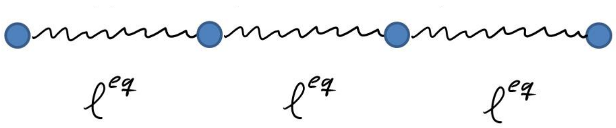
Exercises Week 6: Linear elasticity from harmonic oscillators



Consider a sequence of connected harmonic springs with point masses m and equilibrium lengths ℓ^{eq}. Initially, when no force is applied at the outer edges, the masses are assumed to be located at equilibrium positions x_i, where $x_{i+1} = x_i + \ell^{eq}$.

Assume that a force F is applied to the system and that the system has reached a new equilibrium configuration where the masses are separated by distances $\ell - \ell^{eq} = F/k$ (The spring constants are assumed to be the same for all springs).

1) What is the new spring constant k^L when $N = L^{eq}/\ell^{eq}$ springs are connected? and what is relation between $L - L^{eq}$ and $\ell - \ell^{eq}$?

If there is no dissipation in the system the springs will continue to vibrate and the position of the point masses at a time t will be given by a displacement $u(x_i, t)$ from the original position x_i,

$$x_i \mapsto x_i + u(x_i, t). \tag{1}$$

2) Use Newton's second law to write down an equation of motion in $u(x_i, t)$ for the individual point masses.
3) Show that the equations of motion can be written as a matrix equation

$$\ddot{\mathbf{u}} = \mathbf{A}\mathbf{u}, \tag{2}$$

where $u_i = u(x_i, t)$ and the matrix is of size N by N and will have a tri-block structure. Find the eigenvectors and eigenvalues of the matrix (numerically) and show how the leading eigenvectors (those with the largest eigenvalues) look like for $N = 3, 10, 100, \ldots$.

If we now assume that $\ell^{eq} \ll 1$ (or more precisely, we should assume that u varies slowly on the scale of ℓ^{eq}), we can write a continuum version of the equation of motion.

4) Derive a continuum version of the equation of motion. In the derivation you can convert the mass to a density by assuming that the springs has a fixed cross-sectional area.
5) From the continuum equation find the velocity of the elastic waves.

Exercises from the book

6.5, 6.6, 6.8