Continuum Mechanics 2016:
Solutions to Exercise Set 1

Exercise 1: (Linear elasticity from harmonic oscillators)

Solution:

(a) The new spring constant k_N is defined implicitly through

$$F = k_N(L - L_{\text{eq}}). \tag{1}$$

We have $\ell - \ell_{\text{eq}} = F/k$ and $N = L_{\text{eq}}/\ell_{\text{eq}} = L/\ell$, so that

$$L - L_{\text{eq}} = N(\ell - \ell_{\text{eq}}) = \frac{NF}{k}. \tag{2}$$

Upon comparing eqs. (1) and (2), we find

$$k_N = \frac{k}{N}. \tag{3}$$

(b) We denote the displacements $u_i \overset{\text{def}}{=} u(x_i, t)$. Newton's second law, where f_i denotes the sum of forces on particle i, yields

$$m\ddot{u}_i = f_i = k\left((u_{i+1} - u_i) - (u_i - u_{i-1})\right). \tag{4}$$

Thus,

$$\ddot{u}(x_i, t) = \frac{k}{m}\left(u(x_{i+1}, t) - 2u(x_i, t) - u(x_{i-1}, t)\right). \tag{5}$$

(c) We have $x_{i\pm1} = x_i \pm \ell_{\text{eq}}$. Taylor expansion of $u(x_{i\pm1}, t)$ yields

$$u(x_i \pm \ell_{\text{eq}}, t) = u(x_i, t) \pm \ell_{\text{eq}} \frac{\partial u(x,t)}{\partial x}\bigg|_{x=x_i} + \frac{\ell_{\text{eq}}^2}{2}\frac{\partial^2 u(x,t)}{\partial x^2}\bigg|_{x=x_i} + \mathcal{O}(\ell_{\text{eq}}^3). \tag{6}$$

Now, since this holds $\forall x_i$, we let $x_i \to x$, and since x is now also a continuous variable, the time derivative in eq. (5) becomes partial. Inserting eq. (6) in eq. (5) and omitting the $\mathcal{O}(\ell_{\text{eq}}^3)$ terms since $\ell_{\text{eq}} \ll 1$, we obtain

$$\frac{\partial^2 u}{\partial t^2} = \frac{k\ell_{\text{eq}}^2}{m}\frac{\partial^2 u}{\partial x^2}. \tag{7}$$

(d) This the wave equation with the wave velocity $v = \ell_{\text{eq}}\sqrt{k/m}$. Realize this by inserting the general solution $u(x,t) = \tilde{u}(x \pm vt)$.

Exercise 2: (6.5 from Lautrup [1]) Find the eigenvalues λ_i and eigenvectors $\mathbf{v}_i$ of

$$\boldsymbol{\sigma} = \begin{bmatrix} \tau & \tau & \tau \\ \tau & \tau & \tau \\ \tau & \tau & \tau \end{bmatrix}. \tag{8}$$

Solution: Straightforward calculation gives the eigenvalues $\lambda_1 = \lambda_2 = 0$ and $\lambda_3 = 3\tau$, and the corresponding eigenvectors $\mathbf{v}_1 = [1, -1, 0]$, $\mathbf{v}_2 = [1, 0, -1]$ (or any linear combination) and $\mathbf{v}_3 = [1, 1, 1]$. *Normalized* eigenvectors (as given in the book) are obtained by $\hat{\mathbf{v}}_i = \mathbf{v}_i/|\mathbf{v}_i|$.

Exercise 3: (6.6 from Lautrup [1]) *Alternative formulation:* Show that a stress tensor that is diagonal in all coordinate systems, have identical nonzero entries.

