{q^2}{q^2 - t},
\]

where \((1/\pi)\text{Im} \Pi(t)\) is the vacuum polarization spectral function (which is positive definite)

\[
\frac{1}{\pi} \text{Im} \Pi(t) = \frac{\alpha}{\pi} \frac{1}{3} (1 + \frac{2m^2}{t^2}) \sqrt{1 - \frac{4m^2}{t}} \theta(t - 4m^2).
\]

The effective potential seen by the electron is more attractive than the Coulomb potential and as a consequence the S levels are lowered by an amount which turns out to be (without including the reduced mass correction)

\[
\mathcal{L} \text{ (vac. pol.)} = \frac{\alpha}{\pi} (Z\alpha)^4 m (-1/30) = -27.13 \text{ MHz}.
\]

Experimentally, there are two accurate direct measurements of the \(n = 2\) Lamb-shift in hydrogen: Triebwasser, Dayhoff and Lamb [109] * \(\mathcal{L}_{\text{exp}} = 1057.86 \pm 0.06 \text{ MHz}\). Robiscoe, Shyn [110] (revised) [160] \(\mathcal{L}_{\text{exp}} = 1057.90 \pm 0.06 \text{ MHz}\)

There are also three independent measurements of the interval \(2P_{3/2} - 2S_{1/2}\) in hydrogen, which combined with the theoretical value for the fine structure interval \(\Delta \mathcal{E} = 2P_{3/2} - 2P_{1/2}\), give indirect determinations of the Lamb-shift:

Kaufman, Lamb, Lea and Leventhal [111]

\((\Delta \mathcal{E} - \mathcal{L})_{\text{exp}} = 9911.38 \pm 0.03, \mathcal{L} = 1057.65 \pm 0.05\).

Shyn, Williams, Robiscoe and Rebane [112]

\((\Delta \mathcal{E} - \mathcal{L})_{\text{exp}} = 9911.25 \pm 0.06, \mathcal{L} = 1057.78 \pm 0.07\).

Vorburger and Cosens [113]

\((\Delta \mathcal{E} - \mathcal{L})_{\text{exp}} = 9911.17 \pm 0.04; \mathcal{L} = 1057.86 \pm 0.06\).

The accuracies of these determinations of the Lamb-shift range from 66 ppm to 47 ppm. The terms which so far have been calculated are given in Table 1.1. A detailed analysis of the different contributions can be found in two articles** by Erickson and Yennie [114, 115]. Of particular

*Corrected by Robiscoe and Shyn [160]

**For a discussion of the nuclear recoil corrections see also Grotch and Yennie [116]
Table 1 1

Compilation of contributions to the Lamb-shift $2S_{1/2} \to 2P_{1/2}$ in hydrogen* ($\alpha' = 137.036.08(26)$)

<table>
<thead>
<tr>
<th>Order</th>
<th>Description and references</th>
<th>Correction (units $\pi (Z_a)^4/m$)</th>
<th>Numerical value (MHz)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Z\alpha^4 m$</td>
<td>2nd order self-energy [101-106]</td>
<td>$\left(-2 \log(Z_a)^4/m + \frac{1}{2} \log K_0(2,0) \frac{1}{K_0(2,1)} (1 - 3m/M)\right)$</td>
<td>1009 920</td>
</tr>
<tr>
<td>$Z\alpha^4 n$</td>
<td>2nd order magnetic moment [063]</td>
<td>$\frac{1}{2}(1 - 2.75 m/M)$</td>
<td>67 720</td>
</tr>
<tr>
<td>$Z\alpha^4 m$</td>
<td>2nd order vacuum polarization [107, 1081]</td>
<td>$-\frac{1}{5}(1 - 3m/M)$</td>
<td>-27 084</td>
</tr>
<tr>
<td>$Z\alpha^4 m$</td>
<td>2nd order binding [125, 126 115]</td>
<td>$(Za)(3\pi + 11_{128}) + \frac{1}{2} \log 2 + \frac{5}{192})$</td>
<td>7 140</td>
</tr>
<tr>
<td>$Z\alpha^4 m$</td>
<td>4th order binding + higher order [127-129, 115, 158]</td>
<td>$(Za)^4(a + b \log(Za)^3 + c \log^3(Za)^3)/(\alpha)^3 \pi \times 0.56**$</td>
<td>0.372 - 0.00491</td>
</tr>
<tr>
<td>$Z\alpha^4 m$</td>
<td>Higher, $\alpha^2 \times Z_a$ and $\alpha$ uncertainty</td>
<td>$\pm 0.65868$</td>
<td></td>
</tr>
<tr>
<td>$Z\alpha^4 m$</td>
<td>4th order self-energy [119, 123]</td>
<td>$3(a/\pi) 0.470$</td>
<td>0.444</td>
</tr>
<tr>
<td>$Z\alpha^4 m$</td>
<td>4th order magnetic moment [406, 407]</td>
<td>$(a/\pi) 0.328$</td>
<td>0.102</td>
</tr>
<tr>
<td>$Z\alpha^4 m$</td>
<td>4th order vac polarization [130, 512]</td>
<td>$(a/\pi) 41/54$</td>
<td>0.239</td>
</tr>
<tr>
<td>Red mass uncertainty</td>
<td></td>
<td>$\pm 0.00341$</td>
<td></td>
</tr>
<tr>
<td>$(Za)^4 m$</td>
<td>Recoil corrections [131 132 116]</td>
<td>$(Z/m)(a + b \log(Za)^3)$</td>
<td>0.359</td>
</tr>
<tr>
<td>$(Za)^4 m$</td>
<td>Proton size [114, 116 158]</td>
<td>$\frac{1}{2} \pi^2 m R_p \pi^2$</td>
<td>0.125 $\pm 0.00634$</td>
</tr>
<tr>
<td>$mR_p(e)^2 m$</td>
<td>Proton structure uncertainty</td>
<td>$\pm 0.00063$</td>
<td></td>
</tr>
</tbody>
</table>

Total = $1057.911 \pm 0.012$

*This table is an updated version of the compilation made by A. Peterman [R 13]. The value of the Bethe logarithm $\log(K_0(2,1)/K_0(2,0))$ is that evaluated by Schwartz and Tiemann [106]. The constants $a$, $b$, $c$, $a_1$, and $b_1$ are the following:
$a = -\frac{9}{2} - 4 - 4 \log^2 2 - 2 - 0.28 \times 0.5$, $b = -\frac{45}{22} - \log 2$, $c = -3/4$, $b_1 = -1/4$, $a_1 = 2 \log(K_0(2,1)/K_0(2,0)) + 97/12$

**This formula is valid for the 1S state. The state dependence gives rise to tiny corrections, of the order of $+ 0.014$ MHz

*This is interest to us, because of recent changes in their evaluation, is the fourth order self-energy contributions which we discuss next.

The QED quantity which is involved in the calculation of the fourth order self-energy contribution to the Lamb-shift is the slope of the Dirac form factor of the electron to order $\alpha^2$. With the electron vertex definition
Fig. 1.4. Feynman diagrams contributing to the 4th order electron vertex. These are the diagrams which contribute to the slope of the Dirac form factor of the electron (see section 1.1) and to the anomalous magnetic moment of the electron (see section 4.1).

\[ \bar{u}(p+q) \Gamma^\mu(p+q,p) u(p) = \bar{u}(p+q) \left\{ \gamma^\mu F_1(q^2) + \frac{i}{2m} \sigma^{\mu\nu} q_\nu F_2(q^2) \right\} u(p), \]

we are interested in the evaluation of

\[ \sigma^{(4)} = m^2 \frac{dF_1^{(4)}(q^2)}{dq^2} \bigg|_{q^2 = 0}. \]

There are seven Feynman diagrams contributing to \( \sigma^{(4)} \). They are shown in fig. 1.4 and their contributions to the slope can be found in table 1.2. The first attempt to this extremely intricate calculation was made by Weneser, Bersohn and Kroll [117] who gave analytic expressions for the contributions from the diagrams of figs. 1.4b, 1e, 4f, 4g and bounds for the others. Later on, Sotoc [118] made a complete analytic calculation of \( \sigma^{(4)} \). Motivated by serious discrepancies between
Table 1.2

<table>
<thead>
<tr>
<th>Graphs</th>
<th>Analytic results</th>
<th>Numerical results</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \rho ) ( 1.4a )</td>
<td>(-\frac{13}{36} \log \lambda^2 \pi^2 \left( -\frac{1181}{1728} \pi^4 + 7 \times 10^2 \pi^2 \log 2 - \frac{7}{8} \zeta(3) = 2.3925 \right) ) (refs. [122, 159])</td>
<td>(-\frac{13}{36} \log \lambda^2 + 2.37 \pm 0.02 ) (ref [119])</td>
</tr>
<tr>
<td>( \delta ) ( 1.4b )</td>
<td>( \frac{13}{36} \log \lambda^2 \pi^2 \left( \frac{319}{864} - \frac{31}{432} \pi^2 = -1.710 \right) ) (refs. [118, 159])</td>
<td>(-\frac{13}{36} \log \lambda^2 \pi^2 - 1.69 \pm 0.02 ) (ref [119])</td>
</tr>
<tr>
<td>( \rho ) ( 1.4c )</td>
<td>( \frac{1}{12} \log \lambda^2 \pi^2 + \frac{1}{72} \log \lambda^2 \pi^2 \left( \frac{1511}{1728} \pi^4 - 10 \log 2 + \frac{1}{8} \zeta(3) = -1.9328 \right) ) (ref [121])</td>
<td>(-\frac{1}{12} \log \lambda^2 \pi^2 + \frac{1}{72} \log \lambda^2 \pi^2 - 1.91 \pm 0.02 ) (ref [119])</td>
</tr>
<tr>
<td>( \rho ) ( 1.4d )</td>
<td>( \frac{1}{12} \log \lambda^2 \pi^2 + \frac{1}{72} \log \lambda^2 \pi^2 \left( \frac{1109}{1728} \pi^4 = 1.688 \right) ) (refs. [117, 118, 121])</td>
<td>(-\frac{1}{12} \log \lambda^2 \pi^2 - \frac{1}{72} \log \lambda^2 \pi^2 - 1.9 \pm 0.05 ) (ref [120])</td>
</tr>
<tr>
<td>( \rho ) ( 1.4f )</td>
<td>( \frac{1}{12} \log \lambda^2 \pi^2 + \frac{1}{72} \log \lambda^2 \pi^2 \left( \frac{1097}{1728} \pi^4 + 0.0316 \right) ) (refs. [117, 118, 120])</td>
<td>(1.03 \pm 0.0002 ) (ref [119])</td>
</tr>
</tbody>
</table>

All the diagrams have been evaluated in the Feynman gauge. For the analytic results of the vacuum polarization, self-energy and add t diagrams we have corrected the overall sign error in refs. \[117, 118\]. All the results are given in units \((a/\pi)^3\)

The new value of the fourth order self-energy contribution to the Lamb-shift\(^*\),

\[
\frac{\alpha}{\pi} (Z \alpha) \frac{m}{2} \left\{ \frac{4819}{5184} - \frac{49}{432} \pi^2 + \frac{1}{2} \pi^2 \log 2 - \frac{3}{4} \zeta(3) = 0.46994 \right\}
\]

\(^*\)For a review of the comparison between theory and experiment before Appelquist and Brodsky's reevaluation of \(\sigma\) see Broost [R 3].

\(^\ast\)For a description of the techniques used in their calculation see section 4.2

\(^\ast\)The recent analytic result of Barberi, Mignaco and Remiddi [159] confirms the results of Peterman [122] and disagrees with ref [124].

\(^\dagger\)The complete 4th order imaginary parts of \(F_1\) and \(F_2\) have been calculated for all \(q^2\) by Barberi, Mignaco and Remiddi (Nuovo Cimento, to be published)
Table 1.3
Determinations of $\alpha^{-1}$ obtained from bound systems*

<table>
<thead>
<tr>
<th>Bound system</th>
<th>Discussion, section</th>
<th>Value of $\alpha^{-1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hydrogen, $\Delta E_H$</td>
<td>1.2</td>
<td>137.035 45(39)</td>
</tr>
<tr>
<td>Hydrogen, $(\Delta E_H - L_H)(I)$</td>
<td>1.2</td>
<td>137.035 70(27)</td>
</tr>
<tr>
<td>Hydrogen, $(\Delta E_H - L_H)(II)$</td>
<td>1.2</td>
<td>137.035 44(23)</td>
</tr>
<tr>
<td>Hydrogen, hfs</td>
<td>1.3</td>
<td>137.035 91(35)</td>
</tr>
<tr>
<td>Muonium, hfs</td>
<td>3.1</td>
<td>137.036 17(30)</td>
</tr>
</tbody>
</table>

*For reference, the value of $\alpha^{-1}$ obtained from the a.c. Josephson effect is $\alpha^{-1} = 137.03608(26)$.

when added to the other contributions in Table 1.1 leads to a theoretical value [123]

$$\mathcal{L}_{th} = 1057.911 \pm 0.012 \text{ MHz}.$$  

This includes the recent result of Erickson [158]. The theoretical and experimental values for the Lamb shift in hydrogen and hydrogenic atoms is given in Table 1.4.

1.2. The fine structure $\Delta E(2P_{3/2} - 2P_{1/2})$ in hydrogen

Let us denote $\Delta E_H$, the $n = 2, P_{3/2} - P_{1/2}$ level interval in atomic hydrogen (see fig. 1.1). The theoretical value of $\Delta E_H$ is well known

$$\Delta E_H = \frac{Ry(Za)^2}{16} \left\{ \left[ 1 + \frac{5}{8}(Za)^2 \right] \left( 1 + \frac{m}{M} \right)^{-1} - \left( \frac{m}{M} \right)^2 \left( 1 + \frac{m}{M} \right)^{-3} + 2a_e \left( 1 + \frac{m}{M} \right)^{-2} + \frac{a}{n}(Za)^2 \log Za \right\}.$$  

The first term in the parenthesis $1 + \frac{5}{8}(Za)^2$, is the well known Dirac solution.* The appearance of the reduced mass factor $(1 + m/M)^{-1}$ is explained in detail in Grotch and Yennie [116]. The $(m/M)^2$ term, calculated by Barker and Glover [133], is the effect of the Dirac moment, of the electron and the proton. The $a_e$ term is the effect of the electron anomaly and is the first term of radiative origin. Its contribution is roughly $0.1\%$, whereas the last term, which is also a radiative correction, contributes only about 1 ppm. These radiative corrections have been calculated by the authors of refs. [127] and [128]. Bounds to the next uncalculated terms, $O(\alpha/\pi)(Za)^2$, have been estimated by Erickson.** Since these terms are comparable to terms of order $O(\alpha^2 (m/M); \alpha(m/M)^2; \text{etc.})$, it seems more natural to expand the theoretical expression for $\Delta E_H$ in powers of $m/M$ and keep only the first power in this parameter. Thus we get

$$\Delta E_H = \frac{Ry(Za)^2}{16} \left\{ \left[ 1 + \frac{5}{8}(Za)^2 \right] \left( 1 - \frac{m}{M} \right) + 2a_e \left( 1 - 2\frac{m}{M} \right) + \frac{2a}{n}(Za)^2 \log Za \right\}.$$  

From this equation, and using the 'without quantum electrodynamics value' (WQED) $\alpha^{-1} = 137.03608(26)$ obtained from the a.c. Josephson effect,*** we get

$$\Delta E_H(\text{th.}) = 10969.03 \pm 0.04 \text{ MHz}.$$  

Comparing the value of $\Delta E_H(\text{th.})$ with the theoretical value for the Lamb-shift (see section 1.1) we have that

*See e.g., Bethe and Salpeter [R. 2]

**Quoted by Brodsky and Parsons [134]

***See Taylor, Parker and Langeber [R. 161].
Table 1.4

Lamb-shift in hydrogenic systems in MHz units

<table>
<thead>
<tr>
<th>System</th>
<th>Theory</th>
<th>Experiment</th>
<th>Refs</th>
</tr>
</thead>
<tbody>
<tr>
<td>( l (n = 2) )</td>
<td>1057.911 ± 0.012</td>
<td>1057.90 ± 0.06</td>
<td>[160]</td>
</tr>
<tr>
<td>( l (n = 3) )</td>
<td>1057.88 ± 0.06</td>
<td>1057.86 ± 0.06</td>
<td>[109, 160]</td>
</tr>
<tr>
<td>( l (n = 4) )</td>
<td>314.896 ± 0.003</td>
<td>314.810 ± 0.052</td>
<td>[170]</td>
</tr>
<tr>
<td>( l (n = 2) )</td>
<td>133.084 ± 0.001</td>
<td>133.18 ± 0.59</td>
<td>[161]</td>
</tr>
<tr>
<td>( le^{+}(n = 2) )</td>
<td>1059.271 ± 0.025</td>
<td>1059.28 ± 0.06</td>
<td>[162] (revised)</td>
</tr>
<tr>
<td>( le^{+}(n = 3) )</td>
<td>14044.765 ± 0.613</td>
<td>14045.4 ± 1.2</td>
<td>[163]</td>
</tr>
<tr>
<td>( le^{+}(n = 4) )</td>
<td>4184.42 ± 0.18</td>
<td>4183.17 ± 0.54</td>
<td>[164]</td>
</tr>
<tr>
<td>( le^{+}(n = 2) )</td>
<td>1769.088 ± 0.076</td>
<td>1776.0 ± 7.5</td>
<td>[165]</td>
</tr>
<tr>
<td>( l^{+}(n = 3) )</td>
<td>1768.0 ± 5.0</td>
<td>1764.0 ± 12</td>
<td>[166]</td>
</tr>
<tr>
<td>( l^{+}(n = 4) )</td>
<td>6276.341 ± 9.07</td>
<td>63031.0 ± 3270</td>
<td>[167]</td>
</tr>
</tbody>
</table>

The frequencies are the same as those quoted by G.W. Erickson (ref [158] and private communication), but using \( \alpha^{-1} = \alpha^{-1}(\text{WQED}) = 137.03602 \) instead of \( \alpha^{-1} = 137.03602 \). The Lamb-shift values for superheavy elements \((l > 1)\) can be found in ref [158].

\[
(\Delta E_H - \mathcal{L}_H)_{\text{th}} = 2P_{3/2} - 2S_{1/2} = 9911.13 \pm 0.04 \text{ MHz}
\]

As we pointed out in the precedent section, there are three independent measurements of the interval \(2P_{3/2} - 2S_{1/2}\) in hydrogen \([111-113]\). We can combine these experiments with the theoretical prediction of \(\Delta E_H - \mathcal{L}_H\) to obtain values for the fine-structure constant \(\alpha\):

1. the Vorburger-Cosens measurement \([113]\)

\[
\Delta E_H - \mathcal{L}_H = 9911.17 \pm 0.04 \text{ MHz}
\]

combined with the theoretical prediction yields a 2.0 ppm accurate value of \(\alpha^{-1}\):

\[
\alpha^{-1}_{(v)} = 137.035 70(27)
\]

2. the weighted average of the values of refs. \([115]\) and \([112]\):

\[
\Delta E_H - \mathcal{L}_H = 9911.21 \pm 0.035 \text{ MHz}
\]

combined with \(\Delta E_H - \mathcal{L}_H\), yields

\[
\alpha^{-1}_{(w)} = 137.035 43(23) \quad (1.7 \text{ ppm})
\]

The values \(\alpha^{-1}_{(v)}\) and \(\alpha^{-1}_{(w)}\) are derived purely from radiation theory and experiment, and they are in agreement with the determination of \(\alpha^{-1}\) from phase coherence effects in superconductors. At the present time, the accuracy on \(\alpha^{-1}\) obtained with the new theoretical expression for the interval \(2P_{3/2} - 2S_{1/2}\) is about the same as that on the value of \(\alpha^{-1}\) derived from the hydrogen hyperfine structure discussed in the next section.

