Exercises from the book

15.3, 15.4

Darcy's Law for flows in a Hele-Shaw Cell

The Navier-Stokes equation for an incompressible, isotropic and homogeneous fluid when the gravitational field is not important has the form

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\frac{1}{\rho_0}\nabla p + \frac{\mu}{\rho_0}\nabla^2 \mathbf{u}, \qquad \nabla \cdot \mathbf{u} = 0 \tag{1}$$

where ρ_0 is the fluid density and μ is the dynamic viscosity.

For a steady flow we have that $\partial \mathbf{u}/\partial t \approx 0$ and in the limit of $Re \ll 1$, the full Navier-Stokes equation reduces to the Stokes equation

$$\nabla^2 \mathbf{u} - \frac{1}{\mu}\nabla p = 0. \tag{2}$$

The flow is still assumed to satisfy the incompressibility condition,

$$\nabla \cdot \mathbf{u} = 0. \tag{3}$$

If we apply the operator $\nabla \cdot$ on both sides of Eq. (2) we have by the incompressibility that the pressure satisfies the Laplace equation

$$\nabla^2 p = 0 \tag{4}$$

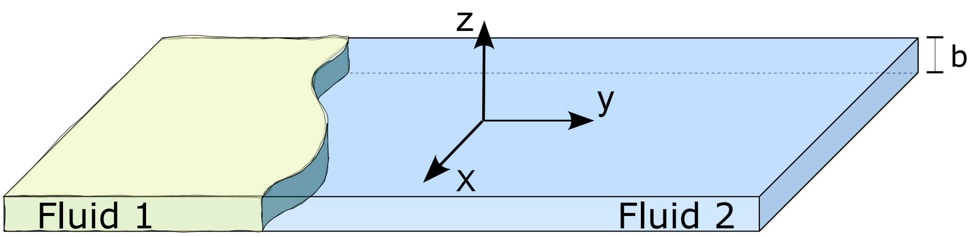

Figure 1: Hele-Shaw cell made from two plates separated by a gap of size b. The fluid 1 is injected into fluid 2 from the left.

A Hele-Shaw cell is a common experimental system in which a fluid is trapped in an infinitesimal small gap between two parallel flat plates. The cell is assumed to be located in the x-y plane and to be of thickness b in the z-direction. We shall now consider an experiment where the cell contains two immiscible fluids separated by an interface located at a position $y = h(x,t)$. One fluid is located at $y < h(x,t)$ and the other at $y > h(x,t)$, see Fig. 1.

If the thickness b is sufficiently small, one can derive a set of approximate equations for the flow in the Hele-Shaw cell. We do this by averaging the flow inside the gap of the cell. The averaged incompressibility equation can be written on the form

$$\frac{1}{b}\int_0^b \nabla\cdot\mathbf{u}\ \mathrm{dz} = \partial_x\tilde{u}_x + \partial_y\tilde{u}_y + (u_z|_{z=b} - u_z|_{z=0}) = \partial_x\tilde{u}_x + \partial_y\tilde{u}_y. \tag{5}$$

We have here used the assumption that the velocity of the fluid vanishes at the boundaries of the cell. Since all the velocity components vanish at the boundaries, we expect that the maximum flow rate is attained in the middle of the gap between the two plates. It is therefore assumed that the velocity components can be approximated by a parabolic shape in the z-coordinate

$$\mathbf{u}(x,y,z) = -\mathbf{v}(x,y)z(z-b) \tag{6}$$

where $\mathbf{v}(x,y)$ is a gap averaged strength.

In general the change in fluid flow velocity is predominant in the z-direction, where it changes over an infinitesimal thickness from zero at the boundaries to a maximal flow between the plates. Therefore the Laplacian term in Eq. (2) is almost entirely given by the second order derivate with respect to z alone i.e. $\nabla^2\mathbf{u} \approx \partial_z\partial_z\mathbf{u}$. It then follows by inserting Eq. (6) in Eq. (2) that

$$\mathbf{v} = -\frac{1}{2\mu}\nabla p. \tag{7}$$

Performing a gap average of the velocity component u_j we end up with Darcy's law

$$\tilde{u}_j = \frac{1}{b}\int_0^b u_j(x,y,z)\ \mathrm{dz} = \frac{v_j b^2}{6} = -\frac{b^2}{12\mu}\partial_j p \tag{8}$$

In the gap averaged quantities we therefore end up with the following set of equations (the three dimensional problem is now reduced to a two dimensional problem of a flow in a plane)

$$\nabla^2 p = 0, \qquad \tilde{\mathbf{u}} = -\frac{b^2}{12\mu}\nabla p \tag{9}$$

Solution to the pressure field for a flat interface

We now consider the case where the fluids move with a steady velocity and form an interface which at a time t is located at $y = h(x,t) = V_0 t$, i.e. the fluids are driven by an appropriate external pressure gradient applied at the remote boundaries $y \to \pm\infty$.

