Loop Equation and Some of Its Applications

by

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- **Lecture 1. Pedagogical Introduction and Historical Overview**

Schwinger–Dyson equations, Wilson loops, path and area derivatives, loop space, loop equations, $1/N$-expansion, loop-space Laplacian, UV regularization, iterative solution and perturbation theory, applications of LEs

- **Lecture 2. Matrix Models: complete realization of the program**

Hermitian one-matrix model, loop equation, solution in $1/N$, iterative procedure and moments, two-matrix model

- **Lecture 3. Cusped Loop Equation and Super Yang–Mills**

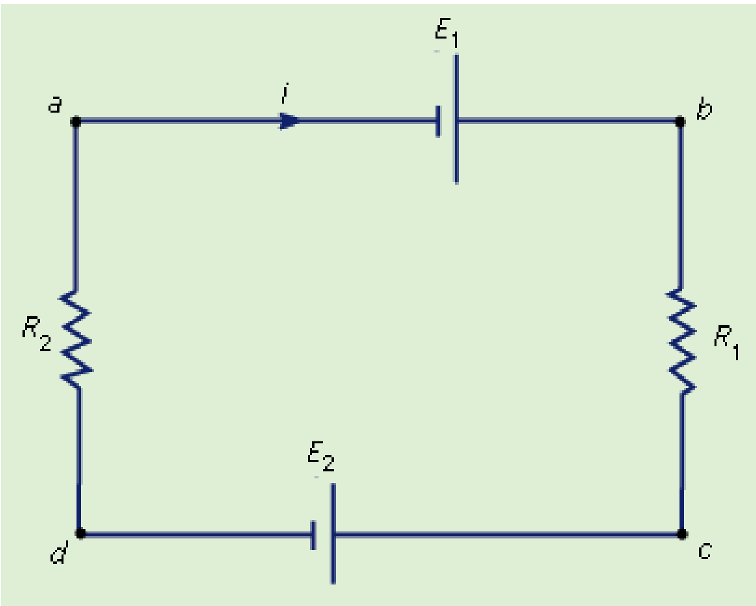
renormalization of Wilson loops, cusp anomalous dimension, relation to twist-two operators, SYM Wilson loops, explicit two loops and the anomaly terms, cusped LE, SUSY extension, anomalous dimension from LE

What is the loop equation?

Encyclopedia Britannica says:

Kirchhoffs circuit rules (in Kirchhoffs circuit rules (physics):

The second rule, the loop equation, states that around each loop in an electric circuit the sum of the emf s (electromotive forces, or voltages, of energy sources such as batteries and generators) is equal to the sum of the potential drops, or voltages across each of the resistances, in the same loop. All the energy imparted by the energy sources to the charged particles that carry the current...



Loop equation in QCD

Schwinger–Dyson equation for Wilson loops

$$\nabla^{ab}_{\mu} F^{b}_{\mu\nu}(x) \overset{\text{w.s.}}{=} \hbar \frac{\delta}{\delta A^{a}_{\nu}(x)}$$

can be translated as $N \to \infty$ to the loop equation

Migdal, Yu.M. (1979)

$$\partial^{x}_{\mu} \frac{\delta}{\delta \sigma_{\mu\nu}(x)} W(C) = \lambda \oint_{C} \mathrm{d}y_{\nu}\, \delta^{(d)}(x-y)\, W(C_{yx})\, W(C_{xy})$$

which includes path and area derivatives

Applications of the loop equation

- QCD string is not Nambu–Goto Migdal, Y.M. (1979)
- Area law is a self-consistent solution Migdal, Y.M. (1980)
- Complete solution in 2d Kazakov, Kostov (1980)
- Elfin string is a formal solution Migdal (1981)
- Stochastic quantization Parisi, Wu (1981)
- Reduced models Eguchi, Kawai (1982)
- 2d gravity Kazakov (1990)
- Matrix models Ambjorn, Chekhov, Kristjansen, Y.M. (1993)
- IIB Model Ishibashi, Kawai, Kitazawa, Tsuchia (1997)
- $\mathcal{N} = 4$ super Yang–Mills Drukker, Gross, Ooguri (1999)

Schwinger–Dyson equations

Schwinger–Dyson equations are a quantum analog of the classical equation of motion.

To derive the Schwinger–Dyson equations, let us utilize the fact that the measure $\mathcal{D}\varphi(x)$ in Euclidean averages

$$\langle\, F[\varphi]\,\rangle \;=\; Z^{-1}\int \mathcal{D}\varphi(x)\, \mathrm{e}^{-S[\varphi]}F[\varphi] \tag{1}$$

is invariant under an arbitrary shift of the field

$$\varphi(x) \;\rightarrow\; \varphi(x) + \delta\varphi(x)\,. \tag{2}$$

This invariance is obvious since the functional integration goes over all the fields, while the shift (2) is just a transformation from one field configuration to another.

Since the measure is invariant, the path integral in the average does not change

$$\int \mathrm{d}^d x\, \delta\varphi(x) \int \mathcal{D}\varphi\; \mathrm{e}^{-S[\varphi]} \left[-\frac{\delta S[\varphi]}{\delta\varphi(x)} F[\varphi] + \frac{\delta F[\varphi]}{\delta\varphi(x)} \right] \;=\; 0. \tag{3}$$

Since $\delta\varphi(x)$ is arbitrary, Eq. (3) results in the following quantum equation of motion

$$\frac{\delta S[\varphi]}{\delta\varphi(x)} \stackrel{\text{w.s.}}{=} \hbar\frac{\delta}{\delta\varphi(x)}, \tag{4}$$

where the dependence on Planck's constant $\hbar$ is explicit. It appears this way since the action S is divided by $\hbar$ in Eq. (1).

The symbol "w.s." is to emphasize that it is valid in the weak sense, i.e. under the averaging sign. The variation of the action can always be substituted by the variational derivative when integrated over fields. We arrive at the functional equation

$$\left\langle \frac{\delta S[\varphi]}{\delta\varphi(x)} F[\varphi] \right\rangle = \hbar \left\langle \frac{\delta F[\varphi]}{\delta\varphi(x)} \right\rangle . \tag{5}$$

It is quite similar to that which Schwinger considered.

Commutator terms

To show how Eq. (5) reproduces the free propagator, let us choose

$$F[\varphi] \;=\; \varphi(y)\,. \tag{6}$$

Substituting into Eq. (5) and calculating the variational derivative, one obtains

$$\left(-\partial^2 + m^2\right)\langle\varphi(x)\,\varphi(y)\rangle \;=\; \hbar\left\langle\frac{\delta\varphi(y)}{\delta\varphi(x)}\right\rangle \;=\; \hbar\,\delta^{(d)}(x-y)\,, \tag{7}$$

as it should.

The LHS emerges from the variation of the free classical action

$$\frac{\delta S_{\text{free}}}{\delta\varphi(x)} \;=\; \left(-\partial^2 + m^2\right)\varphi(x) \tag{8}$$

while the RHS, which results from the variational derivative, emerges in the operator formalism from the canonical commutation relations

$$\delta(x_0 - y_0)\,[\boldsymbol{\varphi}(x_0,\vec{x})\,,\dot{\boldsymbol{\varphi}}(y_0,\vec{y})] \;=\; \mathrm{i}\,\delta^{(d)}(x-y)\,. \tag{9}$$

The RHS of Eq. (7) (and Eq. (5)) is called the commutator term. The variational derivative plays the role of the conjugate momentum in the operator formalism. The calculation of this variational derivative in Euclidean space is equivalent to differentiating the T-product and using canonical commutation relations in Minkowski space.

When $\hbar \to 0$, the RHS of Eq. (4) vanishes and it reduces to the classical equation of motion

$$\frac{\delta S[\varphi]}{\delta \varphi(x)} = 0 \,. \tag{10}$$

This implies that the path integral over fields has a saddle point as $\hbar \to 0$ given by Eq. (10).

Euclidean averages are associated with Wick-rotated T-products. The Euclidean average is associated with the vacuum expectation value of $\langle 0 \,|\, \boldsymbol{T} F[\boldsymbol{\varphi}] \,|\, 0 \rangle$ in Minkowski space.

Wilson Loops

Non-Abelian phase factor

$$U(C) = \boldsymbol{P}\, \mathrm{e}^{\mathrm{i}g\int_C A_\mu(x)\mathrm{d}x^\mu} \stackrel{\mathrm{def}}{=} \prod_{x\in C} \left(1 + \mathrm{i}gA_\mu(x)\mathrm{d}x^\mu\right)$$

parallel transporter in non-Abelian Yang–Mills field

$\operatorname{tr} U(C)$ is gauge-invariant for closed C

Wilson loop v.e.v. (average in Euclidean formulation)

$$W(C) = Z^{-1} \int \mathcal{D}A_\mu \mathcal{D}\bar{\psi}\mathcal{D}\psi \cdots\ \mathrm{e}^{\mathrm{i}S} \frac{1}{N} \operatorname{tr} U(C)$$

Importance of the Wilson loops (large N):

- observables are expressed via sum-over-path of $W(C)$
- dynamics is entirely reformulated via $W(C)$

$W(C)$ obeys the loop equation (closed equation on loop space)

Typical loops essential in the sum-over-path are cusped

Schwinger–Dyson equations for Wilson loop

The dynamics of (quantum) Yang–Mills theory is described by the Schwinger–Dyson equation

$$-\nabla^{ab}_{\mu} F^{b}_{\mu\nu}(x) \overset{\text{w.s.}}{=} \hbar \frac{\delta}{\delta A^{a}_{\nu}(x)} \tag{11}$$

which is analogous to Eq. (4) for the scalar field, and is understood in the weak sense, i.e. for the averages

$$-\langle \nabla^{ab}_{\mu} F^{b}_{\mu\nu}(x)\, Q\,[A]\, \rangle = \hbar \left\langle \frac{\delta}{\delta A^{a}_{\nu}(x)} Q\,[A] \right\rangle . \tag{12}$$

We have not added the variations of gauge-fixing and ghost terms in the Yang–Mills action. They are mutually canceled for gauge-invariant functionals $Q[A]$ (like the Wilson loops).