In Table 1.3 we have listed various determinations of \(\alpha^{-1}\) obtained from different sources. This table clearly shows the consistency of QED in describing bound systems.

1.3 The hyperfine structure

In hydrogenic atoms, the interaction of the magnetic moment of the orbital electron with the magnetic moment of the nucleus leads to a splitting of a fine-structure level with fixed orbital an-
polar momentum $l$ and fixed total angular momentum $j$ into hyperfine structure (hfs) levels. To a first approximation, the energy separation between the two outermost levels is given by the Fermi formula \[ \Delta \nu = \frac{16}{3} (Z \alpha) \mu_p \text{Ry} \approx 1420 \text{ MHz} \]

where $\mu_p$ is the proton magnetic moment and $\mu_B$ the Bohr magneton \[ \mu_B = e\hbar/2mc. \]

The measurement of the hyperfine-splitting of the hydrogen ground state is probably the most accurate number which is presently known in experimental physics [136]

\[ \Delta \nu_{\exp}(pe) = 1420.4057517864(17) \text{ MHz}. \]

It corresponds to an accuracy of 1.2 parts in $10^{14}$! More than a test of QED the hfs has become a yardstick to measure our progress in theoretical physics. As we shall see below there is at present a gap of seven orders of magnitude between theory and experiment.

The framework for the formal treatment of the corrections to the Fermi formula is the Bethe-Salpeter equation. The dimensionless parameters which appear are:

- $m/M$, electron to proton mass ratio;
- $R/a_0$, ratio of nuclear to atomic sizes;
- $\alpha$, the fine structure constant;
- $Z \alpha$, the strength of the Coulomb potential.

A sequence of approximations in the corrections to $\Delta \nu(\text{Fermi})$ is established as follows. To first approximation, with \[ \Delta \nu(\text{hfs}) = \Delta \nu(\text{Fermi})(1 + \delta), \]

the proton is taken as a fixed point Coulomb potential \[ m/M \rightarrow 0 \text{ and } R/a_0 \rightarrow 0; \]

radiative corrections are also neglected, and only relativistic corrections are taken into account. This yields the Breit correction [138]

\[ \delta_{\text{Breit}} = \frac{3}{2}(Z \alpha)^2. \]

At the next level, one still takes $m/M \rightarrow 0$ and $R/a_0 \rightarrow 0$ but radiative corrections are taken into account as successive powers of $\alpha$, the dependence on $(Z \alpha)$, which arises when binding is taken into account, is not however simply a power series in $(Z \alpha)$. Altogether, radiative plus binding corrections take the following form \[ \delta_{\text{rad}} + \delta_{\text{binding}} = a_1 \frac{\alpha}{\pi} + c_{10} \alpha(Z \alpha) + a_2 \left( \frac{\alpha}{\pi} \right)^2 + \frac{\alpha}{\pi} (Z \alpha)^2 \{c_{27} \log^2(Z \alpha)^{-2} + c_{37} \log(Z \alpha)^{-2} + c_{27}\} + + a_3 \left( \frac{\alpha}{\pi} \right)^3 + c_{181} \alpha \frac{\alpha}{\pi} Z \alpha + \ldots \]

*An excellent detailed exposition of the successive orders of approximation can be found in Badeau and Feshbach [132]
The coefficients $a_1$, $a_2$, $a_3$ are those which give the corresponding order contribution to the anomalous magnetic moment of the electron:

$$a_1 = \frac{1}{2}$$

$$a_2 = \frac{197}{144} \pi^2 - \frac{\pi^2}{2} \log 2 + \frac{3}{4} \xi(3) = -0.3285$$

$$a_3 = 1.49 \pm 0.20$$

The coefficient $c_{10}$ was first calculated by Kroll and Pollack [139] and Karplus, Klein and Schwinger [140].

$$c_{10} = \frac{5}{2} + \log 2.$$  

The coefficients $c_{22}$ and $c_{21}$ have been calculated by Layzer [141], Zwanziger [142], and Brodsky and Erickson [137]. The latter authors have also calculated the dominant contribution to the coefficient $c_{20}$. The results are

$$c_{22} = -2/3, \quad c_{21} = \frac{281}{360} - \frac{8}{3} \log 2, \quad c_{20} = 18.4 \pm 5$$

The higher order terms have not been calculated as yet. We recall that the coefficients $c_{21}$ and $c_{20}$ are state dependent.** In fact, the values for the $n = 2$ S-level have also been calculated [141, 142, 37, 143].

The next level of approximations takes into account the finite mass and structure of the proton. The corresponding corrections can be classified as follows:

1. **reduced mass corrections**
2. **nonrelativistic size contributions** $\delta_{NR}$
3. **additional recoil terms of order $\alpha(m/M)\delta_{RE}$**
4. **proton polarization corrections** $\delta_p$

The reduced mass correction gives simply a factor $[M/(m+M)]^3$ in the Fermi formula. Estimates of nonrelativistic size contributions were made by Brown and Arfken [144], and, more elaborated, by Zemach [145]. The calculation of Zemach, in which the nonrelativistic approximation to the wave function was used, has been analyzed in detail by Grotch and Yennie [116] within the framework of an effective potential model. It amounts to a correction

$$\delta_{NR} = -2m \alpha R_p,$$

where

$$R_p = \int |u - r| \rho_M(u) \rho_p(r) \, d^3u \, d^3r$$

and $\rho_p(r), \rho_M(r)$ denote the charge and magnetic distribution of the proton. Grotch and Yennie [116], using the expression

$$\rho_p(r) = \rho_M(r) = \Lambda^3/8G^2 \, e^{-r}$$

*See section 4

**See e.g. Brodsky and Erickson [137] for a discussion.
with $A = 0.91M$ as suggested by the experimental determination of the proton form factors obtain

$$R_p = 1.02F,$$

which amounts to a correction

$$\delta_{NR} = -38.2 \text{ ppm}$$

to the Fermi formula. Additional recoil corrections of relative order $\alpha m/M$ have also been evaluated by Arnowitt [146] using the method of Karplus, Klein and Schwinger [147] and by Newcomb and Salpeter [148] using the Bethe-Salpeter equation. These corrections, which in the case of muonium can be calculated exactly*, have been done for a point proton with an anomalous magnetic moment i.e., the vertex corresponding to the absorption of a virtual photon of energy-momentum $q$ by the proton is put equal to

$$\Gamma^\mu = \gamma^\mu + \frac{\mu}{2M} i \sigma^{\mu\nu} q_\nu$$

The result is logarithmically divergent, due to the nonrenormalizability of the Pauli interaction. However, as was shown by the calculations of Iddings and Platzman [149], the divergence disappears when the form factors of the proton are taken into account. Iddings and Platzman calculate the corrections to $\Delta \nu$ (Fermi) arising from two-photon-exchange (elastic contribution). In fact, they calculate the contribution to the hfs from the difference between a coupling with form factors and the point-like coupling vertex quoted above. This, when added to the result of the calculations of Arnowitt; and Newcomb and Salpeter gives a finite correction of

$$\delta_p^{(\text{elastic})} = 3.6 \text{ ppm}.$$

There are further corrections arising from the polarizability of the proton, i.e. contributions to the hfs from two-photon-exchange graphs with virtual hadronic states other than the proton itself. The possibility of calculating these corrections from experimental data on inelastic electron scattering from protons was first pointed out by Iddings [150]. The argument is analogous to Cottingham's formulation of the neutron-proton mass difference [151] in terms of the proton structure functions. However, in the case of the hfs of hydrogen, what is needed are the spin-dependent structure functions of the proton. These are accessible from experiments on inelastic scattering of polarized electrons from polarized protons**. The polarizability contribution due to inelastic contributions can be written

$$\delta_p^{(\text{inelastic})} = \frac{\alpha m}{\pi M} \frac{1}{2(1 + \kappa_p)} (\Delta_1 + \Delta_2),$$

where $\kappa_p$ is the anomalous magnetic moment of the proton, and $\Delta_1, \Delta_2$ the contributions from the two spin-dependent structure functions $G_1(q^2, \nu), G_2(q^2, \nu)$ defined in such a way that for real photons ($q^2 = 0$),

*See section 3.

**For a discussion of the spin dependent nucleon structure functions where earlier references can be found, see Doncel and de Rafael [152]
$G_1(0, \nu) = (\nu/8\pi^2a)(\sigma_\nu(\nu) - \sigma_p(\nu))$

where $\sigma_\nu(\nu)$ and $\sigma_p(\nu)$ are the total photoabsorption cross sections for photon-proton parallel (A) and antiparallel (P) spin configurations. Estimates of $\Delta_1$ and $\Delta$ have always lead to small contributions

$\delta_p(\text{inelastic}) \sim 1 - 2 \text{ ppm}$.

The possible contributions to the term $\Delta_1$ have been discussed in detail by Drell and Sullivan [154]. These authors exploited the information provided by physical Compton scattering ($q^2 = 0$) and concluded that in all cases the contribution to $\delta_p$ from the $\Delta_1$ term amount to no more than 1 - 2 ppm. It has been recently pointed out by de Rafael [169] that, from positivity inequalities, the contribution $\delta_p(\Delta_2)$ can be bounded in terms of the spin independent structure functions $W_1$ and $W_2$ of the proton. An estimate of the bounds from the region covered by present inelastic electron proton scattering experiments gives

$2 \text{ ppm} < \delta_p(\Delta_2) < 3 \text{ ppm}$.

From the comparison between the experimental result [136]

$\Delta \nu_{\exp}(pe) = 1420.4057517864(1.7') \text{ MHz}$,

and the Fermi formula with all corrections discussed above incorporated except the proton polarization correction, and using the WQED value of $\alpha^{-1}$,**

$\alpha^{-1} = 137.035(11)$, one obtains

$\frac{\Delta \nu_{\exp}}{\Delta \nu_{\text{th}}} = 2.5 - 4.0 \text{ ppm} \quad \delta_p \text{ inelastic}$

consistent with the estimates of $\delta_p(\text{inelastic})$ [153, 155, 169].

One can also use the theoretical expression of $\Delta \nu$ (hfs) to obtain the fine structure constant by comparison with the measured $\Delta \nu$ (hfs). The value is

$\alpha^{-1} \text{ (hfs)} = 137.035(91(35))$

consistent with other determinations of $\alpha^{-1}$ (see table 1.3).

2. Positronium

Positronium is the atom consisting of an electron and a positron. It was discovered by Deutsch in 1951. Positronium is clearly a fundamental system to test QED in particular our understanding of the binding mechanism as described by the Bethe-Salpeter formalism [201].

*The n-N S-waves contribution has been estimated by Guenn [153] to be 1 ppm and for N N amplitudes to give a contribution smaller than 1 ppm. The importance of hadronic continuum contributions has been particularly emphasized by Drell and Sullivan [154], and more recently by Chernik, Struminski and Zimovii [155], using a quasipotential method developed by Logunov and Tavkhelidze [146] and Faustov [157].

**See Taylor, Parker and Langenberg [166].

***See ref. [201] For a review of the earlier experiments, see Martin Deutsch [202]. For a more recent review, see [R 11].
At the present time, the only measurements performed in the positronium system are on the fine-structure splitting of the ground \((n = 1)\) state \([205, 208]\), and on the decay rates of orthopositronium (the \(3S_1\) state) \([209]\) and parapositronium (the \(1S_0\) state). The first measurement of the decay rate of parapositronium \(\Gamma_p\) was obtained from a measurement value of \(\Gamma_p/\Gamma_o\) \([210]\), where \(\Gamma_o\) denotes the decay rate of orthopositronium, and using the value of \(\Gamma_o\) quoted in ref \([209]\) (see also refs. [R. 11 and 220]). A direct measurement of \(\Gamma_p\) has only recently been reported \([208]\). A compilation of results is given in table 2.1.

2.1. The fine-structure interval of the ground state of positronium

The qualitative features of the positronium energy levels, as predicted by the Schrödinger equation, are roughly 1/2 those of hydrogen because of the reduced mass for positronium which is \(m_e/2\). Hence the ionization energy of the ground \(n = 1\) state of positronium which is 6.8 eV.

The fine-structure in positronium is of order \(\alpha^2\) Ry and the features here are very different from those of hydrogen. Besides the dipole interaction, as in the hydrogen hfs, there is an exchange interaction due to the virtual annihilation of the e−e+ system in the triplet state into one \(\gamma\) \([211, 213, 221, 222]\). In fact, the largest contribution comes from this virtual annihilation interaction which pushes the \(3S_1\) level upwards with respect to the \(1S_0\) level. The contributions to the triplet-singlet splitting of the positronium ground state have been recently calculated up to terms of relative order \(\alpha^2 \log \alpha\) \([214]\). The result is

\[
\Delta \nu = \alpha^2 \text{Ry} \left[ \frac{7}{6} - \frac{\alpha}{\pi} \left( \frac{16}{9} + \log 2 \right) - \frac{3}{4} \alpha^2 \log \alpha + O(\alpha^2) \right].
\]

The contributions of relative order \(\alpha^2 \log \alpha\) represent recoil corrections arising from low momentum components of the wave function associated with the Bethe-Salpeter equation for positronium. The techniques used in the calculation by Fulton, Owen and Repko \([214]\) are an extension of those previously described by Karplus and Klein \([213]\) and by Fulton and Martin \([215]\). The terms of relative order \(\alpha^2\) have not been calculated as yet. The theoretical prediction from the calculated terms is thus

\[
\Delta \nu_{\text{th}} = 2.03415 \times 10^5 \text{ MHz},
\]

to be compared with the most recent measurement \([208]\).

Table 2.1

<table>
<thead>
<tr>
<th>Observable</th>
<th>Experiments</th>
<th>Theory</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fine-structure of ground state ((S_1 - 1S_0))</td>
<td>((2.03380 \pm 0.00040) \times 10^5 \text{ MHz} [206])</td>
<td>(2.03427 \times 10^5 \text{ MHz} [214])</td>
</tr>
<tr>
<td>Decay-rate orthopositronium ((3S_1))</td>
<td>((0.7262 \pm 0.0015) \times 10^{-10} \text{ sec}^{-1} [209, R.11])</td>
<td>(0.7211 \times 10^{-10} \text{ sec}^{-1} [216])</td>
</tr>
<tr>
<td>Decay-rate parapositronium ((1S_0))</td>
<td>((0.799 \pm 0.011) \times 10^{10} \text{ sec}^{-1} [208])</td>
<td>(0.798 \times 10^{10} \text{ sec}^{-1} [218, 219])</td>
</tr>
</tbody>
</table>
\[ \Delta \nu_{\text{exp}} = (2.034 \pm 0.00012) \times 10^8 \text{ MHz} \] (60 rpm)

2. The annihilation rates of orthopositronium and parapositronium:

The annihilation rate of orthopositronium (the \(^1S_0\) \(e^+e^-\) state) has been measured to an accuracy of 0.2% [209] (see also ref. [R. 11]).

\[ \Gamma_o(\text{exp}) = (0.7262 \pm 0.0015) \times 10^7 \text{ sec}^{-1}. \]

The theoretical value for the 3\(\gamma\) ray annihilation, which includes only the lowest order contribution, was calculated by Ore and Powell [216].

\[ \Gamma_o(\text{th}) = \frac{\alpha^6}{\pi} m \frac{2}{9} (\pi^2 - 9) = (0.72112 \pm 0.00001) \times 10^7 \text{ sec}^{-1} \]

Recently, the correction term of relative order \(\alpha\) due to the interference of the lowest order diagrams with the higher order diagrams involving photon-photon scattering has also been calculated [217], using numerical integration techniques. The effect is to lower the orthopositronium decay rate

\[ \Gamma_o(\text{th}) = \frac{\alpha^6}{\pi} m \frac{2}{9} (\pi^2 - 9) \left[ 1 - \frac{\alpha}{\pi} (0.741 \pm 0.017) + \right] \]

It must be noted, however, that the photon-photon scattering correction is only a part of the complete correction of relative order \(\alpha\) The calculation of the other terms, is clearly necessary for an adequate comparison with the experimental value.