Solution: Let the stress tensor be given by $\boldsymbol{\sigma} = \operatorname{diag}(\sigma_x, \sigma_y, \sigma_z)$ in the initial coordinate system. Consider a rotation by an angle θ about some fixed axis, and assume first that this is the z-axis. The associated rotation matrix is given by

$$\mathbf{R} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{9}$$

The stress tensor in the rotated coordinate system is given by

$$\boldsymbol{\sigma}' = \mathbf{R}\boldsymbol{\sigma}\mathbf{R}^\top \tag{10}$$

$$= \begin{bmatrix} \sigma_x \cos^2\theta + \sigma_y \sin^2\theta & (\sigma_x - \sigma_y)\sin\theta\cos\theta & 0 \\ (\sigma_x - \sigma_y)\sin\theta\cos\theta & \sigma_x \sin^2\theta + \sigma_y\cos^2\theta & 0 \\ 0 & 0 & \sigma_z \end{bmatrix} \tag{11}$$

This should also be diagonal, $\boldsymbol{\sigma}' = \operatorname{diag}(\sigma_{x'}, \sigma_{y'}, \sigma_{z'})$. Therefore the off-diagonal entries must be zero $\forall\theta$, and so $\sigma_x = \sigma_y$. By symmetry, this argument applies to rotation about both the x- and y-axes as well, and thus $\sigma_x = \sigma_y = \sigma_z = \operatorname{tr}(\boldsymbol{\sigma})/3 = -p$.

Exercise 4: (6.8 from Lautrup [1])

(a) Show that the average of a unit vector $\mathbf{n}$ over all directions obeys

$$\langle n_i n_j \rangle = \frac{1}{3}\delta_{ij}. \tag{12}$$

(b) Use this to show that the average of the normal stress acting on a surface element is (minus) the mechanical pressure.

Solution:

(a)
- **Method 1:** Brute force it. A unit vector $\mathbf{n} = \mathbf{n}(\theta, \phi)$ is expressed by

$$\mathbf{n} = \begin{bmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{bmatrix}. \tag{13}$$

  Hence,

$$\mathbf{n}\otimes\mathbf{n} = \begin{bmatrix} \sin^2\theta\cos^2\phi & & \text{(sym.)} \\ \sin^2\theta\sin\phi\cos\phi & \sin^2\theta\sin^2\phi & \\ \sin\theta\cos\theta\cos\phi & \sin\theta\cos\theta\cos\phi & \cos^2\theta \end{bmatrix}. \tag{14}$$

  Carrying out all six integrals (straightforward, but cumbersome!) over all angles (θ, ϕ) and dividing by 4π, we get $\langle \mathbf{n}\otimes\mathbf{n}\rangle = \frac{1}{3}\mathbf{I}$, or eq. (12).
- **Method 2:** Use the insight from Exercise 3. The *average* of a quantity *over all directions* can not itself depend on the direction you evaluate it in. Using our acquired insight, we can write it on the form

$$\langle n_i n_j \rangle = k\delta_{ij}, \tag{15}$$

  where k is an undetermined constant. Taking the trace of both sides yields

$$3k = \langle n_i n_i \rangle = 1 \quad \Longrightarrow \quad k = \frac{1}{3}, \tag{16}$$

  which gives eq. (12).

(b) The normal force on a surface element is given by the traction at the surface, $T_i^{(\mathbf{n})} = \sigma_{ij} n_j$, projected in the normal direction: $f^{(\mathbf{n})} = n_i T_i^{(\mathbf{n})} = n_i \sigma_{ij} n_j$. Taking the average,

$$\left\langle f^{(\mathbf{n})} \right\rangle = \langle n_i \sigma_{ij} n_j \rangle = \langle n_i n_j \rangle \sigma_{ij} = \frac{1}{3}\delta_{ij}\sigma_{ij} = \frac{\sigma_{ii}}{3} = -p. \tag{17}$$

Here we have used that σ_{ij} is constant, and the definition of the (mechanical) pressure, $p = -\operatorname{tr}(\boldsymbol{\sigma})/3 = -\sigma_{ii}/3$.

[1] B. Lautrup. *Physics of Continuous Matter: Exotic and Everyday Phenomena in the Macroscopic World.* CRC Press, second edition, 2011.