It is convenient to use the matrix notation, when Eq. (12) takes the form (in units with $\hbar = 1$)

$$-\left\langle \frac{1}{N} \operatorname{tr} \boldsymbol{P}\, \nabla_{\mu} \mathcal{F}_{\mu\nu}(x)\, \mathrm{e}^{\mathrm{i} \oint_C \mathrm{d}\xi^{\mu} \mathcal{A}_{\mu}} \right\rangle = \left\langle \frac{g^2}{N} \operatorname{tr} \frac{\delta}{\delta \mathcal{A}_{\nu}(x)} \boldsymbol{P}\, \mathrm{e}^{\mathrm{i} \oint_C \mathrm{d}\xi^{\mu} \mathcal{A}_{\mu}} \right\rangle , \tag{13}$$

The variational derivative on the RHS can be calculated by

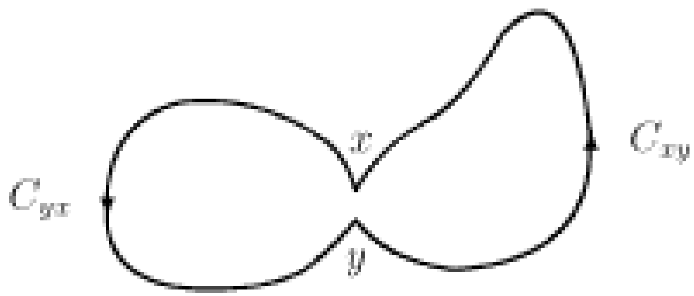
$$\frac{\delta \mathcal{A}^{ij}_\mu(y)}{\delta \mathcal{A}^{kl}_\nu(x)} = \delta_{\mu\nu}\,\delta^{(d)}(x-y)\left(\delta^{il}\delta^{kj} - \frac{1}{N}\delta^{ij}\delta^{kl}\right) \tag{14}$$

The second term in the parentheses in Eq. (14) is because $\mathcal{A}_\mu$ is a matrix from the adjoint representation of $SU(N)$.

By using Eq. (14), we obtain for the variational derivative on RHS of Eq. (13):

$$\mathrm{tr}\,\frac{\delta}{\delta \mathcal{A}_\nu(x)} \boldsymbol{P}\,\mathrm{e}^{\mathrm{i}\oint_C \mathrm{d}\xi^\mu \mathcal{A}_\mu} = \mathrm{i}\oint_C \mathrm{d}y_\nu\,\delta^{(d)}(x-y) \times \left[\frac{1}{N}\mathrm{tr}\,\boldsymbol{P}\,\mathrm{e}^{\mathrm{i}\int_{C_{yx}} \mathrm{d}\xi^\mu \mathcal{A}_\mu}\frac{1}{N}\mathrm{tr}\,\boldsymbol{P}\,\mathrm{e}^{\mathrm{i}\int_{C_{xy}} \mathrm{d}\xi^\mu \mathcal{A}_\mu} - \frac{1}{N^3}\mathrm{tr}\,\boldsymbol{P}\,\mathrm{e}^{\mathrm{i}\int_{C} \mathrm{d}\xi^\mu \mathcal{A}_\mu}\right]. \tag{15}$$

C_{yx} and C_{xy} are parts of C: from x to y and from y to x, respectively. They are closed owing to the delta-function $\Longrightarrow$ x and y should be the same points of *space* but not necessarily of the *contour*.



Finally, we rewrite Eq. (13) as

$$\begin{aligned} & \mathrm{i} \left\langle \frac{1}{N} \operatorname{tr} \boldsymbol{P} \, \nabla_\mu \mathcal{F}_{\mu\nu}(x) \, \mathrm{e}^{\mathrm{i} \oint_C \mathrm{d}\xi^\mu \mathcal{A}_\mu} \right\rangle \\ & \quad = \lambda \oint\limits_C \mathrm{d}y_\nu \, \delta^{(d)}(x-y) \left[\langle \, \Phi(C_{yx}) \, \Phi(C_{xy}) \, \rangle - \frac{1}{N^2} \langle \, \Phi(C) \, \rangle \right], \end{aligned} \tag{16}$$

where we have introduced the *'t Hooft coupling*

$$\lambda = g^2 N \,. \tag{17}$$

Note that the RHS of Eq. (16) is completely represented via the (closed) Wilson loops

$$\Phi(C) = \frac{1}{N} \operatorname{tr} \boldsymbol{P} \, \mathrm{e}^{\mathrm{i} \int_C \mathrm{d}\xi^\mu \mathcal{A}_\mu}$$

Loop space

The RHS of Eq. (16) is completely represented via the (closed) Wilson loops. It is crucial for the loop-space formulation of QCD that the LHS of can also be represented in loop space as some operator applied to the Wilson loop. To do this we need to develop a differential calculus in loop space.

Loop space consists of arbitrary continuous closed loops, C. They can be described in a parametric form by the functions $x_\mu(\sigma) \in L_2$, where L_2 denotes the Hilbert space of functions $x_\mu(\sigma)$, the square of which is integrable over the Lebesgue measure: $\int_{\sigma_0}^{\sigma_1} \mathrm{d}\sigma\, x_\mu^2(\sigma) < \infty$. The functions $x_\mu(\sigma)$ can be discontinuous, generally speaking, for an arbitrary choice of the parameter σ. The continuity of the loop C implies a continuous dependence on parameters of the type of proper length

$$s(\sigma) \;=\; \int\limits_{\sigma_0}^{\sigma} \mathrm{d}\sigma' \sqrt{\dot{x}_\mu^2(\sigma')}\,, \tag{18}$$

where $\dot{x}_\mu(\sigma) = \mathrm{d}x_\mu(\sigma)/\mathrm{d}\sigma$.

The functions $x_\mu(\sigma) \in L_2$ which are associated with the elements of loop space obey the following restrictions.

(1) The points $\sigma = \sigma_0$ and $\sigma = \sigma_1$ are identified: $x_\mu(\sigma_0) = x_\mu(\sigma_1)$ – the loops are closed.

(2) The functions $x_\mu(\sigma)$ and $\Lambda_{\mu\nu} x_\nu(\sigma) + \alpha_\mu$, with $\Lambda_{\mu\nu}$ and α_μ independent of σ, represent the same element of the loop space – rotational and translational invariance.

(3) The functions $x_\mu(\sigma)$ and $x_\mu(\sigma')$ with $\sigma' = f(\sigma),\ f'(\sigma) \geq 0$ describe the same loop – reparametrization invariance.

An example of functionals defined on elements of loop space is the Wilson loop average.

The differential calculus in loop space is built out of the path and area derivatives.

Path and area derivatives

The standard variational derivative $\delta/\delta x_\mu(\sigma)$ is related to the path and area derivatives by the formula

$$\frac{\delta}{\delta x_\mu(\sigma)} \;=\; \dot{x}_\nu(\sigma)\frac{\delta}{\delta\sigma_{\mu\nu}(x(\sigma))} + \sum_{i=1}^{m} \partial_\mu^{x_i}\delta(\sigma-\sigma_i)\,, \tag{19}$$

where the sum on the RHS is present for the case of a functional having m marked (irregular) points $x_i \equiv x(\sigma_i)$. The simplest example of the functional with m marked points is just a function of m variables $x_1,\ldots,x_m$.

Using Eq. (19), the path derivative can be calculated as the limiting procedure

$$\partial_\mu^{x(\sigma)} \;=\; \int\limits_{\sigma-0}^{\sigma+0} \mathrm{d}\sigma' \frac{\delta}{\delta x_\mu(\sigma')}\,. \tag{20}$$

The result is obviously nonvanishing only when ∂_μ^x is applied to a functional with $x(\sigma)$ being a marked point.

It is nontrivial that the area derivative can also be expressed via the variational derivative

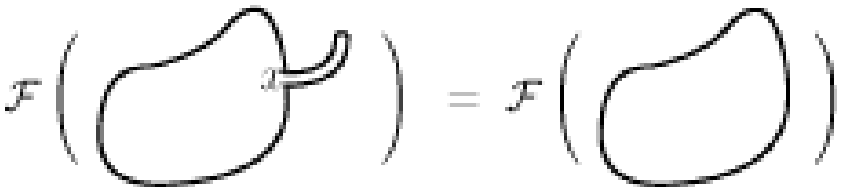
Polyakov (1980)

$$\frac{\delta}{\delta\sigma_{\mu\nu}(x(\sigma))} = \int_{\sigma-0}^{\sigma+0} \mathrm{d}\sigma'(\sigma'-\sigma)\frac{\delta}{\delta x_\mu(\sigma')}\frac{\delta}{\delta x_\nu(\sigma)}\,. \tag{21}$$

The point is that the six-component quantity, $\delta/\delta\sigma_{\mu\nu}(x(\sigma))$, is expressed via the four-component one, $\delta/\delta x_\mu(\sigma)$, which is possible because the components of $\delta/\delta\sigma_{\mu\nu}(x(\sigma))$ are dependent owing to the loop-space Bianchi identity.

Stokes functionals

The path and area derivatives are defined for Stokes functionals which satisfy the backtracking (zig–zag) condition – they do not change when a small path passing back and forth is added to the loop at some point x:



(22)

This condition is equivalent to the Bianchi identity of Yang–Mills theory and is obviously satisfied by the Wilson loop owing to the properties of the non-Abelian phase factor. Such functionals are known in mathematics as Chen integrals.

A simple example of the Stokes functional is the area of the minimal surface, $A_{\text{min}}(C)$. Otherwise, the length $L(C)$ of the loop C is not a Stokes functional, since the lengths of contours on the LHS and RHS of Eq. (22) are different.

If x is a regular point (such as any point of the contour for the Wilson loop), the path derivative vanishes owing to the backtracking condition. For the result to be nonvanishing, the point x should be a *marked* (or irregular) point. A simple example of the functional with a marked point x is

$$\Phi^a[C_{xx}] \equiv \frac{1}{N} \mathrm{tr}\,(t^a \boldsymbol{P}\, \mathrm{e}^{\mathrm{i}\int_{C_{xx}} \mathrm{d}\xi^\mu \mathcal{A}_\mu(\xi)}) \tag{23}$$

with the $SU(N)$ generator t^a inserted at the point x.

The area derivative of the Wilson loop is given by the Mandelstam formula

$$\frac{\delta}{\delta\sigma_{\mu\nu}(x)} \frac{1}{N} \mathrm{tr}\, \boldsymbol{P}\, \mathrm{e}^{\mathrm{i}\oint_C \mathrm{d}\xi^\mu \mathcal{A}_\mu} = \frac{\mathrm{i}}{N} \mathrm{tr}\, \boldsymbol{P}\, \mathcal{F}_{\mu\nu}(x)\, \mathrm{e}^{\mathrm{i}\oint_C \mathrm{d}\xi^\mu \mathcal{A}_\mu}. \tag{24}$$

In order to prove this, it is convenient to choose $\delta C_{\mu\nu}(x)$ to be a rectangle in the (μ,ν)-plane. The sense of Eq. (24) is very simple: $\mathcal{F}_{\mu\nu}$ is a curvature associated with the connection $\mathcal{A}_\mu$.