The theoretical value for the 2\(\gamma\) annihilation rate of parapositronium (\(^1S_0\)) is known up to terms of relative order \(\alpha\) The lowest order term was first calculated by Dirac [218] and the corrections of relative order \(\alpha\) by Harris and Brown [219].

\[ \Gamma_p(\text{th}) = \frac{\alpha^4}{\pi} m \left[ 1 - \frac{\alpha}{\pi} \left( \frac{\pi^2}{4} \right) \right] = 0.798 \times 10^{10} \text{ sec}^{-1} \]

Recently, the first direct measurement of the parapositronium annihilation rate has been made [208]. The experiment, which is set to measure the fine-structure interval of the ground state of positronium, involves the measurement of an induced Zeeman transition between magnetic substates of ground-state positronium. Detection of coincident 2\(\gamma\) ray annihilation rather than detection of the \(\gamma\)-ray energy spectrum, as done in previous experiments, was used. The natural linewidth of the Zeeman transition yields the value of \(\Gamma_p\)

\[ \Gamma_p(\text{exp}) = (0.799 \pm 0.011) \times 10^{10} \text{ sec}^{-1}, \ (1.4\%) \]

where a one-standard-deviation error is given in excellent agreement with the theoretical value.

Muonium

Muonium is the atom consisting of an electron and a positive muon. It was discovered by Hughes and collaborators [301] in 1960. The muonium system provides an excellent ground to test our understanding of electromagnetic binding when two different masses are involved, and eventually
to detect a possible breakdown of electron-muon universality. Recently, precision measurements of the muonium ground state hyperfine structure and the magnetic moment of the muon have yielded a new determination of the fine-structure constant $\alpha$, to an accuracy comparable with that reached in measurements of $\alpha$ using the Josephson effect.

### 3.1. The hyperfine-splitting of the muonium ground state

The lowest order splitting of the singlet and triplet levels of muonium ground state is given by the Fermi formula [135]

$$\Delta \nu = \frac{1}{3} \alpha^2 \text{Ry}(\mu_\mu/\mu_0^e),$$

where $\mu_\mu$ is the muon magnetic moment and $\mu_0^e$ the Bohr magneton

$$\mu_0^e = e\hbar/2m_ec$$

and

$$\mu_\mu = \mu_0^e \left( 1 + a_\mu \right)$$

where $\mu_0^e = e\hbar/2m_ec$ and $a_\mu$ is the celebrated anomalous magnetic moment of the muon.

Qualitatively $\Delta \nu(\mu_e) \sim 3 \Delta \nu(\mu_e)$, as expected from the ratio of muon to proton magnetic moments. Corrections to the Fermi formula have been calculated up to terms of relative order $\alpha(Z\alpha)^2$ for the radiative and binding corrections [302] and up to terms of relative order $(m_e/m_\mu) \alpha^2 \log \alpha$ for the recoil corrections [303]. Altogether, the theoretical expression for the hyperfine-splitting of the muonium ground state $\Delta \nu(\mu_e)$ can be written in the following way

$$\Delta \nu(\mu_e) = \frac{16}{3} \alpha^2 \text{Ry} \frac{\mu_\mu}{\mu_0^e} \left( 1 + \frac{m_e}{m_\mu} \right)^{-3} \left\{ 1 + a_e + (Z\alpha)^2 \frac{3}{2} + \alpha(Z\alpha) \left( \frac{5}{2} \log 2 \right) + \alpha_\mu \right\}$$

$$+ \frac{\alpha}{\pi} (Z\alpha)^2 \left[ \frac{2}{3} \log^2 (Z\alpha)^2 + \left( \frac{281}{360} + \frac{8}{3} \log 2 \right) \log(Z\alpha)^2 + 18.4 \alpha \right] + \delta_\mu$$

The factor $\left( 1 + \frac{m_e}{m_\mu} \right)^{-3}$ is a reduced mass correction. The corrections in curly brackets, except for the $\delta_\mu$ term, are the same as for the hyperfine splitting of the $1S$ level of the hydrogen atom. The term $a_e$ is the anomalous magnetic moment of the electron. The term $\delta_\mu$ represents the relativistic recoil corrections which for muonium, unlike the case of the hydrogen atom, can be calculated exactly. The expression for $\delta_\mu$ reads

$$\delta_\mu = \frac{m_e}{m_\mu} \left[ 1 - 3 \frac{\alpha}{\pi} \left( \frac{m_e}{m_\mu} \right)^2 \right]^{-1} \log \frac{m_\mu}{m_e} - \frac{9}{2} \alpha^2 \log \alpha \left( 1 + \frac{m_e}{m_\mu} \right)^2$$

The leading term is well known. It has been calculated by various authors [304–306]. However, the correction term of relative order $(m_e/m_\mu) \alpha^2 \log \alpha$ which represents recoil effects arising from low-momentum components of the muonium Bethe-Salpeter equation is known only since recently. It has been calculated by Fulton, Owen and Repko [303].

The numerical estimate $\Delta \nu(\mu_e)$ requires two quantities as input which have to be taken from experiments: the fine structure constant $\alpha$, and the ratio $\mu_\mu/\mu_0^e$. The latter can be obtained from

*See section 1.3*
measurements of the ratio of the magnetic moment of the muon to the magnetic moment of the
proton \( \mu_\mu / \mu_\rho \) which recently have been performed to an accuracy of a few ppm by two groups
[307, 308]. It is thanks to these new precise measurements of \( \mu_\mu / \mu_\rho \) that muonium has become
a precision test of QED. The previous measurements \([309, 311]\) of \( \mu_\mu / \mu_\rho \) had errors of 1%–22
ppm, too large to profit fully from the more accurate determinations of \( \Delta \mu(e) \) \([312, 313, 308]\).

(i) Precision measurements of the magnetic moment of the muon

The value of the ratio \( \mu_\mu / \mu_\rho \) reported by a University of Washington-Lawrence Radiation Labora-
tory collaboration \([307]\) is

\[
\frac{\mu_\mu}{\mu_\rho} = 3.183347(9) \text{ (2.8 ppm)}
\]

to be compared with previously reported values (see table 3.1). In terms of the muon mass, this
determination of \( \mu_\mu / \mu_\rho \) implies

\[
\frac{m_\rho}{m_\mu} = 206.7683(9) \text{ (2.9 ppm)}
\]

The ratio \( \mu_\mu / \mu_\rho \) measured by the authors of ref. \([307]\) was performed in three chemical environ-
ments showing no substantial differences. This is in contradiction with the suggestion by Ruderman
\([314]\) that a correction due to the effect of diamagnetic shielding on the muon moment
should exist.

An independent determination of \( \mu_\mu / \mu_\rho \) has been made by Telegdi and collaborators at the
University of Chicago \([308]\). In this experiment, both the hyperfine splitting \( \Delta \mu(e) \) and the
muon magnetic moment \( \mu_\mu \) are determined from measurements of the Zeeman (\( F, M_1 \)) transitions
\( 1,1 \rightarrow (1,0) \) and \( 1,1 \rightarrow (0,0) \) in the region of intermediate coupling. The external magnetic
field is chosen at a “magic” value, for which the frequencies of the two Zeeman transitions become
of first order, field independent. The results are

\[
\Delta \mu(e) = 4463.3022(89) \text{ MHz}
\]

and

\[
\frac{\mu_\mu}{\mu_\rho} = 3.183373(13)
\]

The latter value has been obtained assuming that the bound-state g factors are not affected by
collisions with the host gas atoms. If, on the contrary, as suggested by calculations by Herman*,

| Table 3.1 |

| Observable | Experimental value | Reference |
|------------|-------------------|----------|-----------|
| \( \Delta \mu(e) \) | 3.18338(40) | Columbia ref \([309]\) |
| \( \Delta \mu_\rho \) | 3.183360(70) | Berkeley, ref \([310]\) |
| \( \Delta \mu_\mu \) | 3.183330(44) | Princeton, Penn. ref \([311]\) |
| \( \mu_\mu \) | 3.183347(97) | Washington & RL ref \([307]\) |
| \( \mu_\rho \) | 3.183373(13) | Chicago, ref \([308]\) |

See ref \([308]\), note (2) added in proof.
Table 3.2
Compilation of results on the hyperfine splitting of muonium ground state $\Delta \nu(\mu e)$ (All numbers are in MHz)

<table>
<thead>
<tr>
<th>References</th>
<th>Experiments</th>
<th>Theory*</th>
</tr>
</thead>
<tbody>
<tr>
<td>Yale [317]</td>
<td>4463.145(6)</td>
<td>4463.313(21)</td>
</tr>
<tr>
<td>Yale [318]</td>
<td>4463.302(27) (from Kr data)</td>
<td>(using $\frac{\mu}{\mu_p} = 3.183 337(13)$)</td>
</tr>
<tr>
<td></td>
<td>4463.220(33) (from Ar data)</td>
<td></td>
</tr>
<tr>
<td>Chicago [316]</td>
<td>4463.317(21)</td>
<td>4463.323(19)</td>
</tr>
<tr>
<td>Yale [313]</td>
<td>4463.249(31)</td>
<td>(using $\frac{\mu}{\mu_p} = 3.183 347(9)$)</td>
</tr>
<tr>
<td>Chicago [308]</td>
<td>4463.3022(89)</td>
<td></td>
</tr>
<tr>
<td>Yale [315]</td>
<td>4463.311(12)</td>
<td></td>
</tr>
</tbody>
</table>

*The recent recoil correction of Fulton, Owen and Repko [303] has been included. The value of $\alpha^{-1}$ used is 137.036 02(21).

A pressure shift of $-11$ ppm corresponding to the experimental conditions of ref. [308] is assumed, then

$$\frac{\mu}{\mu_p} = 3.183 337(13),$$

in excellent agreement with the result obtained by the authors of ref. [307].

(ii) Comparison between theory and experiment

The theoretical values for $\Delta \nu(\mu e)$ which are obtained using the recommended value of the fine structure constant

$$\alpha^{-1} = 137.036 02(21),$$

and the values of $\frac{\mu}{\mu_p}$ quoted above are

$$\Delta \nu(\mu e)_{\text{Chicago}} = 4463.313(21) \text{ MHz}$$

$$\Delta \nu(\mu e)_{\text{Wash./LRL}} = 4463.323(19) \text{ MHz}.$$  

The most precise experimental determinations of $\Delta \nu(\mu e)$ have been made by the Chicago group [308] and by the Yale group [315]

Chicago, $\Delta \nu(\mu e) = 4463.3022(89) \text{ MHz};$ ref. [308]

Yale, $\Delta \nu(\mu e) = 4463.311(12) \text{ MHz};$ ref. [315].

Comparing these values to the theoretical predictions given above it can be seen that the Chicago and the later Yale experimental results are within one standard deviation of the theoretical values.

The remarkable accuracy of the $\Delta \nu(\mu e)$ measurement obtained by the Chicago group (2.0 ppm) combined with the new determination of $\frac{\mu}{\mu_p}$ allows, from the theoretical expression of $\Delta \nu(\mu e)$ an independent determination of the fine structure constant $\alpha$. The value thus obtained [308] is

$$\alpha^{-1} = 137.036 17(30)$$

in excellent agreement with both the WQED value $\alpha^{-1} = 137 036 08(26)$ and the recommended value (see table 1.3).

*We acknowledge a helpful discussion on this point with Professor Thomas Fulton
### Table 4.1

<table>
<thead>
<tr>
<th>Contribution</th>
<th>$\alpha_{\text{th}}^e (\times 10^9)$</th>
<th>Total $\alpha_{\text{th}}^e (\times 10^9)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>QED 1. order</td>
<td>1 161 409 0 ± 2.2</td>
<td>1 161 409 0 ± 2.2</td>
</tr>
<tr>
<td>QED 4. order</td>
<td>-1772.3</td>
<td>4131.8</td>
</tr>
<tr>
<td>QED 6. order</td>
<td>18.7 ± 2.5</td>
<td>273 ± 14</td>
</tr>
<tr>
<td>hadronic</td>
<td>0.0</td>
<td>65 ± 5</td>
</tr>
<tr>
<td>Total</td>
<td>1 159 655 4 ± 3.3</td>
<td>1 165 879 ± 15</td>
</tr>
</tbody>
</table>

---

**Part B**

### The Anomalous Magnetic Moments of the Charged Leptons

The anomalous magnetic moments (the “anomaly” $\alpha = (g - 2)/2$) of the charged leptons have for many years offered one of the most interesting and important challenges to both theoretical and experimental quantum electrodynamics. The CERN muon experiments are probably the most outstanding examples of high-precision experiments done with a high energy machine. The calculations of the sixth order QED contributions now in progress around the world (and in one case completed) are the highest order experimentally significant radiative corrections that have been evaluated, and are formidable challenges to algebraic manipulation and numerical integration techniques.

The best experimental value for the electron anomaly has been obtained recently by Rich and Wesley [401]:

$$\alpha_{\text{exp}}^e = (1 159 657 7 ± 3.5) \times 10^{-9}$$

The best experimental value for the muon anomaly is the CERN-storage ring value [907]:

$$\alpha_{\text{exp}}^\mu = (116 616 ± 31) \times 10^{-8}$$

The electron anomaly is a purely quantum electrodynamical quantity (see section 7). The theoretical value is

$$\alpha_{\text{th}}^e = 0.5 (\alpha/\pi) - 0.328 48 (\alpha/\pi)^2 + (1.49 ± 0.20) (\alpha/\pi)^3,$$

where the last term is the sixth order contribution with its theoretical uncertainty (see section 4.2). With $\alpha^{-1} = 137.036 08(26)$ we obtain

$$\alpha_{\text{th}}^e = (1 159 655 4 ± 3.3) \times 10^{-9}$$

The uncertainty has two parts, one from the fine structure constant ($± 2.2$) and one from theory ($± 2.5$). Experiment and theory thus agree within one standard deviation. It is interesting to note that a slight improvement in experiment and theory might lead to a value for the fine structure constant which is better than the Josephson value **.

*The Drell-Pagels-Parrons estimate [402, 403] leads to $0.40 (\alpha/\pi)^3$ and thus it disagrees with the latest experiment, which corresponds to a sixth order term $(1.67 ± 0.33) (\alpha/\pi)^3$.

**At present we obtain (disregarding the theoretical uncertainty) $\alpha_{\text{exp}}^{-1} = 137.035 82(4)$.
The muon anomaly is not a pure QED quantity. The theoretical contribution from QED is (section 5)

\[ a_{\mu}^{\text{QED}} = 0.5 \left( \frac{\alpha}{\pi} \right) + 0.6578 \left( \frac{\alpha}{\pi} \right)^2 + (21.8 \pm 1.1) \left( \frac{\alpha}{\pi} \right)^3 \]

which numerically becomes

\[ a_{\mu}^{\text{QED}} = (1165814 \pm 14) \times 10^{-9}. \]

To this we add the strong interaction contribution (section 6.1)

\[ a_{\mu}^{\text{Hadronic}} = (65 \pm 5) \times 10^{-9} \]

so that the full theoretical value is

\[ a_{\mu}^{\text{th}} = (1165879 \pm 15) \times 10^{-9} \]

in reasonable agreement with experiment. In table 4.1 the different contributions are exhibited.

4 The electron anomaly

The electron anomaly has been calculated completely up to and including the sixth order.

4.1. The second and fourth order contributions

In second order there is only one diagram (fig. 4.1) which gives the famous Schwinger contribution [003]

\[ a_e^{(2)} = \frac{1}{2} \left( \frac{\alpha}{\pi} \right) = (1614090 \pm 22) \times 10^{-9} \]

where we have used the value

\[ \alpha^{-1} = 13703608(26) \]

There are seven diagrams (fig. 1.4) which contribute to \( a_e \) at fourth order in \( e \). The result is rather more complicated than the second order expression. Compared to the rational \( 1/2 \), the transcendentals \( \pi^2 \), \( \pi^2 \log 2 \) and \( \xi(3) \) now appear. These represent special values* of the dilog \( \text{Li}_2(x) \) and the trilog \( \text{Li}_3(x) \). More precisely [405–409]

*See e.g. Iw. win [404].
\[ a_c^{(4)} = \left( \frac{\alpha}{\pi} \right)^2 \left[ \frac{197}{144} + \frac{\pi^2}{12} - \frac{1}{2} \pi^2 \log 2 + \frac{3}{4} \zeta(3) \right] \]

his fundamental calculation by Karplus and Kroll [405] (later revised see refs. [406] and [407]) was the first to demonstrate the consistency of the renormalization procedure in higher orders of perturbation theory. Numerically

\[ a_c^{(4)} = -0.328 \quad 48 (\alpha/\pi)^2 = -1772.3 \times 10^{-5} \]

The electron anomaly should in fact be expressed as an expansion in the two parameters \( a \) and \( m_e/m_\mu \). The leading term in \( m_e/m_\mu \), coming from the second diagram of fig. 4.2, is, however, [410]

\[ \left( \frac{\alpha}{\pi} \right)^2 \left( \frac{1}{45} \right) \left( \frac{m_e}{m_\mu} \right)^2 = 3 \times 10^{-12} \]

which, at the level of approximation needed, can be left out.

4.2. The sixth order contribution

Considering the increase in difficulty between the second and fourth order it is not surprising that the evaluation of the sixth order anomaly is indeed hard. First of all the number of graphs is now 72 (fig. 4.3) and the complexity of the graphs is such that an analytic evaluation in closed form is extremely difficult for virtually all diagrams. Actually diagrams 19-22 (fig. 4.3) have been evaluated in closed form [411]. In addition to the transcendental encountered in \( a_c^{(4)} \) there appear now special values of the polylorithm \( \text{Li}_n(x) \) [404]. As the diagrams lead in general to \( n \)-fold integrals, one expects that the analytic results can involve special values of \( \text{Li}_n(x) \) with \( n \) ranging from 1 to 6. Luckily, however, the diagrams are not more complicated than they allow for numerical evaluation.