In both fluids Eq. (9) must be satisfied, i.e.

$$\nabla^2 p_1 = 0 \qquad \text{and} \qquad \nabla^2 p_2 = 0. \tag{10}$$

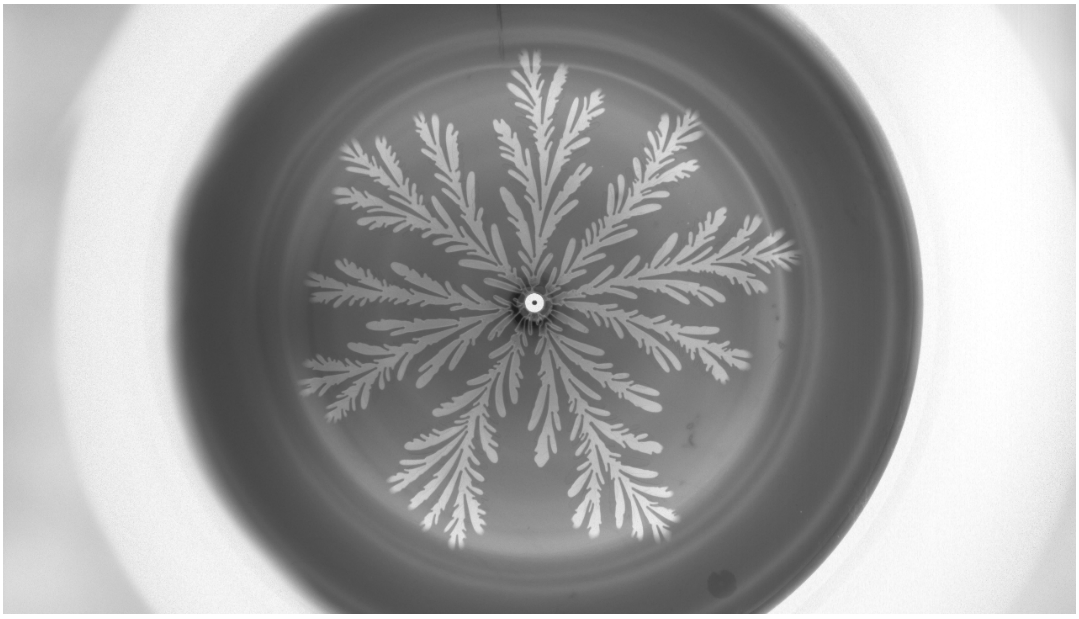

Figure 2: Experimental setup where air is injected into a viscous fluid from a small aperture in the middle of a Hele-Shaw cell. As the viscous fluid is displaced by the air a characteristic ramified pattern emerge.

Moreover, the interface must stay coherent, that is the normal velocities in both fluids when approaching the interface are identical. In addition, the pressure is continuous across the interface[1] i.e.

$$V_0 = -\frac{b^2}{12\mu_1}(\mathbf{n}\cdot\nabla p_1)|_{y\to h^-} = -\frac{b^2}{12\mu_2}(\mathbf{n}\cdot\nabla p_2)|_{y\to h^+} \qquad \text{and} \qquad p_1|_{y\to h^-} = p_2|_{y\to h^+}. \tag{11}$$

Here we have considered a normal vector pointing in the same direction for both fluids at the interface.

Problem 1: For a flat interface separating the two fluids, find solutions to Eq. (10) that satisfy the boundary conditions in Eq. (11).