The functional on the RHS of Eq. (24) has a marked point x, and is of the same type as in Eq. (23). When the path derivative acts on such a functional, the result is given by

$$\partial^x_\mu \frac{1}{N} \mathrm{tr}\, \boldsymbol{P}\, B(x)\, \mathrm{e}^{\mathrm{i} \oint_C \mathrm{d}\xi^\mu \mathcal{A}_\mu} \;=\; \frac{1}{N} \mathrm{tr}\, \boldsymbol{P}\, \nabla_\mu B(x)\, \mathrm{e}^{\mathrm{i} \oint_C \mathrm{d}\xi^\mu \mathcal{A}_\mu}, \tag{25}$$

where

$$\nabla_\mu B \;=\; \partial_\mu B - \mathrm{i}\, [\mathcal{A}_\mu, B] \tag{26}$$

is the covariant derivative in the adjoint representation.

Combining Eqs. (24) and (25), we finally represent the expression on the LHS of Eq. (13) (or Eq. (16)) as

$$\frac{\mathrm{i}}{N} \mathrm{tr}\, \boldsymbol{P}\, \nabla_\mu \mathcal{F}_{\mu\nu}(x)\, \mathrm{e}^{\mathrm{i} \oint_C \mathrm{d}\xi^\mu \mathcal{A}_\mu} \;=\; \partial^x_\mu \frac{\delta}{\delta \sigma_{\mu\nu}(x)} \frac{1}{N} \mathrm{tr}\, \boldsymbol{P}\, \mathrm{e}^{\mathrm{i} \oint_C \mathrm{d}\xi^\mu \mathcal{A}_\mu}, \tag{27}$$

i.e. via the action of the path and area derivatives on the Wilson loop. It is therefore rewritten in loop space.

A summary of these results is presented in the Table as a vocabulary for translation of Yang–Mills theory from the language of ordinary space in the language of loop space.

Vocabulary for translation into loop space

Ordinary space		Loop space	
$\Phi[A]$	Phase factor	$\Phi(C)$	Loop functional
$F_{\mu\nu}(x)$	Field strength	$\dfrac{\delta}{\delta\sigma_{\mu\nu}(x)}$	Area derivative
∇^x_μ	Covariant derivative	∂^x_μ	Path derivative
$\nabla \wedge F = 0$	Bianchi identity		Stokes functionals
$-\nabla_\mu F_{\mu\nu}$ $= \delta/\delta A_\nu$	Schwinger–Dyson equations		Loop equations

Loop equations

By virtue of Eq. (27), Eq. (16) can be represented completely in loop space:

$$\partial_\mu^x \frac{\delta}{\delta \sigma_{\mu\nu}(x)} \langle \Phi(C) \rangle = \lambda \oint_C \mathrm{d}y_\nu \, \delta^{(d)}(x-y) \left\langle \left[\Phi(C_{yx}) \, \Phi(C_{xy}) - \frac{1}{N^2} \Phi(C) \right] \right\rangle , \tag{28}$$

or, using the definitions of the loop averages, as

$$\partial_\mu^x \frac{\delta}{\delta \sigma_{\mu\nu}(x)} W(C) = \lambda \oint_C \mathrm{d}y_\nu \, \delta^{(d)}(x-y) \left[W_2(C_{yx}, C_{xy}) - \frac{1}{N^2} W(C) \right] . \tag{29}$$

This equation is not closed. Having started from $W(C)$, we obtain another quantity, $W_2(C_1, C_2)$, so that Eq. (29) connects the one-loop average with a two-loop one. This is similar to the case of the (quantum) φ^3-theory, whose Schwinger–Dyson equations connect the n-point Green functions with different n. We shall derive this complete set of equations for the n-loop averages later.

However, the two-loop average factorizes in the large-N limit:

$$W_2(C_1, C_2) \;=\; W(C_1)\, W(C_2) + \mathcal{O}\Big(N^{-2}\Big)\,. \tag{30}$$

Keeping the 't Hooft constant λ fixed in the large-N limit, we obtain

Migdal, Y.M. (1979)

$$\partial^x_\mu \frac{\delta}{\delta\sigma_{\mu\nu}(x)} W(C) \;=\; \lambda \oint\limits_C \mathrm{d}y_\nu\, \delta^{(d)}(x-y)\, W(C_{yx})\, W(C_{xy}) \tag{31}$$

as $N \to \infty$.

Equation (31) is a closed equation for the Wilson loop average in the large-N limit. It is referred to as the *loop equation*.

To find $W(C)$, Eq. (31) should be solved in the class of Stokes functionals with the initial condition

$$W(0) \;=\; 1 \tag{32}$$

for loops which are shrunk to points. This is a consequence of the obvious property of the Wilson loop

$$\mathrm{e}^{\mathrm{i} \oint_0 \mathrm{d}\xi^\mu \mathcal{A}_\mu} \;=\; 1 \tag{33}$$

The factorization (30) can itself be derived from the chain of loop equations

$$
\begin{aligned}
&\frac{1}{\lambda}\partial_\mu^x \frac{\delta}{\delta\sigma_{\mu\nu}(x)} W_n(C_1,\ldots,C_n) \\
&\quad = \oint_{C_1} \mathrm{d}y_\nu \, \delta^{(d)}(x-y) \left[W_{n+1}(C_{xy},C_{yx},\ldots,C_n) - \frac{1}{N^2} W_n(C_1,\ldots,C_n) \right] \\
&\qquad + \sum_{j\geq 2} \frac{1}{N^2} \oint_{C_j} \mathrm{d}y_\nu \, \delta^{(d)}(x-y) \left[W_{n-1}\big(C_1C_j,\ldots,\underline{C_j},\ldots,C_n\big) \right. \\
&\qquad\qquad \left. - W_n(C_1,\ldots,C_n) \right] . \qquad (34)
\end{aligned}
$$

Here x belongs to C_1; C_1C_j denotes the joining of C_1 and C_j; $\underline{C_j}$ denotes that C_j is omitted.

Equation (34) looks like the Schwinger–Dyson equations for φ^3-theory. The number of colors N enters Eq. (34) simply as a scalar factor, N^{-2}, likewise Planck's constant $\hbar$ enters there. It is the major advantage of the use of loop space. A "semiclassical" nature of the $1/N$-expansion of QCD is realized explicitly in Eq. (34). Its expansion in $1/N$ is straightforward.

At $N = \infty$, Eq. (34) is simplified to

$$\partial_\mu^x \frac{\delta}{\delta\sigma_{\mu\nu}(x)} W_n(C_1, \ldots) = \lambda \oint_{C_1} \mathrm{d}y_\nu \, \delta^{(d)}(x-y) \, W_{n+1}(C_{yx}, C_{xy}, \ldots) \,. \tag{35}$$

This equation possesses Migdal (1980) a factorized solution

$$\begin{aligned} W_n(C_1, \ldots, C_n) &= \langle \Phi(C_1) \rangle \cdots \langle \Phi(C_n) \rangle + \mathcal{O}\Big(N^{-2}\Big) \\ &\equiv W(C_1) \cdots W(C_n) + \mathcal{O}\Big(N^{-2}\Big) \end{aligned} \tag{36}$$

provided $W(C)$ obeys Eq. (31) which plays the role of a "classical" equation in the large-N limit. Thus, we have given a nonperturbative proof of the large-N factorization of the Wilson loops.

Lattice loop equation

Foerster (1979)
Eguchi (1979)
Weingarten (1979)

The derivation is based on the shift of the link variable in the definition of the lattice Wilson loop average by an infinitesimal traceless Hermitian matrix $\epsilon_\mu(x)$:

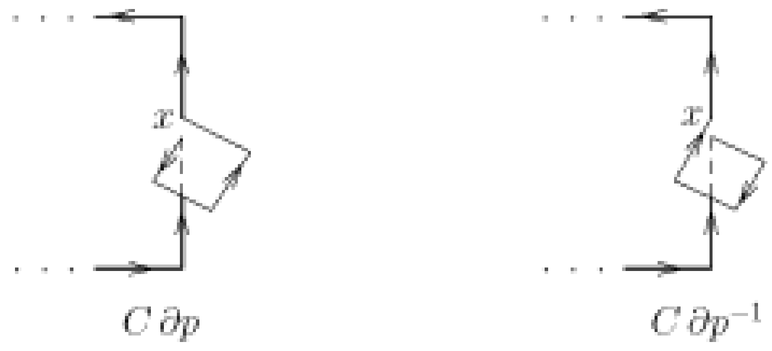
$$U_\mu(x) \to U_\mu(x)\,[1 - \mathrm{i}\epsilon_\mu(x)]\,, \qquad U_\mu^\dagger(x) \to [1 + \mathrm{i}\epsilon_\mu(x)]\,U_\mu^\dagger(x)\,, \quad (37)$$

Similarly to Eq. (31), we obtain

$$\frac{\beta}{2N^2}\sum_p \Big[W(C\,\partial p) - W\Big(C\,\partial p^{-1}\Big)\Big] \;=\; \sum_{l\in C} \delta_{xy}\tau_\nu(l)\,W(C_{yx})\,W(C_{xy})\,. \quad (38)$$

Here the contours $C\,\partial p$ and $C\,\partial p^{-1}$ are obtained from C_{xx} by adding the boundary of the plaquette p (∂p^{-1} denotes that the orientation of the boundary is opposite) and the sum over p goes over the $2(d-1)$ plaquettes involving the link at which the shift of $U_\nu(x)$ is performed.

Contours (a) $C\,\partial p$ and (b) $C\,\partial p^{-1}$ on the RHS of the lattice loop equation (38)



(a) (b)

The sum on the RHS goes over the links belonging to the contour C. The unit vector $\tau_\nu(l) = 0, \pm 1$ denotes the projection of the (oriented) link $l \in C$ on the axis ν ($\tau_\nu(l) =$ 1, -1 or 0 when the directions are parallel, antiparallel, or perpendicular, respectively). The point y is defined as the beginning of the link l if it has positive direction, or as the end of l if it has negative direction. Such an asymmetry arises from the fact that we have performed the right shift (37) of $U_\nu(x)$. The Kronecker symbol δ_{xy} guarantees that C_{yx} and C_{xy} are always closed.

Equation (38) is a lattice regularization of the continuum loop equation (31).