Let us summarize the situation. The 72 diagrams contributing to \( a_c^{(4)} \) fall into six different classes, the choice of which is mainly based on gauge invariance criteria:

Class I. Graphs 1-6, containing photon scattering subgraphs
Class II. Graphs 7-18, containing second order (but not fourth order) vacuum polarization subgraphs
Class III. Graphs 19-22, containing fourth order vacuum polarization subgraphs
Class IV. Graphs 23-28, three-photon exchange
Class V. Graphs 29-48, two-photon exchange
Class VI. Graphs 49-72, one-photon exchange

![Diagrams](image-url)
Fig 4.3. Mass independent sixth order contributions to the lepton vertex.
Table 4.2

History of the calculations of the sixth order contributions to the electron anomaly

<table>
<thead>
<tr>
<th>Graphs</th>
<th>Authors</th>
<th>Year</th>
</tr>
</thead>
<tbody>
<tr>
<td>9; 22</td>
<td>Mignaco and Remiddi [411]</td>
<td>1969</td>
</tr>
<tr>
<td>-6, 22</td>
<td>Aldins et al [412, 413]</td>
<td>1969</td>
</tr>
<tr>
<td>7-22</td>
<td>Brodsky and Kinoshita [414]</td>
<td>1970</td>
</tr>
<tr>
<td>7-22</td>
<td>Calmet and Perrottet [416]</td>
<td>1970</td>
</tr>
<tr>
<td>5+29+30</td>
<td>Levine and Wright [415]</td>
<td>1970</td>
</tr>
<tr>
<td>7-12</td>
<td>De Rújula et al. [417]</td>
<td>1971</td>
</tr>
<tr>
<td>13-25, 29, 30, 39, 40</td>
<td>Calmet [418]</td>
<td>1971</td>
</tr>
<tr>
<td>7, 22</td>
<td>Levine and Wright [419]</td>
<td>1971</td>
</tr>
</tbody>
</table>

The first three classes contain fermion loops and coincide with the class division for the difference \( \alpha_{e} - \alpha_{\mu} \). The last three classes contain no fermion loops and have been classified according to the number of photons crossing the main vertex. This classification has the advantage of being gauge invariant. Inside some of the classes there are gauge invariant subclasses. We leave it to the reader to prove that the following sets of graphs yield a gauge invariant anomaly (after renormalization): 1-6, 7-10, 11+12, 13+14+17+18, 15+16, 19, 20-22, 23-28, 29-48, 49-68, 69-72.

In Table 4.2 the history of the sixth order calculations is given. All six classes have been completely evaluated, although detailed results are not yet available for the last three. The graphs of class III have been evaluated analytically by Mignaco and Remiddi [411] and checked numerically [414-419]. All the other diagrams have only been evaluated numerically. In Tables 4.3-4.6 the results of various calculations are presented. We have chosen to present the results as much as possible in the form and detail in which they were originally published. The analytic results for class III [411] are given in Table 4.3. In Table 4.4 a comparison is made between the results for classes II and III of Brodsky and Kinoshita [414] and Calmet and Perrottet [416]. The agreement is excellent. In Table 4.5 the results of De Rújula, Lautrup and Peterman [417] for the gauge invariant subset (60-72) of the class VI graphs is presented. In Table 4.6 the recent results of Calmet [418] are listed.

We give below the overall results.

Class I

\[ \alpha_{e, \mu}^{(6)} = 0.36(4)(\alpha/\pi)^3 \]

Aldins et al. [412, 413]

Class II

\[ \alpha_{e, \mu}^{(6)} = \begin{cases} 
-0.153(5)(\alpha/\pi)^3 & \text{Brodsky and Kinoshita [414]} \\
-0.151(3)(\alpha/\pi)^3 & \text{Calmet and Perrottet [416]} 
\end{cases} \]

*In ref [416], graphs 13 and 14 should be interchanged with graphs 17 and 18.
Class III

\[ \alpha_{\text{e,III}}^{(6)} = \begin{cases} 
0.055 \ 429 \ (\alpha/\pi)^3 & \text{Mignaco and Remiddi [411]} \\
0.055 \ 46(6) \ (\alpha/\pi)^3 & \text{Brodsky and Kinoshita [414]} \\
0.055(2) \ (\alpha/\pi)^3 & \text{Calmet and Perrottet [416]} 
\end{cases} \]

Class IV, V and VI

\[ \alpha_{\text{e,IV}}^{(6)} + \alpha_{\text{e,V}}^{(6)} + \alpha_{\text{e,VI}}^{(6)} = 1.23(20) \ (\alpha/\pi)^3 \text{ Levine and Wright [419].} \]

The uncertainty in this case is not obtained by statistical methods but it is rather an educated guess.

The overall result for the sixth order electron anomaly is then

\[ \alpha_{\text{e}}^{(6)} = (1.49 \pm 0.20) \ (\alpha/\pi)^3. \]

---

**Table 4.3**

Results of Mignaco and Remiddi [411]. \( \alpha_x = \sum_{n=1}^{\infty} \frac{1}{2^n n^x} = \text{Li}_x(1/2) = 0.51748 \)

<table>
<thead>
<tr>
<th>Graph</th>
<th>Analytic form in units of ((\alpha/\pi)^3)</th>
<th>Numeric value of non-infrared terms in units of ((\alpha/\pi)^3)</th>
</tr>
</thead>
<tbody>
<tr>
<td>19</td>
<td>(-\frac{943}{324} - \frac{4}{135 \pi^2} + \frac{8}{3^3(3)})</td>
<td>0.002 5585</td>
</tr>
<tr>
<td>20+21</td>
<td>(-\frac{1547}{432} - \frac{5}{3 \pi^2} + \frac{2}{3^2 \pi^3} \log^3 - \frac{2}{9^2 \pi^3} \log^2 - \frac{2}{9^3 \pi^3} \log - \frac{4}{9^4 \pi^3} \log^2 + \frac{4}{9^5 \log^2} + \frac{4}{9^6 \log^2} + \frac{4}{9^7 \log^2})</td>
<td>0.054 655</td>
</tr>
<tr>
<td>22</td>
<td>(-\frac{1145}{432} + \frac{161}{162 \pi^2} - \frac{22}{9 \pi^3} \log^3 - \frac{49}{18^2 \pi^3} \log^2 - \frac{4}{9 \pi^3} \log^2 - \frac{4}{9^2 \pi^3} \log^2 + \frac{4}{9^3 \pi^3} \log^2) + (\frac{32}{3} (-\frac{1}{\lambda}) + \log \left( \frac{\lambda}{m_e} \right) ) - (\frac{119}{18} \frac{2}{3 \pi^2}) - (\frac{7}{270 \pi^2})</td>
<td>-0.631 805</td>
</tr>
</tbody>
</table>

**Table 4.4**

Comparison of results for classes II and III

<table>
<thead>
<tr>
<th>Graphs</th>
<th>Brodsky and Kinoshita [414]</th>
<th>Calmet and Perrottet [416]</th>
</tr>
</thead>
<tbody>
<tr>
<td>7+8</td>
<td>-0.0032(3)</td>
<td>-0.0031(10)</td>
</tr>
<tr>
<td>9+10</td>
<td>0.0532(4)</td>
<td>0.0522(10)</td>
</tr>
<tr>
<td>11+12</td>
<td>0.0273(3)</td>
<td>0.0274(5)</td>
</tr>
<tr>
<td>13+14</td>
<td>-0.051(1) - 0.0314 \log(\lambda/m_e)</td>
<td>0.0474(20) + infrared term</td>
</tr>
<tr>
<td>15+16</td>
<td>-0.1161(14)</td>
<td>-0.1151(9)</td>
</tr>
<tr>
<td>17+18</td>
<td>-0.064(3) + 0.0314 \log(\lambda/m_e)</td>
<td>-0.0653(5) + 0.03160(52) \log(\lambda/m_e)</td>
</tr>
<tr>
<td>19</td>
<td>0.002 55(2)</td>
<td>0.002 559(15)</td>
</tr>
<tr>
<td>20+21+22</td>
<td>0.052 91(6)</td>
<td>0.0522(21)</td>
</tr>
</tbody>
</table>
where we have used the analytic class III result, and the weighted average of the class II results. Details of the calculation by Levine and Wright [419] of the three classes IV, V, VI are as yet not available. It is therefore impossible to compare with the three partial calculations that have appeared previously [415, 417, 418] and that overlap with these classes.

All the 72 graphs have now been calculated. However, as “mirror graphs” give the same anomaly, only 41 of the 72 are independent. In view of the complexity of the calculations the necessity for independent checks must be emphasized. So far such checks have only been carried out for classes II and III, and in a few instances for graphs belonging to classes IV, V and VI.

Technically the calculation of the class I, II and III graphs is not different from the calculation of the corresponding graphs for \( \alpha_m - \alpha_e \) (see section 5). For the remaining graphs some simplification can be obtained for those that contain self-energy and vertex insertions by using properly parametrized forms of these insertions. For those that do not contain such insertions (the irreducible ones) there is no way around a full-fledged parametrization of the three interlaced loops. The complexity of the sixth order graphs has necessitated extensive computer use for algebraic manipulations and numerical integrations. The \( \gamma \)-algebra and vector substitutions in the numerator of a graph can lead to hundreds and sometimes thousands of terms. So far four different algebraic programs have been used in connection with the calculations discussed here and in sections 1 to 1.

*The results of Levine and Wright agree however with those of De Rújula et al. (M.J. Levine, private communication)

**See e.g. ref. [417]
The parametrizations of Feynman graphs generally have a singular (or almost singular) behaviour at some parts of the border of the integration region. Gaussian integration methods are very precise for integrands with polynomial type behaviour, but lose rapidly in reliability for integrands with a steep rise towards the border. The singularities can, however, be removed by means of polynomial mappings, thereby smoothing out the function and allowing for Gaussian integration. This technique has been used by Levine and Wright [419] but has the disadvantage that it does not readily allow for an estimate of error. Straightforward Monte Carlo methods do not work well for integrals that have their main contribution from some odd corner of the integration region, because the integrand is not preferentially sampled there. The following technique has proved adequate for many of the integrals met in QED. The integration region (the unit hypercube) is subdivided into a set of subvolumes by dividing the unit interval on each axis. In each subvolume the contribution to the integral and to its variance is estimated by random sampling of the integrand (usually only in two points). Using the variances found one calculates an improved subdivision of the unit intervals and reiterates the above procedure. In this way the function is explored and the interval structure refined such that the interval density adjusts itself to the rate of variation of the integrand, thereby mimicking the total variance. A program implementing this technique was originally devised by C.G. Sheppey [425] at CERN. It was first used by Aldins et al. [412, 413] in a QED context, but has since proved invaluable for many calculations.

4.3 Measurements of the electron anomaly

The best experimental value for the (negatively charged) electron anomalous magnetic moment has so far been obtained by Wesley and Rich [401]

$$ e^\gamma_{\text{exp}} = (1.1596577 \pm 3.5) \times 10^{-9} $$

This value represents an increase of $14 \times 10^{-9}$ with respect to the preliminary measurement by the same authors [426]. The error has decreased by a factor of two.

The first indications that the electron possessed an anomalous magnetic moment were reported in 1947 [427–429] and Kusch and Foley's experimental value [002] turned out to agree with the calculation by Schwinger [003]. Over the years the precision on the measurements has steadily improved [430–433] in particular with the fundamental experiment of Wilkinson and Crane [434–435] in 1963. Their value agreed with the theoretical calculations until Rich [436] and others [437, 438] reanalyzed the experiment and brought out a three standard deviation discrepancy. With the new experiments [426, 401] this discrepancy has, however, again disappeared.** Experiments along similar lines but with less precision have also been done by other groups.***

---

*See ref [446]

**The original value of Wilkinson and Crane was $\gamma_e = 1.159622(27) \times 10^{-9}$ corrected by Rich to $\gamma_e = 1.159549(30) \times 10^{-9}$

***See table 4.7
The technique used by Wesley and Rich [426] is essentially the same as that used by Wilkinson and Crane [434] although the apparatus is completely new and the magnetic field is an order of magnitude stronger. Electrons with energy of around 100 keV are partially polarized by Mott scattering at 90° on a gold foil and subsequently trapped in a magnetic bottle (\(\sim 1000\) G) for a accurately measured interval of time. After being ejected from the bottle the polarization of the electrons is analyzed by means of a second 90° Mott scattering. While trapped the average spin motion of the electrons can be described as a precession on of their polarization relative to their velocity with a frequency

\[
\omega_n = \alpha \omega_0
\]

where \(\alpha\) is the anomalous magnetic moment and \(\omega_0 = eB/m_o c\) (If the magnetic field is not homogeneous or not perpendicular to the velocity of the electrons or if there are electric fields present his formula must be corrected appropriately) as a function of trapping time the polarization and thereby the counting rate will be modulated with this frequency. This permits determination of \(\alpha\). The magnetic field is measured by means of Nuclear Magnetic Resonance (NMR) probes determining the resonance frequency of protons in water. In fact

\[
\omega_n = \frac{\omega_p(H_2O)}{\mu_p/\mu_B}
\]

where \(\mu_p\) is the magnetic moment of the proton in a water sample \((\sim 9)\) and \(\mu_B\) the Bohr magneton. The improvement of the accuracy on the anomaly is essentially due to the hyper magnetic field used by Wesley and Rich.

A promising new experimental technique has been proposed [440] and a preliminary result obtained [441] by a Bonn group. They study polarized electrons scatttaring in a magnetic field at the cyclotron frequency \(\omega_c\). The spin can be flipped by applying a radio frequency (RF) field at the Larmor (spin flip) frequency \(\omega_1 = \omega_c (1 + \alpha)\) and the transition is observed by the accompanying depolarization of the electrons. It is however also possible to observe the beat frequency \(\omega_1 - \omega_2\) corresponding to a simultaneous spin flip and transition between Landau levels (i.e., change of orbit). A measurement of both \(\omega_L\) and \(\omega_L - \omega_c\) leads to a determination of

\[
\frac{\alpha}{1 + \alpha} = \frac{\omega_L - \omega_c}{\omega_L}
\]

The advantage of this experiment over the previous ones is that the anomaly is measured spectroscopically, that the electrons are quite non-relativistic (\(<\) few eV) and both \(\omega_1\) and \(\omega_2\) are determined by the same method in the same field. The preliminary value is

\[
\alpha = 1.159660(300) \times 10^{-9}
\]

and is expected to improve considerably in the future.

The anomaly of the positron was originally measured by Rich and Crane [412], with \(\alpha_{e^-}\), virtually identical with the one used by Wilkinson and Crane [434] (see the description above). They found the result

\[
\alpha_{e^-} = 0.901168(11)
\]
More recently Gilleland and Rich [443] have improved the accuracy by increasing the length of
time the positrons were trapped in the magnetic bottle. The result was
\[ a_\mu^e = 0.031 1602(11). \]
The equality of anomalies for the electron and positron is a test of \textit{CPT} invariance which implies
\[ a_\mu^e = a_e^e. \]
The experimental values for the electron anomaly have been tabulated in table 4.7.

5. The quantum electrodynamics contribution to the difference between the anomalous magnetic
moments of muon and electron

The graphs of the purely quantum electrodynamical contribution to the anomalous magnetic
moment of a charged lepton, electron or muon, can be divided into two groups:
1. Graphs involving only one lepton
2. Graphs involving both leptons
The anomaly for a lepton with mass \( m \) can therefore be written
\[ a = a_1(m) + a_2(m,m'), \]
where \( m' \) is the mass of the other lepton. Since \( a \) is dimensionless, it follows that \( a_1 \) must be mass
independent, and that \( a_2 \) can only depend on the mass ratio. We may then rewrite this equation as
\[ a = a_1 + a_2(m/m'), \]
which specialized to electron and muon becomes
\[ a_\mu = a_1 + a_2(m_\mu/m_e), \]
\[ a_e = a_1 + a_2(m_e/m_\mu). \]
Due to the smallness of the ratio \( m_\mu/m_e \) it is not necessary to evaluate \( a_2(x) \) for all values of \( x \),
but only the asymptotic behaviour for small and large \( x \). It follows from general arguments that
\( a_2(x) \) vanishes as \( x \to 0 \). One may thus, in general, disregard the contribution \( a_2(m_\mu/m_\mu) \) to \( a_e \).
The difference
\[ a_\mu - a_e = a_2(m_\mu/m_e) - a_2(m_e/m_\mu) \]
only involves \( a_2 \) and the evaluation of \( a_\mu - a_e \) is generally easier than the evaluation of the complete anomalies.