Saffman-Taylor instability

Linear stability analysis

We shall now find the solution to the Laplace equation when the flat surface is perturbed by a small amplitude function $\epsilon(t)h(x)$ where $h(x,t) = V_0 t + \epsilon(t)h(x)$ and $\epsilon(t) \ll 1$, that

[1] That is, there is no surface tension at the interface

is, the new field satisfying the Laplace equation is going to be written in terms of an expansion around the solution to the flat interface. Formally the solution is written as

$$p(x,y) = p^{(0)}(x,y) + \epsilon(t)p^{(1)}(x,y) + \mathcal{O}(\epsilon(t)^2), \tag{12}$$

Evaluated at a point on the interface $y = V_0 t + \epsilon(t)h(x)$, we can expand this expression to linear order in ϵ (we omit the argument t of $\epsilon(t)$ to make the following expressions more readable)

$$p(x, V_0 t + \epsilon h(x)) = p^{(0)}(x, V_0 t) + \epsilon h(x)\partial_y p^{(0)}(x,y)|_{y=V_0 t} + \epsilon p^{(1)}(x, V_0 t) + \mathcal{O}(\epsilon^2). \tag{13}$$

Zero order terms: For a flat interface separating the two phases the equation for the harmonic potential or the pressure is given by the translational invariant (in the x-direction) solution computed in Problem 1.

Problem 2: The first order corrections are determined by Fourier transforming the perturbation, $h(x) = \int dk\tilde{h}(k)e^{ikx}$ and $p^{(1)}(x,y) = \int dk\tilde{p}^{(1)}(k,y)e^{ikx}$. Fourier transform the Laplace equations in Eq.(10) with respect to x and find the solutions in fourier space for p_1 and p_2 as function of k and y.

From the problems 1 and 2 you should end up with combined solutions for the zero and first order terms on the form

$$\tilde{p}_1(k,y,t) = -\frac{12\mu_1 V_0}{b^2}y + k_1(t) + \epsilon A_1(k)e^{k(y-V_0 t)} + \mathcal{O}(\epsilon^2) \tag{14}$$

and

$$\tilde{p}_2(k,y,t) = -\frac{12\mu_2 V_0}{b^2}y + k_2(t) + \epsilon A_2(k)e^{-k(y-V_0 t)} + \mathcal{O}(\epsilon^2) \tag{15}$$

The growth of the amplitude of the perturbation is given by the growth in the normal direction of the surface where the surface has a normal $(n_x, n_y) = (-\epsilon h', 1)/\sqrt{1+(\epsilon h')^2}$ and a tangent $(t_x, t_y) = (1, \epsilon h')/\sqrt{1+(\epsilon h')^2}$ vector, respectively. Note that we have a contributions to the surface growth from both the zero and first order terms. The normal velocity of the interface is given by an expression

$$V_n = n_x v_x + n_y v_y = v_y = \partial_t h(x,t) = V_0 + h(x)\partial_t \epsilon(t). \tag{16}$$

Note that $n_x v_x$ is at least $\mathcal{O}(\epsilon^2)$ since n_x is $\mathcal{O}(\epsilon)$ and v_x has no zero order term since there is no x-dependence in the zero order pressure solutions to Eq. (10).

Problem 3: Show that the normal velocity up to first order is given by

$$V_n = V_0 - \epsilon(t)\frac{b^2}{12\mu_i}\partial_y p_i^{(1)} + \mathcal{O}(\epsilon^2). \tag{17}$$

If we now assume that $\epsilon(t) = \epsilon_0 \exp(\omega t)$ and use Darcy's law we have for the first order terms that (we have here compared the zero and first order terms and do only show the first order equation)

$$-\frac{b^2}{12\mu_1}\partial_y \tilde{p}_1^{(1)}(k,y)|_{y=V_0 t} = -\frac{b^2}{12\mu_1}kA_1(k) = \omega\tilde{h}(k) \tag{18}$$

and

$$-\frac{b^2}{12\mu_2}\partial_y \tilde{p}_2^{(1)}(k,y)|_{y=V_0 t} = \frac{b^2}{12\mu_2} k A_2(k) = \omega \tilde{h}(k) \tag{19}$$

In both these equations, the latter equality sign follows from the first order term in the right hand expression of Eq. (16).

Problem 4: We require continuity of the pressure field across the interface, which must be valid for both the zero and first order terms independently. For the first order terms using Eq. (13) balance the pressures on both sides of the interface and use that together with Eqs. (18) and (19) to show that the growth rate for the perturbation is given by

$$\omega = kV_0 \frac{\mu_2 - \mu_1}{\mu_1 + \mu_2}$$

Problem 5: Is the interface between two immiscible fluids stable when a more viscous fluid is displacing a less viscous fluid? Draw a stability diagram showing regions as function of the viscosity of the two fluid where the interface is stable and unstable, respectively (consider different velocities V_0). What would the pattern in Fig. 2 look like, if the same fluid was injected into air?