Loop-space Laplace equation

One more contour integration over y

$$\Delta W(C) = \lambda \oint_C \mathrm{d}x_\mu \oint_C \mathrm{d}y_\mu \, \delta^{(d)}(x-y) \, W(C_{yx}) \, W(C_{xy})$$

Loop-space Laplacian

$$\Delta \equiv \oint_C \mathrm{d}x_\nu \, \partial_\mu^x \frac{\delta}{\delta\sigma_{\mu\nu}(x)} = \int_{\sigma_i}^{\sigma_f} \mathrm{d}\sigma \int_{\sigma-0}^{\sigma+0} \mathrm{d}\sigma' \frac{\delta}{\delta x_\mu(\sigma')} \frac{\delta}{\delta x_\mu(\sigma)}$$

is defined for much wider class of functionals than Stokes

This is important for SUSY extension

It is associated with the second-order Schwinger–Dyson equation

$$\int \mathrm{d}^d x \, \nabla_\mu F^a_{\mu\nu}(x) \frac{\delta}{\delta A^a_\nu(x)} \overset{\text{w.s.}}{=} \hbar \int \mathrm{d}^d x \, \mathrm{d}^d y \, \delta^{(d)}(x-y) \frac{\delta}{\delta A^a_\nu(y)} \frac{\delta}{\delta A^a_\nu(x)}$$

A non-perturbative gauge-invariant regularization Halpern, Yu.M. (1989)

$$\delta^{ab}\delta^{(d)}(x-y) \overset{\text{reg.}}{\Longrightarrow} \left\langle y \left| \left(\mathrm{e}^{a^2\nabla^2/2} \right)^{ab} \right| x \right\rangle$$

To translate this in loop space, we use the path-integral

$$\left\langle y \left| \boldsymbol{R}^{ab} \right| x \right\rangle \;=\; \int\limits_{\substack{r_\mu(0)=x_\mu \\ r_\mu(a^2)=y_\mu}} \mathcal{D}r_\mu(t)\, \mathrm{e}^{-\frac{1}{2}\int_0^{a^2} \mathrm{d}t\, \dot{r}_\mu^2(t)} \; \mathrm{tr}\left[t^a U(r_{yx}) t^b U(r_{xy})\right] \tag{39}$$

with

$$U(r_{yx}) \;=\; \boldsymbol{P}\, \mathrm{e}^{\mathrm{i}\int_x^y \mathrm{d}r^\mu \mathcal{A}_\mu(r)}, \tag{40}$$

where the integration is over regulator paths $r_\mu(t)$ from x to y, for which the typical length is $\sim a$. The conventional measure is implied in (39) so that

$$\int\limits_{\substack{r_\mu(0)=x_\mu \\ r_\mu(a^2)=y_\mu}} \mathcal{D}r_\mu(t)\, \mathrm{e}^{-\frac{1}{2}\int_0^{a^2} \mathrm{d}t\, \dot{r}_\mu^2(t)} \; \mathrm{tr}\left[t^a t^b\right] \;=\; \delta^{ab} \frac{1}{\left(2\pi a^2\right)^{d/2}}\, \mathrm{e}^{-(x-y)^2/2a^2}. \tag{41}$$

Using Eq. (39) and the completeness condition, we obtain as $N \to \infty$

$$\int \mathrm{d}^d x \, \mathrm{d}^d y \, \left\langle y \left| \boldsymbol{R}^{ab} \right| x \right\rangle \frac{\delta}{\delta A_\nu^a(y)} \frac{\delta}{\delta A_\nu^b(x)} \Phi(C)$$
$$= \lambda \oint_C dx_\mu \oint_C dy_\mu \int_{\substack{r_\mu(0)=x_\mu \\ r_\mu(a^2)=y_\mu}} \mathcal{D}r_\mu(t) \, \mathrm{e}^{-\frac{1}{2}\int_0^{a^2} \mathrm{d}t \, \dot{r}_\mu^2(t)} \, \Phi(C_{yx} r_{xy}) \, \Phi(C_{xy} r_{yx}) \, , \tag{42}$$

where the contours $C_{yx} r_{xy}$ and $C_{xy} r_{yx}$ are both closed

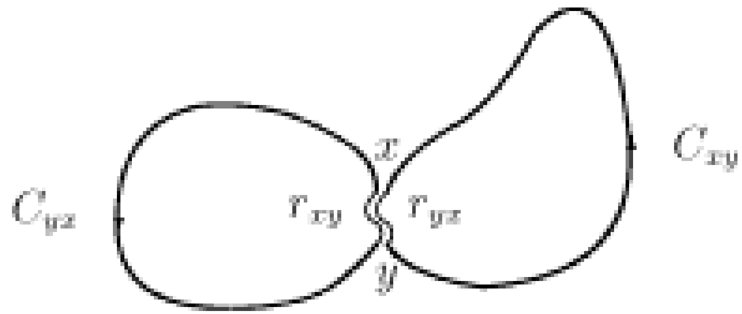


Figure 1: $C_{yx} r_{xy}$ and $C_{xy} r_{yx}$

Averaging over the gauge field and using the large-N factorization, we arrive at the regularized loop-space Laplace equation

Halpern, Y.M. (1989)

$$\Delta W(C) = \lambda \oint_C \mathrm{d}x_\mu \oint_C \mathrm{d}y_\mu \int_{\substack{r_\mu(0)=x_\mu \\ r_\mu(a^2)=y_\mu}} \mathcal{D}r_\mu(t)\, \mathrm{e}^{-\frac{1}{2}\int_0^{a^2} \mathrm{d}t\, \dot r_\mu^2(t)}\, W(C_{yx}r_{xy})\, W(C_{xy}r_{yx}) \tag{43}$$

which manifestly recovers the nonregularized one when $a \to 0$.

The constructed regularization is nonperturbative, while perturbatively it reproduces regularized Feynman diagrams. An advantage of this regularization of the loop equation is that the contours $C_{yx}r_{xy}$ and $C_{xy}r_{yx}$ on the RHS of Eq. (43) are both closed and do not have marked points if C does not have one. Therefore, Eq. (43) is written entirely in loop space.

Smearing of loop-space Laplacian

Smearing of loop-space Laplacian is needed to invert it, i.e. to produce the Green function

Smearing procedure (gets second-order operator from the first order)

$$\begin{aligned} \Delta^{(G)} &= \int\limits_0^1 \mathrm{d}\sigma \int\limits_0^1 \mathrm{d}\sigma'\, G(\sigma,\sigma') \frac{\delta}{\delta x_\mu(\sigma')} \frac{\delta}{\delta x_\mu(\sigma)} \\ &= \int\limits_0^1 \mathrm{d}\sigma \int\!\!\!\!\!/\limits_0^1 \mathrm{d}\sigma'\, G(\sigma,\sigma') \frac{\delta}{\delta x_\mu(\sigma')} \frac{\delta}{\delta x_\mu(\sigma)} + \Delta \end{aligned}$$

with parametric-invariant

$$G(\sigma_1,\sigma_2) = \mathrm{e}^{-\left|\int_{\sigma_1}^{\sigma_2} \mathrm{d}\sigma \sqrt{\dot{x}^2(\sigma)}\right|/\varepsilon} \qquad (\varepsilon \ll L)$$

ε has the meaning of stiffness

Green function of functional Laplacian

Yu.M. (1988)

Loop-space Laplacian can be inverted to produce the Green function (useful for iterative solution)

The functional Laplace equation (with given $J[x]$)

$$\Delta^{(G)} W[x] = J[x]$$

with the proper choice of boundary conditions can be solved to give

$$W[x] = 1 - \frac{1}{2}\int_0^\infty \mathrm{d}A \left\{ \left\langle J[x+\sqrt{A}\xi] \right\rangle_\xi^{(G)} - \left\langle J[\sqrt{A}\xi] \right\rangle_\xi^{(G)} \right\}$$

The average over the loops $\xi(\sigma)$ is given by the path integral

$$\langle F[\xi] \rangle_\xi^{(G)} = \frac{\int_{\xi(0)=\xi(1)} D\xi\, \mathrm{e}^{-S} F[\xi]}{\int_{\xi(0)=\xi(1)} D\xi\, \mathrm{e}^{-S}}$$

with the local action

$$S = \frac{1}{4}\int_0^1 \mathrm{d}\sigma \{ \frac{\varepsilon}{\sqrt{\dot{x}^2(\sigma)}} \dot{\xi}^2(\sigma) + \frac{\sqrt{\dot{x}^2(\sigma)}}{\varepsilon} \xi^2(\sigma) \}$$

It extends the results of Gateux (early 1900's) for functional Laplacian

Iterative solution

In large-N Yang–Mills the regularized $J[x]$ is as above bilinear in W:

$$J^{(G)}[x] = \lambda \int_0^1 \int_0^1 d\sigma_1 d\sigma_2 (1 - G(\sigma_1 - \sigma_2)) \dot{x}^\mu(\sigma_1) \dot{x}^\mu(\sigma_2) \times \int_{r(0)=x(\sigma_1)}^{r(a^2)=x(\sigma_2)} \mathcal{D}r \, \mathrm{e}^{-\frac{1}{2}\int_0^{a^2} d\tau \dot{r}^2(\tau)} \times W(C_{x(\sigma_1)x(\sigma_2)} r_{x(\sigma_2)x(\sigma_1)}) W(C_{x(\sigma_2)x(\sigma_1)} r_{x(\sigma_1)x(\sigma_2)})$$

Iterative solution in λ recovers perturbation theory

All that can be deduced from the general formula

$$\left\langle \mathrm{e}^{\mathrm{i}\sqrt{A}\int d\sigma \dot{p}(\sigma)\xi(\sigma)} \right\rangle_\xi^{(G)} = \mathrm{e}^{-A\int d\sigma \int d\sigma' \dot{p}(\sigma) G(\sigma-\sigma') \dot{p}(\sigma')/2}$$

where $p^\mu(\sigma)$ ($p^\mu(0) = p^\mu(1)$) represents a momentum-space loop

The triple gluon vertex appears from the uncertainty $\varepsilon \times 1/\varepsilon$

Applications of the loop equation

- QCD string is not Nambu–Goto Migdal, Y.M. (1979)
- Area law is a self-consistent solution Migdal, Y.M. (1980)
- Complete solution in 2d Kazakov, Kostov (1980)
- Elfin string is a formal solution Migdal (1981)
- Stochastic quantization Parisi, Wu (1981)
- Reduced models Eguchi, Kawai (1982)
- 2d gravity Kazakov (1990)
- Matrix models Ambjorn, Chekhov, Kristjansen, Y.M. (1993)
- IIB Model Ishibashi, Kawai, Kitazawa, Tsuchia (1997)
- $\mathcal{N} = 4$ super Yang–Mills Drukker, Gross, Ooguri (1999)

Hermitian one-matrix model

The partition function

$$Z_{1\mathsf{h}} \;=\; \int \mathsf{d}\varphi\, \mathsf{e}^{-N\,\mathsf{tr}\,V(\varphi)}.$$

The measure for integrating over Hermitian $N \times N$ matrices:

$$\mathsf{d}\varphi = \prod_{i=1}^{N} \mathsf{d}\varphi_{ii} \prod_{j>i}^{N} \mathsf{d}\,\mathsf{Re}\,\varphi_{ij}\,\mathsf{d}\,\mathsf{Im}\,\varphi_{ij} \tag{44}$$

is invariant under the shift

$$\varphi_{ij} \;\longrightarrow\; \varphi_{ij} + \epsilon_{ij} \tag{45}$$

by an arbitrary $N \times N$ Hermitian ϵ_{ij}.