In the discussion below, we shall only include those graphs which give a non-vanishing contribution after renormalization. Thus we leave out all corrections to external lines. We also leave out all
graphs representing renormalization counter-terms, assuming them to be implicitly included. The difference \( a_\mu - a_e \) has so far been computed up to (and including) the sixth order. It vanishes at second order.
Fig. 5: Mass dependent sixth order contributions to the lepton vertex

1 The fourth order contribution

In fourth order there is only one graph that contributes to $\alpha_2$, namely the one obtained from the Schwinger graph by inserting a vacuum polarization loop in the photon line. Up to terms of the order of $(m_e/m_\mu)^3$ we have the contribution to $\alpha_\mu$

$$\alpha_2^\text{fin} \left( \frac{m_\mu}{m_e} \right) = \left( \frac{\alpha}{\pi} \right)^2 \left\{ \frac{1}{3} \log \frac{m_\mu}{m_e} + \frac{25}{36} + \frac{\pi}{4} m_\mu \right\} 4 \left( \frac{m_\mu}{m_e} \right)^2 \log \frac{m_\mu}{m_e} + \left( \frac{m_\mu}{m_e} \right)^3 \left( \left[ \frac{m_\mu}{m_e} \right] \right)^4$$

The first two terms were evaluated by Suura and Wichmann [501] and by Peterman [502].
Table 5.1

<table>
<thead>
<tr>
<th>Graphs (fig 5.1)</th>
<th>Authors and references</th>
<th>Year</th>
</tr>
</thead>
<tbody>
<tr>
<td>1-6</td>
<td>Kinoshita [505]</td>
<td>1967</td>
</tr>
<tr>
<td>11-24</td>
<td>Lautrup and de Rafael [410]</td>
<td>1968</td>
</tr>
<tr>
<td>19-20</td>
<td>Lautrup and de Rafael [506]</td>
<td>1969</td>
</tr>
<tr>
<td>1-20</td>
<td>Aldins, Brodsky, Duliner and Kinoshita [412, 413]</td>
<td>1969</td>
</tr>
<tr>
<td>11-18</td>
<td>Lautrup, Peterman and de Rafael [507]</td>
<td>1970</td>
</tr>
<tr>
<td>7-10</td>
<td>Lautrup [508]</td>
<td>1970</td>
</tr>
</tbody>
</table>

remaining terms have been obtained by Elend [503] and by Erickson and Liu [504]. These authors have calculated the function $\alpha_2(x)$ for all $x$. To the same accuracy, the contribution to $\alpha_e$ is [410]

$$\alpha_2\left(\frac{m_e}{m_\mu}\right) = \frac{\alpha^2}{\pi} \left\{ \frac{1}{45} \left[ \frac{m_e}{m_\mu} \right]^2 + O\left[ \left( \frac{m_e}{m_\mu} \right)^4 \right] \right\},$$

which is vanishingly small, but has been included as a reminder of the fact that both $\alpha_e$ and $\alpha_\mu$ depend on the mass ratio.

Hence the fourth order difference (fig 4.2) is

$$(\alpha_\mu - \alpha_e)^{(4)} \approx \frac{\alpha^2}{\pi} \left\{ \frac{1}{3} \log \frac{m_\mu}{m_e} - \frac{25}{36} + \frac{\pi^2}{4} \frac{m_e}{m_\mu} - 4\left(\frac{m_e}{m_\mu}\right)^2 \log \frac{m_\mu}{m_e} + \frac{134}{45} \left(\frac{m_e}{m_\mu}\right)^2 \right\} + O\left[ \left( \frac{m_e}{m_\mu} \right)^3 \right].$$

Numerically we have

$$(\alpha_\mu - \alpha_e)^{(4)} = 1.094 \times 10^{-9}$$

5.2. The sixth order contribution

In sixth order there are 24 graphs* contributing to $\alpha_\mu - \alpha_e$. They are shown in fig. 5.1. These graphs naturally are divided into three classes:

I. Graphs 1–6, containing photon-photon scattering subgraphs

II. Graphs 7–20, containing second order (but not fourth order) electron vacuum polarization subgraphs

III. Graphs 21–24, containing fourth order electron vacuum polarization subgraphs.

Not all of these graphs are independent. Some are related to each other by charge conjugation, thereby giving the same anomalous magnetic moment contribution. Two graphs give the same anomalous magnetic moment if they arise from each other by reversing the directions of the muon line. Thus there are 14 independent sets of graphs, the only unpaired graphs being 9, 10, 21, and 24.

From table 5.1 which gives the history of the calculations of these graphs, it is seen that all graphs except those of class I have been evaluated twice (one-graph 21–even three times). The two

*Here we disregard the contribution from $\alpha_\mu\left(\frac{m_e}{m_\mu}\right)$. 
independent results for graphs 7–24 have been tabulated in Table 5.2 and Table 5.3 allowing for a detailed comparison of the values obtained from each graph. The uncertainties arise from the numerical integrations. The agreement is generally very good, although there is a slight difference between the two values for graphs 7+8.

We can now quote the overall results for the three cases**

Class I [412, 413]
\[
\left( \alpha_\mu - \alpha_\text{e}_\text{I} \right)_{\text{II}} = \left( 18.4 \pm 1.1 \right) \left( \alpha/\pi \right)^2
\]

Class II [507, 421, 506, 414]
\[
\left( \alpha_\mu - \alpha_\text{e}_\text{II} \right)_{\text{II}} = \left\{ 8 \left( \pm 0.02 \right) \left( \alpha/\pi \right)^3 \right\} \text{ Table 5.2}
\]

Class III [410, 505, 414]
\[
\left( \alpha_\mu - \alpha_\text{e}_\text{III} \right)_{\text{III}} = \left\{ 4.2414 \left( \alpha/\pi \right)^2 \right\} \text{ Table 5.2}
\]

The overall agreement between the two independent calculations is excellent, the small difference having largely cancelled out in the sum. The final result is

\[
\left( \alpha_\mu - \alpha_\text{e} \right)_{\text{III}} = \left\{ 4.2414 \pm 0.03 \left( \alpha/\pi \right)^3 \right\} \text{ Table 5.3}
\]
Table 5.3
Sixth order results from ref. [414] $f(o) = \frac{2}{3} \left( \log \frac{m_\mu}{m_e} - \frac{25}{12} + \frac{3\pi^3}{4} \frac{m_e}{m_\mu} \right)$

<table>
<thead>
<tr>
<th>Graphs</th>
<th>Semianalytical form in units of $(\alpha/\pi)^3$</th>
<th>Numerical value in units of $(\alpha/\pi)^3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 + 8</td>
<td>$2f(o) = -0.467 + 0.76(1)$</td>
<td>-1.28(1)</td>
</tr>
<tr>
<td>1 + 10</td>
<td>$2f(o) = 0.778 - 0.53(6)$</td>
<td>2.88(6)</td>
</tr>
<tr>
<td>11 + 12 + 15 + 16</td>
<td>$f(o) = -0.654 - 0.53(7)$</td>
<td>-1.96(7)</td>
</tr>
<tr>
<td>13 + 14</td>
<td>$f(o) = -0.564 - \log \frac{1}{m_\mu} - 0.18(6)$</td>
<td>-1.41(3)</td>
</tr>
<tr>
<td>17 + 18</td>
<td>$f(o) = -0.090 + \log \frac{1}{m_\mu} - 0.45(3)$</td>
<td>-0.65(3)</td>
</tr>
<tr>
<td>19 + 20</td>
<td>$2f(o) = -0.650 - 0.18(6)$</td>
<td>0.101(2)</td>
</tr>
<tr>
<td>21</td>
<td>$2f(o) = -0.650 - 0.18(6)$</td>
<td>2.72(2)</td>
</tr>
<tr>
<td>22 + 23 + 24</td>
<td>$2f(o) = -0.650 - 0.18(6)$</td>
<td>1.49(2)</td>
</tr>
</tbody>
</table>

$(a_\mu - a_e)^{(6)} = (20.3 \pm 1.1) (\alpha/\pi)^3 = (254 \pm 14) \times 10^{-9}$.

In view of the fact that the class I contribution is by far the most important one, being an order of magnitude greater than the rest, and at the same time the only part that has not been checked by an independent group, we must emphasize the need for a recalculation. It would also be desirable to lower the uncertainty because the planned experiments on the muon anomaly may reach the precision of $\sim 15 \times 10^{-9}$.

We shall now discuss some of the techniques used in obtaining the sixth order result. The amplitude for an arbitrary vertex graph can be written $-ie \Gamma \mu(p_2, p_1)$, where $p_1$ and $p_2$ are the momenta of the incoming and outgoing lepton. The most general expression between two spinors is

$$\bar{u}_2 \Gamma \mu(p_2, p_1) u_1 = \bar{u}_2 \left\{ (F_1 + F_2) \gamma_\mu - \frac{(p_1 + p_2)_\mu}{2m} F_2 + \frac{(p_2 - p_1)_\mu}{2m} F_3 \right\} u_1$$

which defines the three form factors $F_i[(p_2 - p_1)^2]$. By definition, the contribution to the anomalous magnetic moment is

$$a = F_2(0)$$

A graph can be subjected to two kinds of gauge transformations, external and internal. A set of graphs is invariant under external gauge transformations when

$$(p_2 - p_1) \mu \bar{u}_2 \Gamma \mu(p_2, p_1) u_1 = 0,$$

which implies that $F_3 = 0$. This condition (current conservation) is not in general satisfied for individual graphs. In fig. 5.1 the sets of graphs invariant under external gauge transformations are 1+3+5, 2+4+6, 7+11+14, 8+12+13, 9+15+16, 10+17+18, 19, 20, 21, 22, 23 and 24. The external gauge transformations are, however, not relevant for the purely intrinsic quantity $a$, which will
mly be influenced by internal gauge transformations, i.e., transformations of the photon propagator of the form

$$\frac{g_{\mu\nu}}{k^2} \rightarrow \frac{g_{\mu\nu}}{k^2} + k_\mu k_\nu f(k^2)$$

where $f$ is an arbitrary function. We leave it for the reader to prove that the following set of graphs yield gauge invariant total contributions to the anomaly (after renormalization) $1 + 3 + 5 + 7 + 8 + 9 + 10 + 11 + 12 + 13 + 16 + 14 + 18 + 15 + 16 + 19 + 20 + 21 + 22 + 23 + 24$. It can be seen that the contributions from each set is infrared convergent.

It is an interesting coincidence that the most difficult graphs to evaluate—namely those of class I—are also the graphs that give the largest contribution. These graphs were evaluated by the combined efforts of Aldins, Brodsky, Dufour, and Kinoshita [412–413]. Calling the off-shell photon-photon-scattering amplitude $\Pi_{\mu\nu\rho\sigma}(k_1, k_2, k_3, k_4)$, where $k_1 + k_2 + k_3 + k_4 = 0$, we can write the vertex functions from graphs I, using the notation of Fig. 5-2,

$$\Gamma_\mu(p_2, p_1) = \int \frac{dk_1dk_2}{(2\pi)^3} \Pi_{\mu\nu\rho\sigma}(q, k_1, k_2, k_3) \gamma^\nu(p_2 + k_2 - m_\mu, k_1 - p_1 + m_\mu) \gamma^\rho(p_1 - k_1 + m_\mu) \gamma^\sigma(q),$$

where $q = p_2 - p_1$. An ingenious use of gauge-invariance made the extension of the anomalous magnetic moment much simpler, at the same time explicitly removing the curious logarithmic ultra-violet divergence inherent to the photon-photon scattering amplitude.

Current conservation or gauge invariance gives rise to the identity

$$q^\mu \Pi_{\mu\nu\rho\sigma}(q, k_1, k_2, k_3, k_4) = 0,$$

from which one obtains by differentiation

$$\Pi_{\mu\nu\rho\sigma}(q, k_1, k_2, k_3, k_4) = \frac{\partial}{\partial q^\mu} \Pi_{\lambda\nu\rho\sigma}(q, k_1, k_2, k_3, k_4).$$

Thereby one can write

$$\Gamma_\mu(p_2, p_1) = q^\nu \Gamma_{\nu\mu}(p_2, p_1),$$

with

$$\Gamma_{\nu\mu}(p_2, p_1) = e^2 \int \frac{dk_1dk_2}{(2\pi)^3} \frac{\partial}{\partial q^\mu} \Pi_{\lambda\nu\rho\sigma}(q, k_1, k_2, k_3, k_4) \gamma^\nu(p_2 + k_2, m_\mu, k_1 - p_1 + m_\mu) \gamma^\rho(p_1 - k_1 + m_\mu) \gamma^\sigma(q).$$
In $\Gamma_{\mu\lambda}, p_2$ can be put equal to $p_1$ because one power of $q$ is already taken outside. Differentiation of a graph with respect to an external momentum acts like the insertion of zero momentum photons and decreases the degree of divergence by one, thereby removing the spurious logarithmic divergence. The remainder of the calculation is in principle straightforward although very complicated. Aldins et al. used two different techniques for obtaining the parametric form of the integral, one being the standard Landau method* the other based on a method developed by Nakanishi [510] and Kinoshita [511] (double parametric representation). Part of the reduction to parametric form was done by hand, and part was carried out by means of REDUCE, a programming language for algebraic manipulation developed by Hearn [423]. Finally the (7-dimensional) parametric integral was evaluated numerically by means of the special numerical integration program described at the end of section 4.2. It was found that the class I diagrams contains a logarithmic divergence for $m_e \to 0$ contrary to expectations [505]. Writing

$$\Gamma_{\mu\lambda} = \left(\frac{\alpha}{\pi}\right)^3 \left\{ C_1 \log \frac{m_\mu}{m_e} + C_2 \right\},$$

Aldins et al. found that $C_1$ could be expressed as a 5-dimensional integral with the value

$$C_1 = 6.4 \pm 0.1$$

The remainder $C_2$ was not determined directly but could be inferred from the overall value quoted above

$$C_2 \approx -16 \pm 1$$

The numbers $C_1$ and $C_2$ are surprisingly large. In all other cases purely numerical coefficients turn out to be of the order of unity. This unexpected behaviour also stresses a need for a recalculation of the class I diagrams.

The remaining diagrams are all much simpler to evaluate. All of the class III diagrams are examples of the graph shown in fig. 5.3 where the insertion $G$ is a fourth order vacuum polarization graph. The vertex function from this graph is

$$\Gamma^{(G)}(p_2, p_1) = e^2 \int \frac{d^4 k}{(2\pi)^4} \gamma^\rho (p_2 - k - m_\mu)^{-1} \gamma_\mu (p_1 - k - m_\mu)^{-1} \gamma^\sigma D^{(G)}_{\rho\sigma}(k),$$

where $D^{(G)}_{\rho\sigma}$ is the contribution to the photon propagator form the graph $G$. This quantity can be expressed in terms of a single spectral function.

*See e.g. Bjorken and Drell [509], sec. 18.4
\[ D_{\rho \sigma}^{(G)} = \left( g_{\rho \sigma} - \frac{k_{\rho} k_{\sigma}}{k^2} \right) \frac{\Pi^{(G)}(k^2)}{k^2}, \]
satisfying a once subtracted dispersion relation
\[ \frac{\Pi^{(G)}(k^2)}{k^2} = \frac{1}{\pi} \int_0^\infty \frac{\text{Im} \Pi^{(G)}(t)}{t \; k^2} \; dt. \]

The imaginary part is given by
\[ \text{Im} \Pi^{(G)}(k^2) = \frac{1}{6k^2} \sum_{n \in G} (2\pi)^4 \delta^{(4)}(k - k_n) \langle 0 | J_\mu(0) | n \rangle \langle n | J_\nu(0) | 0 \rangle, \]
where the sum goes over the possible intermediate states in \( G \). Accordingly we can write
\[ \Pi_\mu^{(G)}(p_2, p_1) = \int \frac{dt}{t} \frac{1}{\pi} \text{Im} \Pi^{(G)}(t) \Gamma^{(2)}(p_2, p_1, t), \]
where \( \Gamma^{(2)}(p_2, p_1, t) \) is obtained from the usual second order graph replacing the photon propagator \( \gamma^{-2} \) by a massive transverse propagator
\[ -i \left( g_{\mu \nu} - \frac{k_\mu k_\nu}{k^2} \right) \frac{1}{k^2 - t} . \]

The anomalous magnetic moment satisfies the same equation
\[ a^{(G)} = \int \frac{dt}{t} \frac{1}{\pi} \text{Im} \Pi^{(G)}(t) a^{(2)}(t), \]
where \( a^{(2)}(t) \) is the anomalous magnetic moment obtained from the second order graph replacing the photon propagator by a massive propagator. The \( k_\mu k_\nu \) terms can be disregarded as they only contribute to the renormalization constant \( Z_1 \), and \( a^{(2)}(t) \) is given by the well-known expressions
\[ a^{(2)}(t) = \frac{\alpha}{\pi} K(t) \]
\[ K(t) = \int_0^1 dx \frac{x^2(1-x)}{x^2 + (1-x) m_\mu^2 / t}. \]

These expressions are valid for any kind of insertion \( G \), in particular for the hadronic vacuum polarization (see section 6.1). In the case of fourth order vacuum polarization insertions the function \( \text{Im} \Pi^{(G)}(t) \) was already known from the work of Källen and Sabry [512], and the whole calculation boiled down to a one-dimensional integral which could be evaluated analytically with the result shown in table 5.2.

For the analytic evaluation of the double bubble graphs 19, 21, an expression very well suited is [506]
\[ a^{(1+)} = \int_0^1 dx (1-x) \left[ \Pi^{(G)}(x), \frac{1}{1-x} m_\mu^2 \right] \].

\* \( K(t) \) is bounded, and monotonically decreasing.
which is obtained interchanging the order of integrations in $x$ and $t$ in the general expression for $\tilde{a}^{(G)}$. Here, the $\Pi^{(G)}$-function is simply a product of two second-order $\Pi$-functions.

