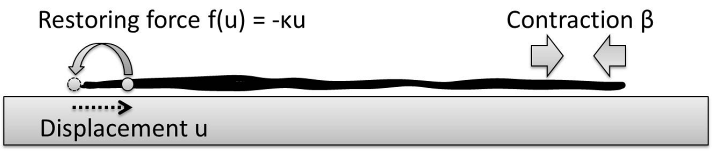
Exercises Week 7

Exercises from the book

7.3, 7.8, 8.5

Mud cracks



We shall here consider a simple 1D model of mud cracks. Mud cracks typically form when drying mud contracts. For the model we imagine a setup where the contraction is restrained by a contact force between the mud and a rigid bottom layer (see figure). As the mud dries and contracts a stress σ_{xx} builds up in the mud. When σ_{xx} exceeds some yield stress σ_Y the mud fractures. In two dimensions the relation between stress and strain can be written on the form

$$\begin{aligned} u_{xx} &= (\sigma_{xx} - \nu\sigma_{yy})/E + \beta \\ u_{yy} &= (\sigma_{yy} - \nu\sigma_{xx})/E + \beta \\ u_{xy} &= (1+\nu)\sigma_{xy}/E \end{aligned}$$

where β is the contraction of the mud. β is similar to a strain and the system is stress free whenever $\beta = u_{xx} = u_{yy}$ (why?).

- How would the stress and strain be related in 1D?
- The restraining force acts in a direction opposite to the displacement u_x and is assumed to have a linear form $f(u_x) = -\kappa u_x$. Write down the force balance equation in 1D for the system.
- Formulate the force balance equation in terms of the strain and solve this equation for the strain. Use the solution to find the maximum stress in the mud as function of the length L of the mud layer.
- Discuss how the maximum stress depends on the contraction β and how the maximum stable length L of the mud depends on σ_Y.

Airy stress function

The Airy stress function $\Phi(x, y)$ (exercise 8.5) satisfies the bi-Laplace equation

$$\Delta^2\Phi(x, y) = 0,$$

where $\Delta \equiv \partial_x^2 + \partial_y^2$. Show that we can write this equation with the help from complex variables on the form

$$\partial_z^2 \partial_{\bar{z}}^2 \Phi(z, \bar{z}) = 0,$$

where $z = x + iy$ and $\bar{z} = x - iy$. Show that we can write a general solution to this equation on the form

$$\Phi(z, \bar{z}) = \mathrm{Re}\Big(\bar{z}\varphi(z) + \eta(z)\Big)$$

where the right hand side is the real part of a combination of two analytic functions φ and η. This form of the solution is very convenient for calculating the stresses under plane deformations.

Stresses in polar coordinates

Using the Airy stress function (exercise 8.5), find expressions for σ_{rr}, $\sigma_{r\theta}$ and $\sigma_{\theta\theta}$ as derivatives of the stress function in polar coordinates (r, θ).