The most general potential

$$V(\varphi) \;=\; \sum_{k} t_k\, \varphi^k\,, \tag{46}$$

where t_k are coupling constants.

The averages are defined by

$$\langle F[\varphi]\rangle_{1\mathsf{h}} \;=\; Z_{1\mathsf{h}}^{-1}\int \mathsf{d}\varphi\,\mathsf{e}^{-N\,\mathrm{tr}\,V(\varphi)}F[\varphi]\,.$$

The Schwinger–Dyson equation

$$\left\langle \frac{\partial\,\mathrm{tr}\,V(\varphi)}{\partial\varphi_{ji}}F[\varphi]\right\rangle_{1\mathsf{h}} \;=\; \left\langle \frac{1}{N}\frac{\partial F[\varphi]}{\partial\varphi_{ji}}\right\rangle_{1\mathsf{h}} \tag{47}$$

results from the invariance of the measure under the shift (45).

The derivatives are calculated by

$$\frac{\partial\varphi_{kl}}{\partial\varphi_{ji}} \;=\; \delta_{il}\delta_{kj}\,.$$

The Feynman graphs can be represented by the double index lines 't Hooft 1974 = zero-dimensional quantum field theory. Solving the Hermitian one-matrix model is equivalent to calculating the number of graphs with a given genus.

The loop equation

Choosing $F[\varphi] = (p-\varphi)^{-1}_{ij}$ in Eq. (47), we obtain the Schwinger–Dyson equation

$$\left\langle \frac{1}{N} \mathrm{tr} \frac{V'(\varphi)}{p-\varphi} \right\rangle_{1\mathrm{h}} = \left\langle \frac{1}{N^2} \mathrm{tr} \frac{1}{p-\varphi} \, \mathrm{tr} \frac{1}{p-\varphi} \right\rangle_{1\mathrm{h}} .$$

Equation (48) can be expressed entirely via the resolvent

$$W(p) = \left\langle \frac{1}{N} \mathrm{tr} \frac{1}{p-\varphi} \right\rangle_{1\mathrm{h}}$$

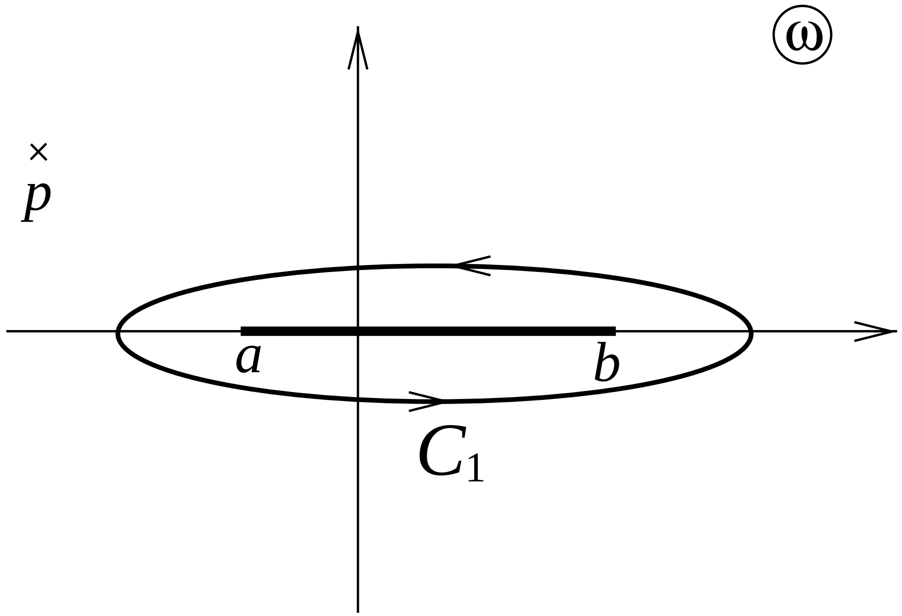
which is a Laplace transform of the "Wilson loop":

$$W(p) = \int_0^\infty \mathrm{d}l \, \mathrm{e}^{-pl} \left\langle \frac{1}{N} \mathrm{tr} \, \mathrm{e}^{l\varphi} \right\rangle_{1\mathrm{h}} .$$

The resulting loop equation reads

$$\int_{C_1} \frac{\mathrm{d}\omega}{2\pi \mathrm{i}} \frac{V'(\omega)}{(p-\omega)} W(\omega) = W^2(p) + \frac{1}{N^2} \frac{\delta}{\delta V(p)} W(p) \,, \qquad (48)$$

where the contour C_1 encloses counterclockwise singularities of $W(\omega)$ leaving outside the pole at $\omega = p$ as depicted in the figure.



Contour C_1 in the ω-plane for integration on the LHS of Eq. (48)

The contour integral on the LHS simply acts as a projector picking up negative powers of p.

At $N = \infty$ the second term on the RHS can be omitted, and the imaginary part of Eq. (48) coincides with the saddle-point equation by Brézin, Itzykson, Parisi, Zuber (1978)

$$V'(p) \;=\; 2 \fint d\lambda \frac{\rho(\lambda)}{p-\lambda} \qquad \boxed{p \in \text{support of } \rho}\,, \tag{49}$$

where the RHS involves the principal part of the integral.

It is written for the spectral density

$$\rho(p) \;=\; \frac{1}{N}\sum_{i=1}^{N}\delta^{(1)}(p-p_i)$$

which becomes a continuous function of p as $N \to \infty$.
It describes the distribution of eigenvalues of the matrix φ.

The first term on the RHS of Eq. (48) is associated with the factorized part of the correlator, while the second term represents the connected part of the two-loop correlator which is $\sim 1/N^2$ as $N \to \infty$. It involves the variational derivative (the loop insertion operator)

$$\frac{\delta}{\delta V(p)} \;=\; -\sum_{k=0}^{\infty} p^{-k-1}\frac{\partial}{\partial t_k} \tag{50}$$

Consequently, Eq. (48) is closed and determines $W(p)$ unambiguously, providing the boundary condition $W(p) \to 1/p$ is imposed as $p \to \infty$.

Note that we obtained a single (functional) equation for $W(p)$. This is due to the fact that $\operatorname{tr} V(\varphi)$ contains a complete set of traces $\operatorname{tr}\varphi^k$. They become independent as $N \to \infty$.

Solution in $1/N$

The loop equation (48) can be solved order by order in $1/N^2$ (= the genus expansion).

The genus zero one-cut solution can be written as (Migdal (1983))

$$W_0(p) = \int_{C_1} \frac{\mathrm{d}\omega}{4\pi \mathrm{i}} \frac{V'(\omega)}{(p-\omega)} \sqrt{\frac{(p-a)(p-b)}{(\omega-a)(\omega-b)}}, \tag{51}$$

where a and b are determined by

$$\int_{C_1} \frac{\mathrm{d}\omega}{2\pi \mathrm{i}} \frac{V'(\omega)}{\sqrt{(\omega-a)(\omega-b)}} = 0, \qquad \int_{C_1} \frac{\mathrm{d}\omega}{2\pi \mathrm{i}} \frac{\omega V'(\omega)}{\sqrt{(\omega-a)(\omega-b)}} = 2.$$

Performing the contour integral in (51) by taking the residues at $\omega = p$ and $\omega = \infty$, we reproduce the one-cut solution for polynomial V. However, Eq. (51) remains valid in the more general case of nonpolynomial V, e.g. having logarithmic singularities. The position of the cut is always such as to avoid these singularities of V.

The multiloop correlators in genus zero can be obtained from $W_0(p)$ given by Eq. (51) applying the loop insertion operator (50). The two-loop correlator Ambjørn, Jurkiewicz, Y.M. (1990)

$$W_0(p,q) \;=\; \frac{1}{4(p-q)^2}\left\{\frac{2pq-(p+q)(a+b)+2ab}{\sqrt{(p-a)(p-b)}\sqrt{(q-a)(q-b)}}-2\right\} \tag{52}$$

depends on the potential V only via a and b but not explicitly (= universality). An analog of this in condensed-matter physics is the correlator of two densities of energy eigenvalues Br ezin, Zee (1993). It does not hold for higher multiloop correlators.

To calculate the $1/N^2$ correction, we substitute

$$W_0(p,p) \;=\; \frac{(a-b)^2}{16(p-a)^2(p-b)^2}$$

extracted from Eq. (52) into the RHS of Eq. (48). $W_1(p)$ can be obtained by solving a linear equation which, in turn, determines F_1.

For this method of solving the Hermitian one-matrix model the free energy generates all multiloop correlators at a given genus.

Iterative solution

The iterative procedure Ambjørn, Chekhov, Krisjansen, Y.M. (1993) of solving the loop equation is based on the genus-zero solution (51). Inserting the genus expansion of $W(p)$ and F:

$$W(p) = \sum_{h=0}^{\infty} \frac{1}{N^{2h}} W_h(p)\,, \qquad F = \sum_{h=0}^{\infty} \frac{1}{N^{2h}} F_h \quad \text{with} \quad W_h(p) = \frac{\delta F_h}{\delta V(p)}\,,$$

into Eq. (48), we obtain the following equation for $W_h(p)$ at $h \geq 1$:

$$\int_{C_1} \frac{\mathrm{d}\omega}{2\pi \mathrm{i}} \frac{V'(\omega)}{(p-\omega)} W_h(\omega) - 2\, W_0(p)\, W_h(p) = \sum_{h'=1}^{h-1} W_{h'}(p)\, W_{h-h'}(p) + \frac{\delta}{\delta V(p)} W_{h-1}(p)\,.$$

It expresses $W_h(p)$ entirely in terms of $W_{h'}(p)$ with $h' < h$. This makes it possible to solve Eq. (53) iteratively genus by genus.