For the class II diagrams it follows by the same arguments as above that the anomaly for a graph is given by

$$a = \int \frac{dt}{4m_e^2} \frac{1}{\pi} \text{Im} \Pi^{(2)}(t) \alpha(t)$$

where $\Pi^{(2)}(t)$ is the second order vacuum polarization by electrons, and $\alpha(t)$ is the anomaly from the fourth order graph obtained from a sixth order graph by replacing the photon propagator containing the vacuum polarization insertion by a massive propagator

$$-i\left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right) \frac{1}{k^2 - t}$$

The $k_\mu k_\nu$ terms can be disregarded because they are essentially gauge terms that cancel within the following sets of graphs: 7+8+9+10, 11+15, 12+16, 13, 14, 17, 18, 19, 20. Thus at the expense of one extra integration relative to the fourth order calculation one can evaluate the anomaly from the class II graphs [507, 421, 414]. It is even possible to perform the $t$-integration analytically in the limit $m_e/m_\mu \to 0$. For a graph (or set of graphs) for which $\alpha(0)$ exists* one has to order $m_e/m_\mu$

$$a = \frac{\alpha}{\pi} \left\{ \frac{2}{3} \log \frac{m_\mu}{m_e} - \frac{5}{9} \right\} \alpha(0) + \frac{1}{3} \alpha' + O \left( \frac{m_e}{m_\mu} \right)$$

where

$$\alpha' = \int_0 \frac{dt}{4m_e^2} \{\alpha(t) - \alpha(0)\} + \int \frac{dt}{t} \alpha(t)$$

does not depend on the mass ratio. The quantity $\alpha(t)$ is a multidimensional integral (up to 5-dimensions) of the form

$$\alpha(t) = \int \int \ldots \int \frac{A_n}{(B+Ct)^n}$$

which can be integrated analytically over $t$. For $t = 0$, $\alpha(t)$ is the ordinary fourth order anomaly from the graph (or set of graphs) obtained by removing the vacuum polarization.** These fourth order anomalies have been given by Peterman, so that it is trivial to obtain the analytic form of the coefficient of $\log(m_\mu/m_e)$ (see table 5.2).

The final stage of the class II calculations is a numerical integration of the parametric integral again using the Sheppey program described at the end of section 4.2. The reason for the difference in the quoted uncertainties of the two different calculations of the class II diagrams is probably

*If $\alpha(0)$ does not exist (as for graphs 9-18) it is always possible to separate $\alpha(t)$ into a regular part for which the analysis can be carried out as shown, plus an irregular part which can be evaluated explicitly.

**If $\alpha(0)$ exists. Otherwise see the previous footnote.
due to the fact that the analytic integration over $t$ was carried out in refs. [506 and 507] (table 1.2) but not in ref. [414] (table 5.3).

5 Other contributions to the muon anomaly

5.1. The hadronic contributions

The importance of hadronic vacuum polarization insertions in the second order muon vertex (see fig. 5.3) was first pointed out by Bouchiat and Michel [601]. They remarked that resonances in the $\pi^-\pi^+$ system can lead to an enhancement of the vacuum polarization corrections which could be observable in precision measurements of $\alpha_\mu$. The first estimates of this effect gave $\alpha_\mu^{\text{hadrons}} \sim 10^{-7}$, i.e., far below the error in the measurement of $\alpha_\mu$ at that time which was [604] $\pm 5 \times 10^{-8}$. Since then, a new precision measurement of $\alpha_\mu$ has been made [607, 608], and simultaneously, colliding beam experiments have been performed at Novosibirsk [605, 606] and Orsay [607-609] which yield precious information on the hadronic contributions to $\alpha_\mu$.

The hadronic spectral function which appears in the Kallen-Lehmann representation of the photon propagator (see the discussion of class III diagrams in section 5.2) can be directly obtained from measurements of the total $e^+e^-$ annihilation cross-section into hadrons. To lowest order in $\alpha$ (see fig. 5.3) and $t \geq 4m_\pi^2$,

$$\sigma_{e^+e^- \to \text{hadrons}}(t) = \frac{4\pi^2\alpha}{t} \text{Im} \Pi^{(1)}(t),$$

where $t$ is the total $e^+e^-$ centre-of-mass energy squared. The hadronic contributions to $\alpha_\mu$, due to vacuum polarization insertions in the second order muon vertex, can thus be directly related to the annihilation cross-section measured in the colliding beams experiments. Assuming that the dispersion integral for the hadronic vacuum polarization only requires one subtraction (charge renormalization) we have

$$\alpha_\mu^{\text{hadrons}} = \frac{1}{4\pi^2} \int \frac{dt}{4m_\pi^2} \sigma_{e^+e^- \to \text{hadrons}}(t) K_\mu(t),$$

where $K_\mu(t)$ is a purely QED function which results from the combination of the two fermion propagators and the $\gamma$ propagator of a "photon" with squared mass $t$ in the muon vertex of fig. 4.1

$$K_\mu(t) = \int_0^1 dx \frac{x^2(1-x)}{x^2+(1-x)t/m_\mu^2}.$$

The function $K_\mu(t)$ is positive definite in the integration region $4m_\pi^2 \leq t \leq \infty$ and therefore $\alpha_\mu^{\text{hadrons}}$ must be positive. Notice that, for large values of $t$, $K(t)$ decreases as $t^{-1}$

$$K_\mu(t) = \frac{1}{3} \frac{m_\mu^2}{t} + O[(m_\mu^2/t)^2 \log(t/m_\mu^2)].$$

*It dependent calculations were also made by Durand [602] and later, using new information on vector mesons, by Kinoshita and Oakes [603], and by Bowcock [604].
It appears thus that the high-energy contributions to $\alpha_\mu(\text{hadrons})$ are depressed by the factor $K_\mu(t)$. The $\alpha_\mu(\text{hadrons})$ integral is dominated by the low-energy region and, in particular, by those values of $t$ corresponding to the mass squared of resonances which have the quantum numbers of the photon.

A calculation of $\alpha_\mu(\text{hadrons})$ using the results of the Orsay colliding beams experiments has been done by Gourdin and de Rafael \cite{610}. The total contribution to $\alpha_\mu(\text{hadrons})$ was separated into an isoscalar contribution $I = 0$ and an isovector contribution $I = 1$. The isoscalar part $\alpha_\mu(I=0)$ was estimated using vector meson dominance, taking into account the contributions from the $\omega$ and $\phi$ mesons:

$$\alpha_\mu(I=0) \simeq \sum_{\nu=0,\phi} \frac{3}{\pi} K_\nu(M_\nu^2) \frac{\Gamma(V \to e^+e^-)}{M_\nu} = (6.1 \pm 1.2 + 5.0 \pm 0.8) \times 10^{-9}.$$  

The isovector part $\alpha_\mu(I=1)$ was estimated assuming that the $I = 1$ hadronic system is dominated by the $\pi^-\pi^+$ $P$-wave. The evaluation of the integral was made using the expression proposed by Gounaris and Sakurai for the pion form factor \cite{611}. This expression, which is based on a generalized effective-range formula for $\pi^-\pi^+$ scattering, takes into account finite width effects and fits well the Orsay data ranging from 644 MeV to 886 MeV with the following values for the mass and the width of the $\rho$-meson:

$$M_\rho = (770 \pm 4) \text{ MeV}, \quad \Gamma_\rho = (111 \pm 6) \text{ MeV}.$$  

The result obtained is

$$\alpha_\mu(I=1) = (54 \pm 3) \times 10^{-9}.$$  

Altogether $\alpha_\mu(\text{hadrons})$ is estimated to be

$$\alpha_\mu(\text{hadrons}) = (65 \pm 5) \times 10^{-9}.$$  

Notice that the error quoted above only reflects the uncertainty in the Orsay data used as input. The uncertainties due to the extrapolation of the Gounaris and Sakurai expression for the pion form factor below 644 MeV and above 886 MeV, as well as the uncertainty in neglecting other contributions to the isoscalar part than those coming from the $\omega$ and $\phi$ resonances could be larger than $5 \times 10^{-9}$. Clearly, more experiments with colliding beams in the region just above the two pion threshold and at high energies will be extremely welcome to reduce these uncertainties.

*In particular, this explains why hadronic vacuum polarization insertions in the second order electron vertex can be neglected. They give a contribution

$$\alpha_e(\text{hadrons}) \simeq \frac{1}{12\pi} \int_{4m_e^2}^{t_0} \frac{dt}{t} \sigma_{e^+e^- \to \text{hadrons}(t)}.$$  

In fact, using the inequality $(t_0/t)K_\mu(t_0) < K_\mu(t) < \frac{1}{3} m_\mu^2/t$ for $t_0 < t$, we get

$$\alpha_e(\text{hadrons}) \leq \frac{1}{12\pi} \int_{4m_e^2}^{t_0} \frac{dt}{t} \sigma_{e^+e^- \to \text{hadrons}(t)} \alpha_\mu(\text{hadrons}) = 3.7 \times 10^{-4} \alpha_\mu(\text{hadrons}).$$
There have been some attempts to bound the hadronic vacuum polarization contributions to $\alpha$. An analysis by Bell and de Rafael [612] shows that a proposed theoretical bound [613], based on the hypothesis of current-field identity [614], does not add usefully to the strict vector meson dominance (V.M.D.) estimate of $\alpha_{\mu}^{\text{hadrons}}$. In fact, the estimated bound depends on a V.M.D. approximation, and this moreover on a quantity for which that approximation is less reliable than for $\alpha_{\mu}^{\text{hadrons}}$ itself. It is possible, however, to bound $\alpha_{\mu}^{\text{hadrons}}$ in a different way by a quantity which governs all sufficiently low energy vacuum polarization effects [612]. Indeed, the hadronic contribution to the photon propagator is

$$P^{(H)}(q^2) = \int_{4m^2_\pi}^\infty \frac{dt}{t} \frac{1}{\pi} \text{Im} \Pi(t)$$

and we have

$$\alpha_{\mu}^{\text{hadrons}} \leq \frac{\alpha}{\pi} \frac{m^2_\pi}{3} P^{(H)}(0) \leq \frac{\alpha}{\pi} \frac{1}{3} m^2_\mu \left(1 - \frac{q^2}{4m^2_\mu}\right) P^{(H)}(q^2)$$

for all negative $q^2$. Large momentum transfer electron-electron scattering experiments can be used to set a limit on $P^{(H)}(q^2)$ for particular values of $q^2$. Present experimental results lead to an upper bound

$$\alpha_{\mu}^{\text{hadrons}} < 9 \times 10^{-6}$$

this limit, however, could be lowered down with improved empirical knowledge on electron-electron scattering at large momentum transfer and electron-positron annihilation at all available energies.

6.2 The weak interaction contribution

To give a definite prediction for the contribution from weak interactions is not possible because of the inherent theoretical difficulties with higher order weak corrections. Estimates can however be made.

If the weak interactions are of the local four-fermion V-A type the diagram in fig. 6.1 will give rise to an anomaly for the muon. The corresponding diagram for the electron is obtained by interchanging $e$ and $\mu$. Power counting indicates that the anomaly will be quadratic in the cut-off and by dimensionality arguments one would expect

$$a_\mu^F \approx C G_F^2 \Lambda^2 m^2_\mu$$

where $G_F$ is the Fermi coupling constant, $\Lambda$ is a cut-off and $C$ a numerical constant. This constant surprisingly turns out to be zero [615]. Therefore, the dominant terms to be expected from the four-fermion interaction is**

$$a_\mu^F \approx C G_F^2 m^4_\mu \log(\Lambda/m_\mu) \approx 10^{-12}$$

*Using the experimental input quoted in refs [607–609], the strict V.M.D. estimate of $\alpha_{\mu}^{\text{hadrons}}$ is $6(\pm 5) \times 10^{-6}$ see refs [610] and [612].

**C. Jarlskog [634] has recently made a careful evaluation of the four-fermion contribution. It appears that the quadratic terms
Fig. 6.1. The four-fermion V–A contribution to the anomalous magnetic moment of the muon.

Fig. 6.2. Weak contribution of the muon anomaly via the intermediate vector-boson W

much too small to be of importance. If, however, the weak interactions are mediated by an intermediate charged boson, W, the weak anomaly will only be of first order in $G_F = \sqrt{2} g^2/m_W^2$. The corresponding diagram is shown in fig. 6.2. The history of this diagram has been particularly controversial [615–622]. Only the last two calculations (Brodsky and Sullivan [621], Burnett and Levine [622]) agree on the expression*

$$a^W_\mu = \frac{G_F m^2_\mu}{8\pi^2\sqrt{2}} \left(2(1-K_W) \log \xi + 10/3\right).$$

Here $K_W$ is the anomalous magnetic moment of the W and $\xi$ is the regularizer used in the $\xi$-limiting procedure [623]. The overall factor is

$$\frac{G_F m^2_\mu}{8\pi^2\sqrt{2}} \approx 10 \times 10^{-9}$$

The other factor is in principle unknown. Brodsky and Sullivan take $\xi = \alpha$ (the fine structure constant) and $K_W = 0$, obtaining $a^W_\mu \approx 10 \times 10^{-9}$

while Burnett and Levine get $a^W_\mu = -20 \times 10^{-9}$

using $\xi = (m_W/\Lambda)^2 = (2 \text{ GeV}/300 \text{ GeV})^2$ and also $K_W = 0$. **

**6.3 Measurements of the muon anomaly**

The best experimental value for the muon anomaly has been obtained at CERN [007, 008] and the value is†

$$a^\text{exp}_\mu = 0.001 \pm 0.0016 (3\text{1})$$

---

*The case $A_W = 0$ considered by Schaffer [620] also agrees with the previous expression.

**If the anomalous magnetic moment of W is $K_W = 1$ instead of 0, as suggested by a recently revived theory [636], the weak anomaly becomes finite, positive and small $\sim 3 \times 10^{-9}$ (see refs. [637–639]).

†This value contains results for both $\mu^-$ and $\mu^+$, thus assuming CPT. Separating $\mu^+$ and $\mu^-$ contributions one has $a^\text{exp}_\mu - a^\text{exp}_{\mu^+} = (50 \pm 75) \times 10^{-9}$ (see ref. [R 1]) as a test of CPT.
The first determination of the muon anomaly was the precision measurement by Garwin et al. [624] of the magnetic moment of the muon. Combined with the measurement of the muon mass anomaly, the anomaly could be calculated. The precision (see Table 6.1) was however not sufficient to see the effect of the fourth order term (section 5.1) which differs from the corresponding term in the electron anomaly due to vacuum polarization by electrons. The subsequent CERN experiment [625, 604, 626, 627] although much more precise, did still not allow a definite conclusion to be reached about the fourth order term. The second CERN experiment which gave the final number quoted above, however, established that the muon anomaly differed from the electron anomaly by the amount predicted by theory. It was even so precise that it was necessary to include the 11th order term in the theoretical value in order to obtain full agreement with experiment.

The CERN experiments measure the anomaly directly (as in the electron experiments) from the precession of the spin of the muon relative to its momentum. In a uniform magnetic field B the precession rate (in the laboratory) is [635]

$$\omega_\alpha = \frac{e}{m_\mu c} B$$

independent of the velocity of the muons. The spin (Larmor) precession frequency of muons at rest in vacuum is

$$\omega_\mu = \frac{1}{1+\alpha} \frac{e}{m_\mu c} B$$

such that \(\alpha/(1+\alpha) = \omega_\alpha/\omega_\mu\). The quantity \(\omega_\mu\) cannot be determined directly, but one can measure the precession frequency of protons in water \(\omega_p\) by means of a nuclear magnetic resonance (NMR) magnetometer in the same magnetic field. Combined with the measurements [629, 630] of the ratio \(\lambda = \omega_p/\omega_\mu\) of muon to proton precession frequencies in water we find

$$\frac{\alpha}{1+\alpha} = \frac{\omega_\alpha}{\omega_p} \cdot \frac{1}{\lambda + \epsilon}$$

where we have put \(\omega_p/\omega_\mu = 1/(1+\epsilon)\). The Rudermann [631] correction \(\epsilon\), representing the diamagnetic shielding of the field of the muon in water, has recently been experimentally shown to be negligible [629].

The frequency \(\omega_\alpha\) is measured by observing the decay of muons in a storage ring with an almost...
uniform magnetic field (~ 17 kG). The muons (~ 1.3 GeV) are obtained from the nearly forward decay of π⁺ created when a proton beam (~ 10 GeV) hits a target inside the ring. The muons are on the average longitudinally polarized (~ 26%) when they become trapped, but while they are circulating before they decay the spin will turn relatively to the momentum with the frequency ω. The decay rate in the forward direction will be modulated with this frequency. The top end of the electron spectrum seen at the inside of the ring corresponds to near forward decays and will thus also be modulated with ω. For further details we refer the reader to for instance refs. [007, 008 and R. 1]

A new gμ—2 experiment [632, 633, R. 1] is being planned at CERN as a continuation of the previous experiments. Ingenious new features lead to an expected overall improvement of the uncertainty by a factor of 20.

Finally let us mention that Henry et al. [628] have also measured aμ with a technique resembling that of the measurements of the electron anomaly (section 4.3). The result deviates from theory by two standard deviations, but the precision is rather low (see table 6.1).

7. "Exotic" contributions to the electron and muon anomalies

In this section we present a summary of the various speculative contributions to the anomalies. It is generally impossible to give definite predictions for such contributions due to the lack of knowledge of the coupling constants and masses of the hypothetical particles or fields involved. Instead we shall turn the argument around and use the agreement between theory and experiment to put limits on such parameters.

7.1 Generalities

If a theory gives a certain contribution Δa to the anomaly of a charged lepton, this quantity is restricted* by the inequality

\[ |Δa + a_\text{th} - a_\text{exp}| \leq C \sqrt{(σ_\text{th})^2 + (σ_\text{exp})^2}, \]

where Δa ± σth and Δa ± σexp are the theoretical and experimental lepton anomalies with associated one standard deviation uncertainties.** The constant C is related to the confidence limit of the bound. We shall choose a 95% confidence limit with

\[ C = 1.96. \]

We have (sections 4.3 and 6.2)

\[ a_\text{exp}^e = (1 159 657.7 ± 3.5) \times 10^{-9}, \quad a_\text{th}^e = (1 159 655.4 ± 3.3) \times 10^{-9} \]

\[ a_\text{exp}^\mu = (1 166 160 ± 310) \times 10^{-9}, \quad a_\text{th}^\mu = (1 165 879 ± 15) \times 10^{-9} \]

such that the inequality above becomes

*We assume that two or more exotic contributions do not conspire to cancel each other

**We have added these uncertainties quadratically although this is not a unique choice
\[-7.1 \times 10^{-9} \leq \Delta \alpha_e \leq 11.7 \times 10^{-9}\]

\[-325 \times 10^{-9} \leq \Delta \alpha_\mu \leq 887 \times 10^{-9}\]

Observe that the muon bound is dominated by the experimental uncertainty.

