1 Steady Hagen-Poiseuille Flow

We consider a pipe containing an incompressible Newtonian fluid, as shown in figure 1. The flow is driven by a uniform body force (force per unit volume) along the symmetry axis, generated by imposing a pressure at the inlet. This is known as Hagen-Poiseuille flow, named after the two scientists who solved the problem experimentally in the 19th century. It is one of the few theoretical viscous analysis that can be carried out analytically.

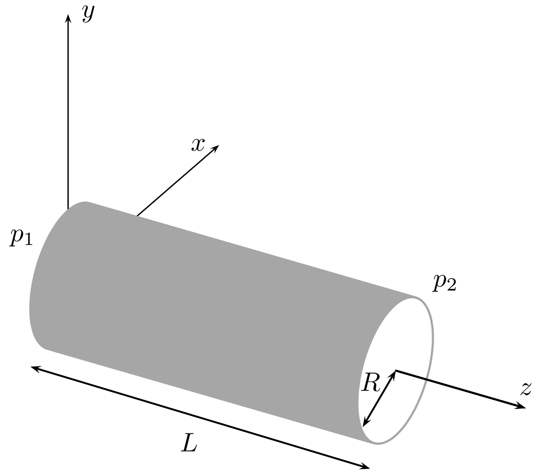

Figure 1: A pipe with constant radius R of length L contains a Newtonian fluid with constant density ρ, dynamic viscosity μ and kinematic viscosity $\nu = \mu/\rho$. The pressure at the inlet (outlet) is denoted p_1 (p_2). Gravitational effects are neglected.

We are interested in finding the steady-state laminar flow field (Re $\lesssim$ 2000) and pressure, so we assume a long pipe ($L \gg R$) such that the velocity profile is purely axial, $v_r = v_\theta = 0$. The governing equations in polar coordinates for our axisymmetric system ($\partial_\theta = 0$) thus read

$$\partial_z v_z = 0 \qquad \text{(continuity)} \tag{1.1}$$

$$\partial_r p = 0 \qquad \text{(r-momentum, N.S.)} \tag{1.2}$$

$$-\partial_z p + \mu \nabla^2 v_z = \rho v_z \partial_z v_z \qquad \text{(z-momentum, N.S.)} \tag{1.3}$$

with no-slip boundary condition $v_z(R) = 0$. The incompressiblity gives us that $v_z = v_z(r)$ and from (1.2) we conclude that $p = p(z)$, so (1.3) becomes

$$-\partial_z p + \mu(\partial_{rr} v_z + r^{-1}\partial_r v_z) = 0. \tag{1.4}$$

The last equation (1.4) balances the pressure force and the viscous damping in our system.

Question 1 Apply ∂_z to (1.4) and find an expression for $p(z)$.

Question 2 Use the expression for $p(z)$ in the governing equation (1.4) and solve the resulting ODE for $v_z(r)$. Comment on the shape and its properties. How about viscosity, do we see the effects of that?

Question 3 Using the expression for $v_z(r)$, find the mass flow rate $\dot{m} = \int_A \rho \mathbf{v}_z \cdot \mathrm{d}\mathbf{A}$ and the shear stress. Does the latter behave as we expect?

2 Unsteady Hagen-Poiseuille Flow

Having found the steady-state value, we now want to do a full analysis of the flow and assume it is initially at rest. A constant pressure p_1 is imposed at the inlet at $t = 0$, which sets the fluid in motion.

The only change to the governing equations is that we need to add the time derivative to (1.3), so we now have

$$\rho \partial_t v_z = -\partial_z p + \mu \nabla^2 v_z \tag{2.1}$$

$$= \mathcal{P} + \mu \frac{1}{r} \partial_r (r \partial_r v_z) \tag{2.2}$$

subject to the no-slip boundary conditions

$$v_z(R,t) = 0 \quad \text{for all } t > 0 \tag{2.3}$$

$$v_z(r,t) = 0 \quad \text{for all } t \leq 0 \tag{2.4}$$

This is an initial boundary value problem involving a Laplacian with axisymmetric geometry and the Hankel transform is a useful tool for such systems. Specifically, if $v_z(r)$ satisfies the Dirichlet conditions in some closed closed interval $[0, R]$, then its finite Hankel transform of zero order and its inverse are defined by[1]

$$v_z^*(k_i,t) = \mathcal{H}_0[v_z(r,t)](k_i,t) = \int_0^R \mathrm{d}r\, v_z(r,t) r J_0(k_i r) \tag{2.5}$$

$$v_z(r,t) = \frac{2}{R^2} \sum_{i=1}^{\infty} v_z^*(k_i,t) \frac{J_0(k_i r)}{J_1^2(k_i R)}, \quad 0 \leq r \leq R \tag{2.6}$$

where J_i is the ith order Bessel function of first kind and k_i the positive roots of $J_0(k_i R) = 0$. The usefulness of (2.5) to our problem ultimately comes from the standard recurrence relation for Bessel functions, all of which is outlined in appendix A.