The iterative procedure simplifies if the moments

$$\left.\begin{aligned} M_k &= \int_{C_1} \frac{\mathrm{d}\omega}{2\pi \mathrm{i}} \frac{V'(\omega)}{(\omega-a)^{k+1/2}(\omega-b)^{1/2}}, \\ J_k &= \int_{C_1} \frac{\mathrm{d}\omega}{2\pi \mathrm{i}} \frac{V'(\omega)}{(\omega-a)^{1/2}(\omega-b)^{k+1/2}}. \end{aligned}\right\} \tag{53}$$

are introduced instead of the coupling constants t_j. The moments M_k and J_k depend explicitly only on t_j with $j \geq k+1$.

The main motivation for introducing the moments (53) is that $W_h(\lambda)$ depends only on $2 \times (3h-1)$ lower moments ($2 \times (3h-2)$ for F_h).

To find F_h, we first solve Eq. (53) for $W_h(\lambda)$ and then use the last equation in (53). The result in genus one reads

Ambjørn, Chekhov, Y.M. (1992)

$$F_1 = -\frac{1}{24} \ln (M_1 J_1) - \frac{1}{6} \ln (b-a).$$

The genus-two results are also explicitly obtained

Ambjørn, Chekhov, Krisjansen, Y.M. (1993)

Hermitian two-matrix model

An obvious extension of the Hermitian one-matrix model is the model of two Hermitian matrices φ_1 and φ_2:

$$Z_{2\text{h}} = \int \mathrm{d}\varphi_1 \, \mathrm{d}\varphi_2 \, \mathrm{e}^{N \, \mathrm{tr}\, [-V(\varphi_1) - V(\varphi_2) + \varphi_1 \varphi_2]} \, .$$

The Wilson loop average and the one-link correlator are defined by

$$W(\lambda) = \left\langle \frac{1}{N} \mathrm{tr} \, \frac{1}{\lambda - \varphi_1} \right\rangle_{2\text{h}} ,$$

$$G(\nu, \lambda) = \left\langle \frac{1}{N} \mathrm{tr} \, \left(\frac{1}{(\nu - \varphi_1)} \frac{1}{(\lambda - \varphi_2)} \right) \right\rangle_{2\text{h}} .$$

The definition of $W(\lambda)$ is similar to Eq. (48) while $G(\nu, \lambda)$ is absent in the one-matrix model.

Expanding $G(\nu, \lambda)$ in $1/\nu$, we obtain

$$\left. \begin{aligned} G(\nu, \lambda) &= \frac{W(\lambda)}{\nu} + \sum_{n=1}^{\infty} \frac{G_n(\lambda)}{\nu^{n+1}} , \\ G_n(\lambda) &= \left\langle \frac{1}{N} \mathrm{tr} \, \left(\varphi_1^n \frac{1}{\lambda - \varphi_2} \right) \right\rangle_{2\text{h}} . \end{aligned} \right\}$$

In the large-N limit, the correlator $G(\nu,\lambda)$ obeys the following loop equation:

$$\int\limits_{C_1} \frac{\mathrm{d}\omega}{2\pi\mathrm{i}} \frac{V'(\omega)}{(\nu-\omega)} G(\omega,\lambda) \;=\; W(\nu)\, G(\nu,\lambda) + \lambda G(\nu,\lambda) - W(\nu)\,,$$

where the contour C_1 encircles counterclockwise the cut (or cuts) of the function $G(\omega,\lambda)$ as above.

The solution for $W(\lambda)$ versus $V(\lambda)$ is determined by the equation

$$\sum_{k\geq 1} k\, t_k G_{k-1}(\lambda) \;=\; \lambda W(\lambda) - 1$$

which is just the $1/\nu$ term of the expansion of Eq. (54) in $1/\nu$.

The functions $G_n(\lambda)$ are expressed via $W(\lambda)$ using the recurrence relation

$$\left.\begin{aligned} G_{n+1}(\lambda) \;&=\; \int\limits_{C_1} \frac{\mathrm{d}\omega}{2\pi\mathrm{i}} \frac{V'(\omega)}{(\lambda-\omega)} G_n(\omega) - W(\lambda)\, G_n(\lambda)\,, \\ G_0(\lambda) \;&=\; W(\lambda) \end{aligned}\right\}$$

which is obtained by expanding Eq. (54) in $1/\lambda$. If $V(\lambda)$ is a polynomial of degree K, Eq. (54) contains $W(\lambda)$ up to degree K and the solution is algebraic Gava, Narain (1991), Alfaro (1993), Staudacher (1993).

For a cubic potential, this equation for $W(\lambda)$ is cubic and determines the critical index of the susceptibility $\gamma_0 = -1/3$. This is in contrast to the Hermitian one-matrix model where the loop equation is quadratic in $W(\lambda)$. We see that matter changes Kazakov (1986) the critical behavior of pure quantum gravity.

The correlator $G(\nu, \lambda)$ is symmetric in ν and λ for any solution of Eq. (54). This symmetry requirement can be used directly to determine $W(\lambda)$ alternatively to Eq. (54).

Multimatrix models

It is possible to further extend the Hermitian two-matrix model by considering a chain of matrices with the nearest-neighbor interaction:

$$Z_{q\mathrm{h}} = \prod_{i=1}^{q} \mathrm{d}\varphi_i \exp\left\{ N\, \mathrm{tr}[-\sum_{i=1}^{q} V(\varphi_i) + \sum_{i=1}^{q-1} \varphi_i\varphi_{i+1}] \right\}.$$

In the limit of $q \to \infty$, we obtain an infinite chain associated with discretization of a one-dimensional theory.

The Hermitian q-matrix model possesses unitary continuum limits with $\gamma_0 = -1/(q+1)$. In the $q \to \infty$ limit, this gives $\gamma_0 \to 0$.

A multidimensional extension Kazakov, Migdal (1992)

$$Z_{\mathrm{KM}} = \int \prod_{x,\mu} \mathrm{d}U_\mu(x) \prod_x \mathrm{d}\varphi_x \, \mathrm{e}^{-S_{\mathrm{KM}}[U,\varphi]}$$

with the action

$$S_{\mathrm{KM}}[U,\varphi] = N \, \mathrm{tr}[-\sum_{x,\mu} \varphi_{x+a\widehat{\mu}} U_\mu(x) \varphi_x U_\mu^\dagger(x) + \sum_x V(\varphi_x)].$$

φ_x and $U_\mu(x)$ are $N \times N$ Hermitian and unitary matrices, respectively, with x labeling lattice sites on a d-dimensional hypercubic lattice. The integration over the gauge field $U_\mu(x)$ is over the Haar measure on $SU(N)$ at each link of the lattice.

It is of the type of Wilson's lattice gauge theory with adjoint matter but without the action for the gauge field.

The large-N solution can be obtained from the loop equation Dobroliubov, Semenoff, Y.M. (1993).

Renormalization of smooth Wilson loops

For smooth loops

Gervais, Neveu (1980)

Polyakov (1980)

Vergeles, Dotsenko (1980)

$$W(g;C) = \mathrm{e}^{-\mathrm{const.}\,L(C)/a}\, W_{\mathrm{R}}(g_{\mathrm{R}};C)$$

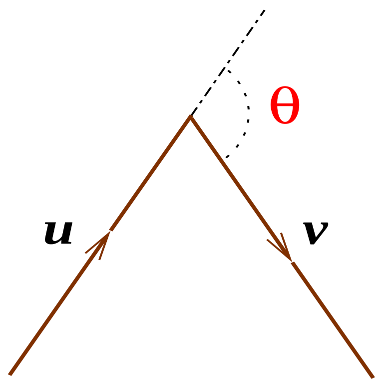
where W_{R} is finite after the charge renormalization $g \Longrightarrow g_{\mathrm{R}}$
and a is a certain (gauge-invariant) UV cutoff

The exponential comes from the renormalization of the mass of a heavy test particle propagating along the loop.
It does not emerge in dimensional regularization

Renormalization of cusped Wilson loops

An additional logarithmic divergency appear for cusped loops

Polyakov (1980)



Segment of a closed loop near the cusp.
θ is the cusp angle formed by the vectors u and v:
$\cosh \theta = \frac{u \cdot v}{\sqrt{u^2}\sqrt{v^2}}$

The cusped Wilson loop is multiplicatively renormalizable

Brandt, Neri, Sato (1981)

$$W(g; \Gamma) = Z(g, ; \theta) W_{\mathrm{R}}(g_{\mathrm{R}}; \Gamma)$$

where (the divergent factor of) $Z(g; \theta)$ depends on the cusp angle θ

This is true if Γ has no light-cone segments

Cusp anomalous dimension

The definition

$$\gamma_{\text{cusp}}(g;\theta) = -a\frac{\mathrm{d}}{\mathrm{d}a}\ln Z(g;\theta)$$

The limit of large θ

Korchemsky, Radyushkin (1987)

$$\gamma_{\text{cusp}}(g;\theta) \stackrel{\theta\to\infty}{\longrightarrow} \frac{\theta}{2}f(g)$$

The same function f appear in the anomalous dimensions of twist two conformal operators with large spin

Relation to twist-two operators

Anomalous dimensions of twist-two operators

$$O_J^{(F)} = \frac{1}{N}\,\mathrm{tr}\,F_{\mu\cdot}\,(\nabla_\cdot)^{J-2}\,F_{\mu\cdot}$$

$$O_J^{(\Psi)} = \bar{\Psi}\gamma_\cdot\,(\nabla_\cdot)^{J-1}\,\Psi$$

with Lorentz spin J (measurable in deep inelastic)

Also

$$O_J^{(\Phi)} = \frac{1}{N}\,\mathrm{tr}\,\Phi\,(\nabla_\cdot)^{J}\,\Phi$$

in $\mathcal{N}=4$ SYM.