Many of the exotic contributions to be discussed below depend on a mass parameter \(\Lambda\) which can be simply a high mass cut-off or the mass of a hypothetical heavy particle. The exotic contributions vanish in general when \(\Lambda \to \infty\) and in most cases we have a quadratic dependence on \(1/\Lambda\) i.e. for \(\Lambda \gg m\)

\[\Delta \alpha = A(m/\Lambda)^2,\]

where \(m\) is the lepton mass and \(A\) is a quantity consisting of coupling constants, numerical constants and perhaps a slowly varying function of the masses (Let us stress, however, that there are examples discussed below that do not have this form.) The bounds on \(\Delta \alpha\) lead to lower limits on \(\Lambda\) The superscripts + and – correspond to \(\Delta \varepsilon > 0\) and \(\Delta \alpha < 0\), respectively

\[\begin{align*}
\Lambda_e^+ &\geq 4.8 \text{ GeV} \sqrt{A_e} \\
\Lambda_e^- &\geq 5.9 \text{ GeV} \sqrt{-A_e}.
\end{align*}\]

\[\begin{align*}
\Lambda_\mu^+ &\geq 113 \text{ GeV} \sqrt{A_\mu} \\
\Lambda_\mu^- &\geq 183 \text{ GeV} \sqrt{-A_\mu}.
\end{align*}\]

These limits are somewhat better for negative exotic contributions because they make worse agreement between theory and experiment.

Although the electron experiment is about 90 times more precise than the muon experiment, the latter in general puts more stringent limits on exotic contributions. This is mainly due to a large mass of the muon which allows it to probe much smaller distances than the electron.

A very interesting case arises when we assume*

\[\begin{align*}
\Lambda_e &= \Lambda_\mu = \Lambda \\
A_e &= A_\mu = A
\end{align*}\]

in which case we obtain

\[\begin{align*}
\Lambda^+ &\geq 113 \text{ GeV} \sqrt{A} \\
\Lambda^- &\geq 183 \text{ GeV} \sqrt{-A}
\end{align*}\]

Since now

\[\Delta \alpha_e = (m_e/m_\mu)^2 \frac{A}{\Lambda},\]

*These equations could be called \(e-\mu\)-universality but we shall refrain from doing so here because of the ambiguous meaning of this terminology.
we obtain an induced bound on the exotic electron contributions from the bound on the corresponding muon contribution

\[-8 \times 10^{-12} \lesssim \Delta a_e \lesssim 21 \times 10^{-12}.\]

We can conclude quite generally that for any exotic contribution satisfying \( \Delta a = \pi/(m/\Lambda)^2 \), \( A_e = A_\mu, \Delta_e = \Delta_\mu \), the agreement between theory and experiment for the muon anomaly guarantees that the electron anomaly is not influenced by it. This is the meaning of the usual statement: the electron anomaly is a pure QED quantity.

7.2. Modifications of quantum electrodynamics

In order to measure the "goodness" of QED, modifications of a general nature not specifying special interactions or particles have been attempted. As shown by Kroll [701] such ad-hoc modifications are severely restricted by local current conservation giving rise to Ward-type identities. A modification of the lepton propagator must be accompanied by a change of the vertex function, and Kroll showed that these almost completely cancel each other out, the only exception being the propagators involved in closed Fermion loops. In the case of the anomaly, modifications of the lepton propagator can first show up in fourth order and may be of the form

\[ \Delta a \approx K(a/\pi)^2 (m/\Lambda)^2 \]

where \( K \) is a numerical constant and \( \Lambda \) is a cut-off characterizing the modification. For \( K = \pm 1 \) we obtain

\[ \Lambda_e^+ \gtrsim 11 \text{ MeV} \quad \Lambda_e^- \gtrsim 14 \text{ MeV} \]

\[ \Lambda_\mu^+ \gtrsim 260 \text{ MeV} \quad \Lambda_\mu^- \gtrsim 425 \text{ MeV} \]

The anomalous magnetic moment is therefore not particularly sensitive to modifications of the charged lepton propagator.

Vertex modifications, not due to propagator modifications, will give rise (besides the obvious possibility of an intrinsic anomaly) to second order effects of the form

\[ \Delta a = K \frac{\pi}{\alpha} \frac{m^2}{\Lambda^2} \]

For \( K = \pm 1 \) we obtain**

**Notice however, that the effect of a modification of the electron propagator on the muon anomaly (via the diagram in fig. 6.2) has not been investigated. One might expect an enhancement for \( \Lambda_e < m_\mu \) such that the muon experiment could limit \( \Lambda_e \) better than the electron experiment.

**A recently [713] suggested modification of the charge form factor of the muon to explain slight deviations from \( \mu-e \) universality

\[ F_\mu(q^2) = 1 - b + \frac{\alpha}{1-q^2/\Lambda_\mu^2} \]

with \( b > 0.04 \) leads to

\[ \Delta a_\mu = -\frac{\alpha^2}{\pi} \frac{m_\mu^3}{\Lambda_\mu^4} \]

and thereby

\[ \Lambda_\mu > 1.4 \Lambda_\text{NU} \]
\[ \lambda^+ \gtrsim 230 \text{ MeV} \quad \lambda^- \gtrsim 280 \text{ MeV} \]
\[ \lambda^\mu \gtrsim 5.4 \text{ GeV} \quad \lambda^- \gtrsim 8.8 \text{ GeV} \]

Every modification of the photon propagator corresponds to a modification of the spectral function of the photon and the influence on the anomaly is most easily expressed by the formula (see section 5.2)

\[ \Delta \alpha = \frac{\alpha}{\pi} \int_0^\infty \frac{dt}{t} \mspace{1mu} \text{Im} \Delta \Pi(t) \mspace{1mu} K(t) \]

where

\[ K(t) = \int_0^1 dx \mspace{1mu} \frac{x^2(1-x)}{x^3 + (1-x)t/m^2} \approx \frac{1}{3} \frac{m^2}{t} \mspace{1mu} \text{for} \mspace{1mu} t \gg m^2 \]

Since most modifications involve large \( t \) values we can use the asymptotic form and write

\[ \Delta \alpha = \frac{1}{3} \frac{\alpha}{\pi} \left( \frac{m}{\lambda^2} \right)^2 L \]

where \( L/\Lambda^2 \) is defined by

\[ \frac{L}{\Lambda^2} = \int_0^\infty \frac{dt}{t^2} \mspace{1mu} \text{Im} \Delta \Pi(t) \]

For \( L = \pm 1 \) we obtain

\[ \Lambda^+ \gtrsim 3.1 \text{ GeV} \quad \Lambda^- \gtrsim 5.1 \text{ GeV} \]

Observe that photon propagator modifications obey \( \Delta \alpha \approx (m_e/m_\mu)^2 \mspace{1mu} \Delta \alpha \mspace{1mu} \eta \mspace{1mu} \text{and therefore they cannot influence significantly the electron anomaly} \]

If the space of quantum mechanical states has a positive definite metric \( \text{Im} \Delta \Pi(t) \) (and thereby \( L \) must itself be positive definite. The traditional photon propagator modification

\[ \frac{g_{\mu\nu}}{k^2} \rightarrow \frac{g_{\mu\nu}}{k^2} - \frac{g_{\mu\nu}}{k^2 - \Lambda^2} \]

does not satisfy this requirement (it leads to \( L = -1 \)). Lee and Wick [805–809], however, have recently proposed a theory to handle the problems associated with an indefinite metric.

### 7.3 Suggested couplings of leptons to exotic particles

(i) The most often occurring case is that of a neutral boson coupled to the charged leptons [703–707]. If the mass of the external lepton is \( m \), the mass of the internal lepton \( M \) (which may or may not be equal to \( m \)), the mass of the boson is \( \Lambda \), and its coupling is \( t \) (see fig. 7.1) we have for the case of scalar, pseudoscalar, vector and pseudovector coupling the following general expression.

*See e.g. Feinberg and Lederman [R 7]
Fig. 7.1. Neutral boson exchange.

\[
\Delta \alpha = \frac{f^2}{4\pi^2} \frac{m^2}{\Lambda^2} L,
\]

\[
L = \int_0^1 dx \frac{Q(x)}{(1-x)(1-(m/\Lambda)^2 x) + (M/\Lambda)^2 x},
\]

where \(Q(x)\) is a polynomial in \(x\) dependent on the type of coupling. We list it for the four standard cases.

1) scalar \(Q_s(x) = \frac{1}{2} x^2 (1+\epsilon-x)\)

2) pseudoscalar \(Q_{ps}(x) = \frac{1}{2} x^2 (1-\epsilon-x)\)

3) vector \(Q_v(x) = x(1-\lambda)(x-2(1-\epsilon)) + \frac{1}{2} x^2 (1+\epsilon-x) \lambda^2 (1-\epsilon)^2\)

4) pseudovector \(Q_{pv}(x) = x(1-\tau)(x-2(1+\epsilon)) + \frac{1}{2} x^2 (1-\epsilon-x) \lambda^2 (1+\epsilon)^2\)

where \(\epsilon = M/m\) and \(\lambda = m/\Lambda\). In the limit of a heavy boson, i.e., \(m, M \ll \Lambda\) we have in the four cases

\[
L_s = \frac{M}{m} \left( \log \frac{\Lambda}{M} - \frac{3}{4} \right) + \frac{1}{6}
\]

\[
L_{ps} = -\frac{M}{m} \left( \log \frac{\Lambda}{M} - \frac{3}{4} \right) + \frac{1}{6}
\]

\[
L_v = \frac{M}{m} - \frac{2}{3}
\]

\[
L_{pv} = -\frac{M}{m} - \frac{2}{3}
\]

These expressions are valid for all \(m, M \ll \Lambda\).* We specialize them to the following cases:

*Also assuming \((M/\Lambda)^3 \ll m/M\)
a) If a neutral boson exists coupled to only one charged lepton [703 - 706] we have $M = m = m_e$ or $m_e$ and

$$ L_s = \log \frac{\Lambda}{m} - \frac{7}{12} $$

$$ L_{ps} = -\log \frac{\Lambda}{m} + \frac{11}{12} $$

$$ L_{\nu} = \frac{1}{3} $$

$$ L_{p\nu} = -\frac{5}{3} $$

As the logarithmic is slowly varying, we can conclude that if the coupling $f$ is the same to muon and electron, the electron anomaly is free of the influence of such bosons (see section 7 1). No absolute limit can, however, be put upon the mass because of the lack of knowledge of the coupling constants.

For a neutral vector boson [708], $W^*$, one finds by combining the $\nu$ and $p\nu$ case (the cross terms do not contribute)

$$ \Delta \alpha_W = -\frac{4}{3} \frac{m^2}{4\pi^2 A^2} $$

If one writes $f^2/\Lambda^2 = KG_t / \sqrt{2}$ where $K$ is a numerical constant one obtains

$$ \Delta \alpha_W = 3 \times 10^{-9} K $$

and one finds the rather uninteresting limit

$$ K \lesssim 10 $$

b) If a neutral leptonic boson coupled to $e$ and $\mu$ exists [707]

$$ e^- \mu^+ \to Z^0 $$

then we have for electron and muon respectively the dominant terms

$$ L_s^e \approx \frac{1}{6} $$

$$ L_s^\mu \approx \frac{m_\mu}{m_e} \left( \log \frac{\Lambda}{m_\mu} - \frac{3}{4} \right) $$

$$ L_{ps}^e \approx \frac{1}{6} $$

$$ L_{ps}^\mu \approx \frac{m_\mu}{m_e} \left( \log \frac{\Lambda}{m_\mu} - \frac{3}{4} \right) $$

$$ L_{\nu}^e \approx \frac{2}{3} $$

$$ L_{\nu}^\mu \approx \frac{m_\mu}{m_e} $$

$$ L_{p\nu}^e \approx -\frac{2}{3} $$

$$ L_{p\nu}^\mu \approx \frac{m_\mu}{m_e} $$
Observe that for the electron we have a strong enhancement. The induced limit on $\Delta \alpha_e$ does not obey the inequality stated in section 7.1 but rather the following less restrictive inequalities:

\[
\begin{align*}
-9.6 \times 10^{-9} F \\ -26 \times 10^{-9} F \\ -6.4 \times 10^{-9} \\ -2.4 \times 10^{-9}
\end{align*}
\]

\[
\lesssim \Delta \alpha_e \lesssim \begin{cases} 
26 \times 10^{-9} F & s \\
9.6 \times 10^{-9} F & ps \\
2.4 \times 10^{-9} & \nu \\
6.4 \times 10^{-9} & pv
\end{cases}
\]

where $F = \log \frac{\Lambda}{m_\mu} - \frac{3}{4}$.

c) Neutral $\mu$-p resonance. If a neutral lepto-baryonic resonance of non-zero spin exists with a mass of about 1.9 GeV [710], the contribution can be estimated (disregarding strong interactions) from the general formulas given above by putting $m = m_\mu$, $M = m_p$. Assuming spin 1 and a coupling [711] $f(1+\lambda \gamma_5)\gamma_\mu$, we obtain in the not quite justified approximation $\Lambda >> 1$ GeV

\[
\Delta \alpha_\mu = \frac{f^2}{4\pi^2} \frac{m_\mu^2}{\Lambda^3} \left( (1-\lambda^2) \frac{m_p}{m_\mu} - (1+\lambda^2) \frac{2}{3} \right).
\]

In the experimentally favored case $\lambda = 1$ we obtain

\[
\frac{f^2}{4\pi} \lesssim 3 \times 10^{-4}
\]

which is far above the estimate of a lower limit ($10^{-13}$) obtained from the neutrino experiment [711].

(ii) Neutral leptonic resonance. If a neutral $\pi\mu(\sim 430$ MeV) resonance $\nu_\mu^*$ exists [712], and the spin is $1/2$, the dominating contribution in the not quite justified limit $m_\mu, m_\pi << m_{\nu*}$ is

\[
\Delta \alpha_\mu = \frac{f^2}{4\pi^2} \frac{m_\mu}{m_{\nu^*}} \left( \frac{1}{4} (1-\lambda^2) - \frac{1}{6} m_\mu \frac{1+\lambda^2}{m_{\nu^*}} \right)
\]

for a coupling of the form $f(\gamma_5 + \lambda)$. For $\lambda = 0$ we obtain the limit on the coupling constant*

\[
f^2/4\pi \lesssim 6 \times 10^{-5}.
\]

(iii) Magnetic monopoles. If magnetic monopoles exist, one may expect they contribute to the anomaly of the leptons. However, because of their huge coupling, no trustworthy method exist for calculating their contribution. Perturbation theory is not valid, unless the effective coupling turns out to be $g m/M$ where $g$ is the monopole coupling, $m$ the lepton mass, and $M$ the monopole mass. From general considerations, this seems to be the case if the contribution arises via the vacuum polarization. Taking**

\[
L = \frac{g^2}{4\pi^2} \frac{1}{15}
\]

*The lower limit is $1.6 \times 10^{-11}$. Private communication from A. De Rújula.

**This is the value obtained for the vacuum polarization by heavy charged fermion pairs [407].
we obtain (with $M$ the monopole mass)

$$\Delta \alpha \sim \frac{1}{45} \frac{\alpha^2 g^2 (m)}{\pi M} \quad (M)$$

which at best is unreliable. The lower bound on $M$ is 5.4 GeV when $eg/4\pi \approx 1$. No serious significance should however be attributed to this number.

*part C*

High energy experiments

3. Scattering experiments at high momentum transfer

One of the aims of these experiments is to test the validity of QED at small interaction distances. High energy scattering experiments with very large momentum transfer $q$, say $q \geq 2$ GeV, can probe interaction distances $R \sim h/q \leq 10^{-14}$ cm. To obtain such large momentum transfers one has resorted to the colliding beams type of experiment and to the so-called Bethe-Heitler type of experiment.

8.1 Colliding beams

Four types of experiments have been carried out:

(i) Möller scattering $e^- e^- \rightarrow e^- e^-$, see fig. 8.1.

(ii) Bhabha scattering $e^+ e^- \rightarrow e^+ e^-$, see fig. 8.2.

(iii) Annihilation into muon pairs $e^+ e^- \rightarrow \mu^+ \mu^-$, see fig. 8.3.

(iv) Annihilation into photon pairs $ee^- \rightarrow \gamma \gamma$, see fig. 8.4.

(i) Möller scattering was done at the Princeton-Stanford storage ring at total energy of the colliding electrons $E_{CM} = \sqrt{s} = 600$ MeV [801] and 1100 MeV [802]. The experiment measured the angular dependence of the cross section but not the absolute magnitude. The observed angular distribution was compared with the Möller formula [803] modified by radiative corrections [804].

A convenient way to make this comparison is to assume a modification factor for the photon propagator of the form

$$\frac{1}{q^2} \rightarrow \left( \frac{1}{q^2} \right)^\Lambda^2$$

The maximum-likelihood value of $\Lambda^2$ from the 1100 MeV data is

$$1/\Lambda^2 = -0.06 \pm 0.06 \text{ (GeV}^{-2})$$

*The modification with a minus sign has received much attention lately in connection with the problem of divergence difficulties in Physics. In a series of papers, Lee and Weik have reexamined the classical difficulties inherent to such a modification and have presented a new theory of quantum electrodynamics [805 809].*
Fig 8.1. Lowest order Feynman diagrams contributing to Møller scattering.

Fig 8.2. Lowest order Feynman diagrams contributing to Bhabha scattering.

Fig 8.3. The lowest order annihilation graph for $e^+ e^- \rightarrow \mu^+ \mu^-$.