Question 4 Use the main result (A.10), $\mathcal{H}_0[r^{-1}(rf')'](k) = -k^2 \mathcal{H}_0[f](k)$ to write (2.2) in k_i-space. Solve the resulting ODE (e.g., by Laplace transforming it). You should get

$$v_z^*(k_i,t) = \frac{\mathcal{P} R J_1(k_i R)}{\mu k_i^3} \left(1 - e^{-k_i \nu t}\right). \tag{2.7}$$

[1] Just like the Fourier transform has the sinusoidal kernel $\exp(i\omega t)$, the kernel of the Hankel transform consists of the complete set of orthogonal Bessel functions.

Question 5 We now have an expression for $v_z^*(k_i,t)$. Transform it back to real-space using (2.6). Use that

$$R^2 - r^2 = \frac{8}{R}\sum_{i=1}^{\infty} k_i^{-3}\frac{J_0(k_i r)}{J_1(k_i R)}. \tag{2.8}$$

Comment on the final expression. How is this similar to a diffusion process?

Question 6 Rescale the solution with the characteristic parameters R and $v_{z,\max}$ and plot it. It should look like figure 2.

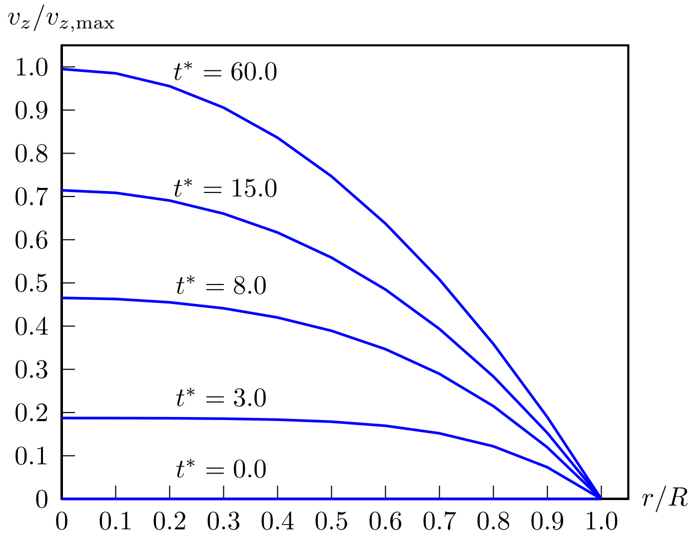

Figure 2: Scaled velocity profile of a laminar flow with Re = 64 at different scaled times, $t^* = t/t_c$, where $t_c = R/v_{z,\max}$ is the characteristic time.

A Hankel Transform of Axisymmetric Laplacian

In this appendix we will briefly touch upon the general properties of the Hankel transform that enables us to solve our axisymmetric problem. We first look at the νth order Hankel transform of differentials, namely (a prime denotes the derivative wrt. r)

$$\mathcal{H}_\nu[r^{\nu-1}(r^{1-\nu}f(r))'](k) = \int_0^\infty \mathrm{d}r\, r^\nu (r^{1-\nu}f(r))' J_\nu(kr) \tag{A.1}$$

$$= [rf(r)J_\nu(kr)]_0^\infty - \int_0^\infty \mathrm{d}r\, r^{1-\nu} f(r)[r^\nu J_\nu(kr)]' \tag{A.2}$$

for $\nu \geq -1/2$. The first term vanishes as $r \to \infty$, because the existence of the Hankel transform requires that $rf(r) \to 0$ in this limit (in our finite case the no-slip boundary condition ensures that the term vanishes at the upper limit). The lower limit is trivially satisfied for well-behaved functions, so the first term vanishes all together. We proceed by invoking the standard recurrence relation for Bessel functions

$$[r^\nu J_\nu(kr)]' = kr^\nu J_{\nu-1}(kr), \tag{A.3}$$

which gives us

$$\mathcal{H}_\nu[r^{\nu-1}(r^{1-\nu}f(r))'](k) = -k\int_0^\infty \mathrm{d}r\, rf(r)J_{\nu-1}(kr) \tag{A.4}$$

$$= -k\mathcal{H}_{\nu-1}[f(r)](r). \tag{A.5}$$

Taking $\nu = 1$ we end up with the identity

$$\boxed{\mathcal{H}_1[f'(r)](k) = -k\mathcal{H}_0[f(r)](k).} \tag{A.6}$$

Following this approach once again together with the relation $[r^{-\nu}J_\nu(kr)]' = -kr^{-\nu}J_{\nu+1}(kr)$ yields

$$\mathcal{H}_\nu[r^{-1-\nu}(r^{1+\nu}f(r))'](k) = k\mathcal{H}_{1+\nu}[f(r)](r) \tag{A.7}$$

and taking $\nu = 0$ yields a second identity

$$\boxed{\mathcal{H}_0[r^{-1}(rf(r))'](k) = k\mathcal{H}_1[f(r)](k).} \tag{A.8}$$

These two relations enable us to Hankel transform the Laplacian in (2.2),

$$\mathcal{H}_0[r^{-1}(rf'(r))'](k) = k\mathcal{H}_1[f'(r)](k) \tag{A.9}$$

$$= -k^2\mathcal{H}_0[f(r)](k). \tag{A.10}$$