Notation: $\nabla_\cdot \equiv \nabla_\mu \xi_\mu \qquad \xi^2 = 0$

— symmetrization and subtraction of traces

$(\nabla_\cdot)^J$ is in fact a (Gegenbauer) polynomial in $\overleftarrow{\nabla}_\cdot$ and $\overrightarrow{\nabla}_\cdot$

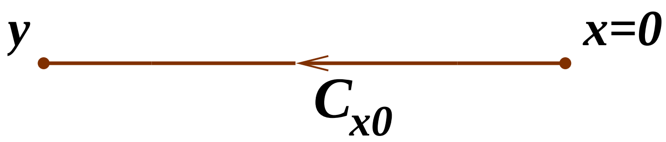
— conformal operators

Brodsky, Frishman, Lepage, Sachrajda (1980)
Y. M. (1981)
Ohrndorf (1982)

Relation to twist-two operators (cont.1)

The relation can be understood from open Wilson loops

$$O(C_{y0}) = \bar{\psi}(y)\boldsymbol{P}\,\mathrm{e}^{\mathrm{i}g\int_0^y \mathrm{d}\xi^\mu A_\mu}\psi(0)$$



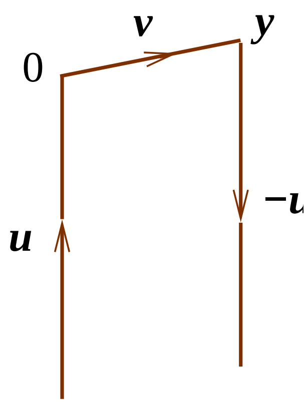
with matter fields attached at the ends

Standard triangular diagrams comes from

$$\left\langle \psi(\infty,\vec{y})O(C_{y0})\bar{\psi}(\infty,\vec{0})\right\rangle \propto W(\sqcap)$$

as mass of matter fields $\to \infty$

$\sqcap$-shaped Wilson loop



Relation to twist-two operators (cont.2)

Remember that the propagator in an external field A_μ

$$\left\langle \psi_i(x)\bar{\psi}_j(y) \right\rangle_\psi \stackrel{\text{large } N}{=} \sum_{C_{yx}} \left[\mathrm{e}^{\mathrm{i}g \int_{C_{yx}} \mathrm{d}\xi^\mu A_\mu} \right]_{ij} \stackrel{\text{mass}\to\infty}{\propto} \left[\mathrm{e}^{\mathrm{i}g \int_{C_{yx}^{(\text{min})}} \mathrm{d}\xi^\mu A_\mu} \right]_{ij}$$

and thus straight vertical lines appear in $\sqcap$

The central segment of $\sqcap$ is near the light-cone (to kill twists higher than 2).
$\sqcap$ has two cusps with $\theta \to \infty$.

This is how the light-cone Wilson loop appear

Light-cone Wilson Loops

For Π-shaped loop (1 light cone)

Korchemsky, Marchesini (1993)

$$W(\Pi) = \mathrm{e}^{-\frac{1}{2} f(\lambda) \ln^2 \frac{T}{a} + \mathrm{const.}(\lambda) \ln \frac{T}{a} + \mathrm{finite}(\lambda)}$$

with the same $f(\lambda)$ as before.
v^μ is along the light cone ($v^2 = 0$) and $y_\mu = v_\mu T$.

For Γ-shaped loop (2 light cones)

Alday, Maldacena (2007)

$$W(\Gamma) = \mathrm{e}^{-\frac{1}{2} f(\lambda) \ln \frac{T}{a} \ln \frac{S}{a} + g(\lambda)(\ln \frac{T}{a} + \ln \frac{S}{a}) + \mathrm{finite1}(\lambda)}$$

both v^μ and u^μ are along the light cones ($v^2 = 0$, $u^2 = 0$) and $y_\mu = v_\mu T$, $x_\mu = u_\mu S$.

Most probably it gives the same $f(\lambda)$ but is not proved

SYM Wilson Loops

Extension to $\mathcal{N} = 4$ SYM Maldacena (1998)

$$W_{\rm SYM}(C) = \left\langle \frac{1}{N} \operatorname{tr} \boldsymbol{P} \, \mathrm{e}^{\mathrm{i} g \oint_C \mathrm{d}\sigma \left(\dot{\xi}^\mu A_\mu + |\dot{\xi}| n^i \Phi_i \right)} \right\rangle$$

with unit vector n^i $(n^2 = 1)$ and 6 scalars Φ_i $(i = 1, \cdots, 6)$

No relative i in Minkowski space

Adjoint Wilson loop

$$\operatorname{tr}_A U = |\operatorname{tr} U|^2 - 1$$

Due to factorization at large N

$$\left\langle \frac{1}{N^2} \operatorname{tr}_A U(C) \right\rangle = \left\langle \frac{1}{N} \operatorname{tr} U(C) \right\rangle^2$$

adjoint fundamental

Same results as in QCD hold and some more

BPS for a straight line inside the light-cone

$$W_{\rm SYM}(|) = 1$$

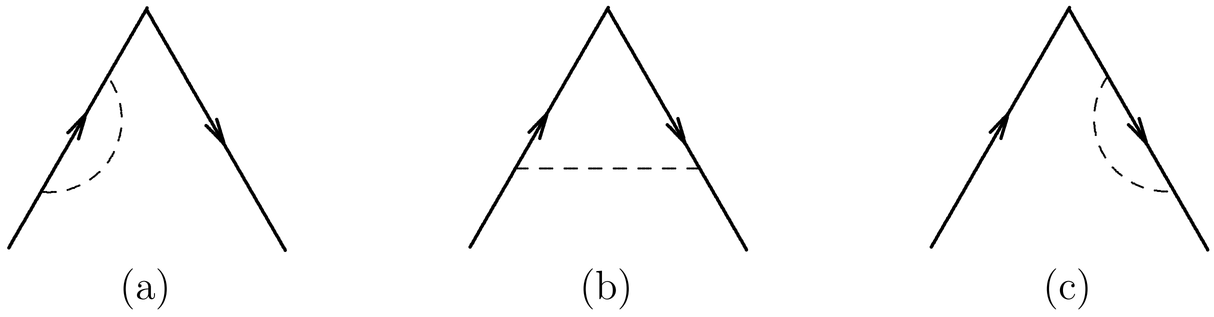
Perturbation Theory

Order λ (one loop)

$$W(\Gamma) = 1 - \frac{\lambda}{2} \int_{-\infty}^{+\infty} d\sigma_1 \int_{-\infty}^{+\infty} d\sigma_2 \left[\dot{x}^{\mu}(\sigma_1)\dot{x}_{\mu}(\sigma_2) - |\dot{x}(\sigma_1)||\dot{x}(\sigma_2)|\right] \times D\left(x(\sigma_1) - x(\sigma_2)\right)$$

with (scalar) propagator in d-dimensions

$$D(x) = -\frac{\Gamma\left(d/2 - 1\right)}{4\pi^{d/2}}[-x^2]^{1-d/2}$$

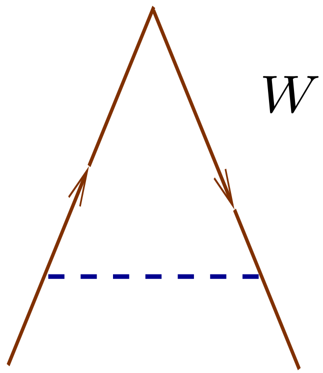


diagrams of order λ

Diagrams (a) and (c) vanish (gluons are cancelled by scalars)

One-loop perturbation theory

Only one diagram is nonvanishing



$$W(\Gamma) = 1 - \frac{\lambda}{4\pi^2}(\cosh\theta - 1)\int \mathrm{d}s \int \mathrm{d}t\, \frac{1}{s^2 + 2st\cosh\theta + t^2}$$

$$= 1 - \frac{\lambda}{4\pi^2}\frac{\cosh\theta - 1}{\sinh\theta}\theta\ \ln\frac{L}{a}$$

$$\xrightarrow{\text{large } \theta} 1 - \frac{\lambda}{4\pi^2}\theta\ \ln\frac{L}{a}$$

which yields $\Longrightarrow$ $f(\lambda) = \dfrac{\lambda}{2\pi^2}$

No mass-renormalization term $-\lambda/4\pi a$ as is in QCD

One-loop perturbation theory (cont.)

Exact formula

$$W(S,T;a,b) = 1 - \frac{\lambda}{4\pi^2}(\cosh\theta - 1)\int_a^S \mathrm{d}s \int_b^T \mathrm{d}t \, \frac{1}{s^2 + 2st\cosh\theta + t^2}$$

$$= \quad 1 - \frac{\lambda}{8\pi^2}\frac{\cosh\theta - 1}{\sinh\theta}\Big(\mathrm{Li}_2(-\frac{T}{S}\mathrm{e}^{\theta}) - \mathrm{Li}_2(-\frac{T}{S}\mathrm{e}^{-\theta}) - \mathrm{Li}_2(-\frac{T}{a}e^{\theta})$$

$$+\mathrm{Li}_2(-\frac{T}{a}e^{-\theta}) - \mathrm{Li}_2(-\frac{b}{S}\mathrm{e}^{\theta}) + \mathrm{Li}_2(-\frac{b}{S}\mathrm{e}^{-\theta}) + \mathrm{Li}_2(-\frac{b}{a}\mathrm{e}^{\theta}) - \mathrm{Li}_2(-\frac{b}{a}\mathrm{e}^{-\theta})\Big)$$

where Li_2 is Euler's dilogarithm

$$\mathrm{Li}_2(z) = \sum_{n=1}^{\infty}\frac{z^n}{n^2} = -\int_0^z \frac{dx}{x}\ln(1-x)$$

which obeys the relation

$$\mathrm{Li}_2\left(-e^{\Omega}\right) + \mathrm{Li}_2\left(-e^{-\Omega}\right) = -\frac{1}{2}\ln^2\Omega - \frac{\pi^2}{6}$$

It is used to extract the double logarithms

Double-Logarithmic Approximation

Again at one loop

$$W(S,T;a,b) = 1 - \frac{\lambda}{4\pi^2}(\cosh\theta - 1)\int_a^S \mathrm{d}s \int_b^T \mathrm{d}t \, \frac{1}{s^2 + 2st\cosh\theta + t^2}$$

The double-logarithmic region of integration, is

$$t\,\mathrm{e}^{-\theta} \lesssim s \lesssim t\,\mathrm{e}^{\theta} \qquad \text{or} \qquad s\,\mathrm{e}^{-\theta} \lesssim t \lesssim s\,\mathrm{e}^{\theta}$$

so write it in DLA

$$W(S,T;a,b) = 1 - \beta \int_b^T \frac{\mathrm{d}t}{t} \int_{\max\{a,t\,\mathrm{e}^{-\theta}\}}^{\min\{S,t\,\mathrm{e}^{\theta}\}} \frac{\mathrm{d}s}{s}$$

$$\Longrightarrow = 1 - 2\beta\theta \ln\frac{T}{b} \qquad \text{very large } S\ , \quad \text{very small } a$$

reproducing the above result

$$\Longrightarrow = 1 - \beta \ln\frac{T}{b} \ln\frac{S}{a} \qquad \text{very large } \theta$$

reproducing the 2 light-cone result

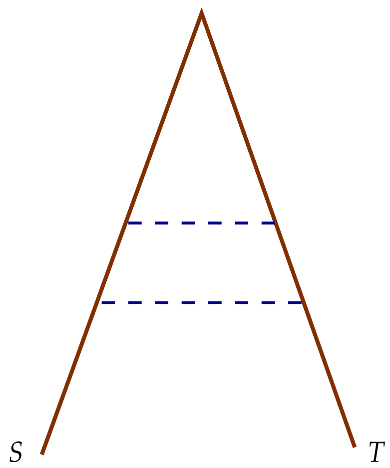
Two-Loop Ladder Diagram

Korchemsky, Radyushkin (1987)

Contribution to cusp anomalous dimension

$$\gamma_{\rm cusp}^{\rm (lad)} = \frac{\lambda^2}{128\pi^4}\frac{(\cosh\theta - 1)^2}{\sinh^2\theta}\int_0^\infty \frac{d\sigma}{\sigma}\ln\left(\frac{1+\sigma\,e^{\theta}}{1+\sigma\,e^{-\theta}}\right)\ln\left(\frac{\sigma+e^{\theta}}{\sigma+e^{-\theta}}\right)$$

$$\to \frac{\lambda^2}{96\pi^4}\left(\theta^3 + \frac{\pi^2}{2}\theta + \mathcal{O}(1)\right)$$



θ^3 should be cancelled by interaction !!!