Fig 8.4. The annihilation graphs corresponding to $e^+ e^- \rightarrow 2\gamma$. 
The corresponding limits of \( \Lambda \) are

\[ \Lambda_- > 4 \text{ GeV and } \Lambda_+ > 2.4 \text{ GeV} \]

at a 95\% confidence level.

(ii) Large angle Bhabha scattering has been performed at the Orsay storage rings* at a total energy \( \sqrt{\mathcal{E}} = 1020 \text{ MeV} \). In this experiment the absolute Bhabha cross section \( \sigma_{e^+e^-} \) was measured ** the determination of \( \sigma_{e^+e^-} \) implies a simultaneous measurement of the number of Bhabha scattering events \( N_{e^+e^-} \) and the luminosity \( L \) of the storage ring, since

\[ N_{e^+e^-} = L \sigma_{e^+e^-} \]

In order to determine the luminosity \( L \), the double bremsstrahlung reaction \( e^+e^- \rightarrow e^+e^- + 2\gamma \) of nonvanishing cross-section \( \sigma_{2\gamma} \) [813–816] was chosen. In the Orsay experiment, the two photons of double bremsstrahlung are emitted at very small angle (of a few mrad). Therefore, a breakdown of QED at large momentum transfer cannot give an appreciable effect on \( \sigma_{2\gamma} \). The simultaneous measurement of the two reactions led to the determination of \( \sigma_{e^+e^-} \).

\[ \sigma_{e^+e^-} = \sigma_{2\gamma} \frac{N_{e^+e^-}}{N_{2\gamma}} \]

Experimentally, \( \sigma_{e^+e^-} \) is obtained from the corrected number of Bhabha events and the value of the luminosity integrated over data taking time,

\[ \sigma_{exp} = [1.97] \pm 0.09 \text{ (statistical) } \pm 0.10 \text{ (systematic) } \times 10^{-31} \text{ cm}^2 \]

the corresponding theoretical cross section [817] obtained from the calculation of the lowest order Feynman diagrams shown in fig. 8.2 and modified so as to take into account the radiative corrections*** and the integration over accepted solid angle and average over the energy spectrum of incident electrons is found to be

\[ \sigma_{th} = 2.13 \times 10^{-31} \text{ cm}^2. \]

When the comparison between theory and experiment is made by means of the cut-off parametrization indicated above, it is found that

\[ \Lambda_- > 2.0 \text{ GeV and } \Lambda_+ > 3.8 \text{ GeV} \]

at a 95\% confidence level.

Recently, electron-positron elastic scattering has also been performed at the Frascati storage ring Adone †. The total energy \( \sqrt{\mathcal{E}} \) ranges from 1.4 to 2.4 GeV. The published results [842] are

* For a technical description of the Orsay storage rings, see e.g. J. L. Augustin [810]
** For a detailed description of this experiment, see ref. [811]. The results were first reported at the Liverpool Conference, 1981 [812]
*** The radiative corrections to \( e^+e^- \to e^+e^- \) can be obtained from the calculation of Tsaï [804]. For \( e^+e^- \to e^+e^- \), the appropriate transcription, relevant to the Orsay experiment, has been made by Tavernlet [818]. They lead to a decrease of the Born cross-section of 7.3\%. Vacuum polarization corrections are negligible (< 0.1\%). The radiative corrections to \( e^+e^- + 2\gamma \) have been recently calculated by Baser (private communication from Dr. I. Buon). They are found to be small and do not alter the conclusions of the Orsay experiment. We wish to thank Dr. Buon for an informative discussion on this point.
† For a technical description of Adone, see e.g. F. Amman et al. [841]
Table 8.1

<table>
<thead>
<tr>
<th>Experiment</th>
<th>Tests</th>
<th>Reference</th>
<th>Total c.m. energy (MeV)</th>
<th>Cut-off (95% c.l.) (GeV)</th>
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<tbody>
<tr>
<td>e⁺e⁻ → e⁺e⁻</td>
<td>space-like photon</td>
<td>[801 and 802]</td>
<td>600 and 1100</td>
<td>( \Lambda_\gamma &gt; 4, \Lambda_\mu &gt; 2.4 )</td>
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<tr>
<td>e⁺e⁻ → e⁺e⁻</td>
<td>space-like photon</td>
<td>[811 and 812]</td>
<td>1020</td>
<td>( \Lambda_\gamma &gt; 2.0, \Lambda_\mu &gt; 3.8 )</td>
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<tr>
<td>e⁺e⁻ → ( \mu^+\mu^- )</td>
<td>time-like photon(^a)</td>
<td>[842]</td>
<td>1400–2400</td>
<td>( \Lambda_\gamma &gt; 6 )</td>
</tr>
<tr>
<td>e⁺e⁻ → 2( \gamma )</td>
<td>space-like photon(^b)</td>
<td>[812]</td>
<td>580, 644 and 704</td>
<td>( \Lambda_\gamma &gt; 1.3 )</td>
</tr>
</tbody>
</table>

\(^a\)See also the results of ref [843].

\(^b\)See also the results of ref [844].

based on an analysis of 3255 wide angle scattering events (WAS). At these energies, the WAS events are suitable to test the validity of QED.

In the Frascati experiment, the reaction chosen as a monitor to determine the luminosity was Bhabha scattering at small angle scattering (SAS). The SAS events involve small momentum transfers and therefore they are unaffected by a possible breakdown of QED at large momentum transfer. The comparison between theory and experiment made by means of the minus sign cut-off parametrization (see table 8.1) leads to the result

\[ \Lambda_\gamma > 6 \text{ GeV} \]

at a 95% confidence level.

(iii) The annihilation into a muon pair \( e⁺e⁻ → \mu^+\mu^- \) has also been performed at Orsay\(^*\) at the three total energies 580 MeV, 664 MeV and 704 MeV. The muons were separated from the pions on the basis of their different range in the thick plate chambers. They obtained 62 events. This experiment tests the time-like propagator of the photon since only the annihilation graph shown in fig. 8.3 is present. From the analysis of the results in terms of a modification of the photon propagator it is found that

\[ \Lambda_\gamma > 1.3 \text{ GeV} \]

at a 95% confidence level.

Production of muon pairs has also been recently performed at Frascati [843]. The results are presented in terms of the ratio of cross-sections

\[
\frac{R}{(e⁺e⁻ → e⁺e⁻)} = \frac{(e⁺e⁻ → \mu^+\mu^-)}{(e⁺e⁻ → c⁺e⁻)}_{\exp} \frac{(e⁺e⁻ → \mu^+\mu^-)}{(e⁺e⁻ → e⁺e⁻)}_{\text{QED}}
\]

as a function of energy. Straight-line interpolation of all the points gives:

\[
R = 0.98 \pm 0.08; \quad 2.5 \leq s \leq 4.0 \text{ GeV}^2
\]

\(^*\)See Perez-y-Jorba [812].
8.5 Lowest order Feynman diagrams contributing to lepton pair photoproduction. In fig. 8.5a there are two diagrams obtained by attaching the exchanged photon to each lepton line.

(iv) The annihilation into two photons $e^+e^- \rightarrow 2\gamma$ has been studied at the Novosibirsk storage rings [819]. The relative deviation of the experimental cross section for $2\gamma$ annihilation from the calculated one is

$$\Delta \sigma/\sigma = 0.1 \pm 0.2$$

Notice that this experiment tests the electron propagator (see fig. 8.4). It is the electron which in this process is off-mass shell and carries a large four-momentum. When a cut-off for the fermion propagator is introduced in the same way as the $(-)$ photon propagator modification, and no corresponding vertex correction*

$$\frac{1}{p^2 - m^2} \rightarrow \frac{1}{p^2 - m^2 - \Lambda^2},$$

it is found that

$$A(\text{electron}) > 1.5 \text{ GeV}$$

at a 95% confidence level.

The reaction $e^+e^- \rightarrow \gamma\gamma$ has also been recently studied at Frascati [844]. The experiment goes up to 1.6 GeV virtual-lepton 4-momentum. A total of 597 events were analyzed. The QED test consists in comparing the experimental ratio of wide-angle to small-angle rates with the theoretical prediction. The two agree within one standard deviation.

A summary of the results obtained from colliding beams experiments has been compiled in table 8.1.

8.2. Bethe-Hettler type experiments

Three types of experiments have been performed:

(i) Wide angle electron or muon pair photoproduction (see fig. 8.5).

(ii) Wide angle bremsstrahlung of electrons or muons (see fig. 8.6).

(iii) Trident production of leptons (see fig. 8.7).

These processes have in common that, at lowest order of perturbation theory, the corresponding Feynman diagrams involve a fermion line which is off-shell and possible modifications to the corresponding propagator can be tested. Here, large off-shell fermion masses are attainable because the proton target, or the low Z nucleus target, can be used to fix the center of mass. The nuclear stru

*See the discussion in section 7.2 concerning the difficulties with this type of modification of a fermion propagator.
Table 8.2

Limits on the breakdown of QED from measurements of symmetric wide angle lepton pair photoproduction. Parametrization

\[ \sigma = \sigma_{\text{BH}} \left[ 1 \pm \left( Q^2 / \Lambda^2 \right)^2 \right] \]

where \( Q \) is the invariant mass of the final lepton pair. For each experiment, the signature retained is the one which yields the lowest cut-off.

<table>
<thead>
<tr>
<th>Experiment</th>
<th>Reference</th>
<th>Invariant mass of lepton pair (MeV/c^2)</th>
<th>Cut-off (95% c.l.) (GeV/c^2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( r + C \rightarrow C + e^+e^- )</td>
<td>[821]</td>
<td>( Q \leq 900 )</td>
<td>( \Lambda_c &gt; 1.6 )</td>
</tr>
<tr>
<td>( r + C \rightarrow C + e^+e^- )</td>
<td>[822]</td>
<td>( Q \leq 444 )</td>
<td>( \Lambda_c &gt; 0.8 )</td>
</tr>
<tr>
<td>( r + p + p + e^+e^- )</td>
<td>[823]</td>
<td>( Q \leq 490 )</td>
<td>( \Lambda_c &gt; 0.7 )</td>
</tr>
<tr>
<td>( r + C \rightarrow C + \mu^+\mu^- )</td>
<td>[827]</td>
<td>( Q \leq 1225 )</td>
<td>( \Lambda_c &gt; 1.5 )</td>
</tr>
<tr>
<td>( r + C \rightarrow C + \mu^+\mu^- )</td>
<td>[828]</td>
<td>( Q \leq 2100 )</td>
<td>( \Lambda_c &gt; 2.3 )</td>
</tr>
<tr>
<td>( r + C \rightarrow C + \mu^+\mu^- )</td>
<td>[829]</td>
<td>( Q \leq 1770 )</td>
<td>( \Lambda_c &gt; 1.9 )</td>
</tr>
</tbody>
</table>

*For earlier experiments on electron pair photoproduction, see refs [824, 825]. For an earlier experiment on muon pair photoproduction, see also ref [826].

\[ \text{Fig. 8 a) Lowest order Feynman diagrams contributing to bremsstrahlung of leptons. The off-shell fermion in the diagram of fig 8.6a is time-like, while in the diagram of fig 8.6b it is space-like.} \]

\[ \text{Fig. 8 b) Lowest order Feynman diagrams contributing to trident production.} \]

\[ \text{Fig. 8 c) Lowest order Feynman diagrams contributing to trident production.} \]

\[ \text{In the diagrams corresponding to figs. 8.5a, 8.6a and 8.7a can be factored out, and is known from measurements of the form factors in elastic electron-proton scattering.} \]

(1) Measurements of symmetric wide angle electron pair photoproduction have been done on carbon at DESY-M11 [821] and CEA [822], and on hydrogen at Daresbury [823]. In these experiments a symmetric detection system with respect to the incident photon beam direction is

*For a recent review of the nucleon form factors, see e.g. Rutherglen [820].
chosen so as to eliminate the interference of the Bethe-Heitler amplitudes (see fig 8.5a) with the virtual Compton amplitude (see fig 8.5b). Wide angle detection is used to minimize the effect of the virtual Compton-scattering cross-section. In these experiments, the invariant mass of the lepton pair $m_{e^+e^-}$ goes up to 900 MeV/c$^2$.

Measurements of symmetric wide angle muon pair photoproduction on carbon have also been reported [826-828]. The data corresponds to invariant masses of the muon pair $m_{\mu^+\mu^-}$ up to 770 MeV/c$^2$.

A traditional way to represent a breakdown of the lepton propagator has been to introduce a cut-off parameter $\Lambda$ such as illustrated in table 8.1. Then, for values of $p^2$ small compared with $\Lambda^2$ and large compared with $m^2$, the Bethe-Heitler cross-section $\sigma_{BH}$ [830, 831] has to be modified as follows [832]

$$\sigma = \sigma_{BH}(1+p^2/\Lambda^2)$$

Although there is certain arbitrariness in parametrizing the breakdown of QED the modification suggested above has the serious inconvenience of violating the Ward-Takahashi identity. An analysis by Kroll [701] shows that a reasonable modification of the Bethe-Heitler type amplitude leads to a modification of the cross-section of the type

$$\sigma = \sigma_{BH} \left[ 1 \pm (Q_m^2/\Lambda^2)^n \right]$$

where $n = 2$ or a higher even number, and $Q_m$ is the invariant mass of the final lepton system. Using this parametrization, with $n = 2$, the experiments mentioned above lead to the results shown in table 8.2.

(ii) Experiments of wide angle bremsstrahlung of electrons on carbon have been done at Cornell [833], and on a hydrogen target at Frascati [834]. An experiment of bremsstrahlung of muons from 9 to 13 GeV/c on a carbon target has also been done by a Harvard-Case-McGill-SLAC collaboration [835]. Notice that in this type of processes the lowest order Feynman diagrams involve

---

Table 8.3

<table>
<thead>
<tr>
<th>Experiment</th>
<th>Reference</th>
<th>Invariant mass of lepton $\gamma$ system (MeV/c$^2$)</th>
<th>Cut-off (95% c 1) (GeV/c$^2$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$e^+e^- + C \rightarrow e^+ + e^- + C + \gamma$</td>
<td>[833]</td>
<td>$Q &gt; 1630$</td>
<td>$\Lambda &gt; 1.5$</td>
</tr>
<tr>
<td>$\mu^+\mu^- + C \rightarrow \mu^+ + \mu^- + C + \gamma$</td>
<td>[835]</td>
<td>$Q &gt; 650$</td>
<td>$\Lambda &gt; 0.73$</td>
</tr>
</tbody>
</table>

See also ref. [834]
off-shell leptons which are both space-like (fig. 8.6b) and time-like (fig. 8.6a). The time-like lepton can be far off the mass shell if the final lepton and photon energies and angles are large. The results of these experiments, which have been summarized in table 8.3 agree well with the predictions of QED.*

(iii) Experiments on trident production (see fig. 8.7) are still on a preliminary stage to serve as precision tests of QED. There are, however, a number of very interesting features which have already been revealed by these experiments. Production of a muon pair by incident electrons at 1.9 GeV/c on a carbon target has been reported by a Northeastern-Austin collaboration [837]; muon pairs were observed at invariant masses ranging from 0.4 GeV/c² to 0.9 GeV/c²; the scattered electron was not observed. The data are consistent with predictions of a simple diffraction model for the virtual Compton amplitude (see fig. 8.7b) which interferes with the time-like QED amplitude (see fig. 8.7a). Within this model, a heavy photon of mass less than 400 MeV in the time-like propagator is excluded by the data.

Production of muon pairs by incident muons at 11 GeV/c on a lead target have been performed at Brookhaven by a Harvard-U Mass.-McGill collaboration [838]. The angles and momenta of the incident muon and of all three final state muons were measured in optical spark chambers. Runs were made with positive and negative incident muons. The total number of events observed 75 ± 10.5 is in good agreement with the theoretical prediction 88 ± 2.6 which includes the interference term between the direct and exchange graphs.** The theoretical prediction without inclusion of the interference term would be 120.9 ± 2.6 events. In the experiment, the invariant mass spectrum of the two identical final particles clearly shows a depression at low invariant masses as predicted by the Pauli exclusion principle for fermions.

Conclusions

Quantum electrodynamics is in very good shape. No serious discrepancies exist between theory and experiment. The last year has seen improvements in several fields. In the high energy region cut-off masses are being pushed into the region of several GeV. In the low energy region the precision on both experiment and theory is approaching one part per million.

The anomalous magnetic moment of the electron has been remeasured with a precision of 3 ppm. Simultaneously, the sixth order radiative correction has been calculated theoretically with the same precision. The two numbers agree beautifully.

By now, the theoretical value of the anomalous magnetic moment of the muon is known with a precision of 13 ppm. The crucial test of this number awaits the next CERN experiment.

The theoretical value for Lamb shift in hydrogen has been improved by analytic calculations of the fourth order slope of the Dirac form factor of the electron, and by a calculations of the $(Z/\alpha)^6$ contribution. The theoretical uncertainty of 12 ppm is five times smaller than the experimental uncertainty, and the agreement between theory and experiment is excellent.

The increased precision of QED calculations and experiment leads to accurate purely radiative values of the fine structure constant $\alpha$. In one case (the $2P_{3/2} - 2S_{1/2}$ splitting in hydrogen) the precision is even comparable to that of the non-QED ac Josephson value.

*See ref. [830], see also ref. [839].

**Notice that in fig. 8.7a, in the case $\mu + A \rightarrow \mu + \mu^* + \mu^- + A$, there are altogether eight Feynman diagrams: the 4 direct ones plus 4 obtained by exchange in the final fermion lines of equal charge. Theoretical discussion of trident experiments can be found in refs. [839] and [840].
Acknowledgements

The authors wish to thank the following persons for discussions, comments and criticism:


Review articles on quantum electrodynamics

The following is a list of review articles on various aspects of quantum electrodynamics which we have found very useful to consult. No attempt at giving a complete list has been made. We apologize for omissions.


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B.E. Lautrup et al., Comparison between theory and experiments in quantum electrodynamics


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