$\Longrightarrow$ not only ladder diagrams are essential

Similar results for the light-cone Wilson loop:

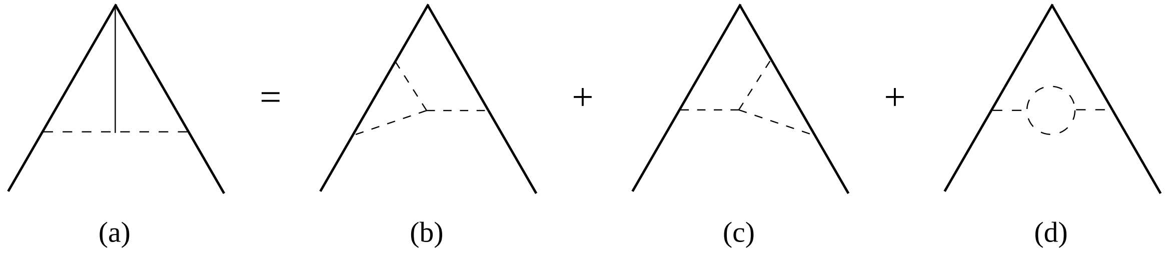
$$\mathcal{G}_{\rm l.c.}^{\rm ladd.} = 1 - \frac{\beta}{2}\ln^2\frac{T}{\varepsilon} + \frac{\beta^2}{12}\ln^4\frac{T}{\varepsilon} - \frac{\beta^2\pi^2}{12}\ln^2\frac{T}{\varepsilon}$$

$\ln^4\frac{T}{\varepsilon}$ is to be cancelled by diagrams with interaction

Surface Term

Olesen, Semenoff, Y.M. (2006)

Cancellation between three-gluon vertex and propagators is *not* complete



Surface term comes from integration by parts

$$\gamma_{\rm cusp}^{\rm anom} = -\frac{\lambda^2}{16\pi^4}\frac{\cosh\theta - 1}{\cosh\theta}\Big(\int_0^\theta + \int_0^{\pi/2}\Big)\frac{d\psi\,\psi}{1-\cosh^2\psi/\cosh^2\theta}\ln\frac{\cosh^2\theta}{\cosh^2\psi}$$

$$\to -\frac{\lambda^2}{96\pi^4}\Big(\theta^3 + \pi^2\theta + \mathcal{O}(1)\Big)$$

Two-loop cusp anomalous dimension

$$\gamma_{\rm cusp} = \frac{\theta}{2}\left(\frac{\lambda}{2\pi^2} - \frac{\lambda^2}{96\pi^2}\right) + \mathcal{O}(\theta^0)$$

reproduces the known results

Loop Equation for scalars

Y.M. (1988)

Wilson loop for scalars

$$W(C) = \left\langle \mathrm{e}^{\mu \int \mathrm{d}\sigma \sqrt{\dot{x}^2(\sigma)}\varphi(x(\sigma))} \right\rangle$$

The proposed loop equation

$$\Delta W(C) = \mu \int \mathrm{d}\sigma \oint \mathrm{d}\sigma' \sqrt{\dot{x}^2(\sigma)} \sqrt{\dot{x}^2(\sigma')} \delta^{(d)} \Big(x(\sigma) - x(\sigma') \Big) W(C)$$

describes a free scalar field.

Δ as averaging

The action of the (smeared) functional Laplacian $\Delta^{(G)}$ on a functional $F[x]$ can be represented as the average

$$\Delta^{(G)} F[x] = \frac{\mathrm{d}^2}{\mathrm{d}r^2} \left\langle F[x + r\xi] \right\rangle_\xi^{(G)} \Big|_{r=0} . \tag{54}$$

This formula can be proven by expanding in r and using

$$\left\langle \xi^\mu(\sigma) \xi^\nu(\sigma') \right\rangle_\xi^{(G)} = \delta^{\mu\nu} G(\sigma, \sigma') . \tag{55}$$

so that

$$\begin{aligned} \left\langle F[x + r\xi] \right\rangle_\xi^{(G)} &= F[x] + \frac{r^2}{2} \int \mathrm{d}\sigma_1 \int \mathrm{d}\sigma_2 \left\langle \xi^\mu(\sigma_1) \xi^\nu(\sigma_2) \right\rangle_\xi^{(G)} \frac{\delta^2 F[x]}{\delta x^\mu(\sigma_1) \delta x^\nu(\sigma_2)} + \\ &= F[x] + \frac{r^2}{2} \int \mathrm{d}\sigma_1 \int \mathrm{d}\sigma_2 \, G(\sigma_1, \sigma_2) \frac{\delta^2 F[x]}{\delta x^\mu(\sigma_1) \delta x^\mu(\sigma_2)} + \mathcal{O}(r^4) \end{aligned}$$

Equation (54) is more convenient for practical calculations than the definition of the loop-space Laplacian.

Cusped Loop Equation

Cusped loop equation for $\mathcal{N}=4$ SYM Drukker, Gross, Ooguri (1999)
for supersymmetric loops $\boldsymbol{C}=\{x_\mu(\sigma),Y_i(\sigma);\zeta(\sigma)\}$ ($\zeta(\sigma)$ denotes the Grassmann odd component)

$$\boldsymbol{\Delta}\ln W(\boldsymbol{C})|_{\boldsymbol{C}=\Gamma} = \lambda\int \mathrm{d}\sigma_1\int \mathrm{d}\sigma_2\,\left(\dot{x}_\mu(\sigma_1)\dot{x}_\mu(\sigma_2)-|\dot{x}_\mu(\sigma_1)||\dot{x}_\mu(\sigma_2)|\right) \times \delta^{(4)}(x_1-x_2)\frac{W(\Gamma_{x_1x_2})W(\Gamma_{x_2x_1})}{W(\Gamma)}$$

where

$$\boldsymbol{\Delta}=\lim_{\eta\to 0}\int \mathrm{d}s\int_{s-\eta}^{s+\eta}\mathrm{d}s'\left(\frac{\delta^2}{\delta x^\mu(s')\delta x_\mu(s)}+\frac{\delta^2}{\delta Y^i(s')\delta Y_i(s)}+\frac{\delta^2}{\delta\zeta(s')\delta\bar{\zeta}(s)}\right)$$

is the supersymmetric extension of the loop-space Laplacian and $\dot{Y}^2=\dot{x}^2$, $\zeta=0$ after acting by $\boldsymbol{\Delta}$.

The RHS $\sim (La)^{-1}$ for smooth loops but $\sim a^{-2}$ for cusped loops (was L/a^3 in QCD)

Cusped Loop Equation (cont.1)

It can be shown for cusped Wilson loops

$$\Delta \ln W(C)|_{C=\Gamma} = -\frac{\mathrm{d}}{\mathrm{d}a^2} \ln W(\Gamma)$$

$$\Longrightarrow \frac{2}{a^2}\gamma_{\text{cusp}}(\theta,\lambda) = \lambda \int \mathrm{d}\sigma_1 \int \mathrm{d}\sigma_2 \left(\dot{x}_\mu(\sigma_1)\dot{x}_\mu(\sigma_2) - |\dot{x}_\mu(\sigma_1)||\dot{x}_\mu(\sigma_2)|\right) \times \delta_a^{(4)}(x_1 - x_2)\frac{W(\Gamma_{x_1x_2})W(\Gamma_{x_2x_1})}{W(\Gamma)}$$

- is observed to order λ by Drukker, Gross, Ooguri (1999)
- is verified to order λ^2 for arbitrary θ Olesen, Semenoff, Yu.M. (2006):

The ladder diagram of order λ^2 comes iteratively from the ladder diagram of order λ

Cusped Loop Equation (cont.2)

The anomaly diagram is reproduced when gluon is attached to the regularizing path $r_{x_1 x_2}$ by the formula Migdal, Yu.M. (1981)

$$\int\limits_{\substack{z(0)=x\\ z(\tau)=y}} \mathcal{D}z(t)\, \mathrm{e}^{-\int_0^\tau \mathrm{d}t\, \dot{z}^2(t)/2} \int_x^y \mathrm{d}z^\mu \delta^{(d)}(z-u)$$
$$= \frac{1}{2}\int_0^\infty \mathrm{d}\tau_1 \int_0^\infty \mathrm{d}\tau_2\, \delta\left(\tau-\tau_1-\tau_2\right)$$
$$\times \frac{1}{(2\pi\tau_1)^{d/2}}\, \mathrm{e}^{-(x-u)^2/2\tau_1} \frac{\overleftrightarrow{\partial}}{\partial u_\mu} \frac{1}{(2\pi\tau_2)^{d/2}}\, \mathrm{e}^{-(y-u)^2/2\tau_2}$$

The loop equation may be useful for next orders in λ

Some comments about large-N QCD

$|\dot{x}|$ can be neglected near the light-cone $\Longrightarrow$ same cusped loop equation as in QCD

This may indicate that γ_{cusp} coincide while the difference is absorbed by charge renormalization

This may be because SUSY is broken by construction
(the presence of a cusp)