Mud Cracks

① $$u_{xx} = \frac{\sigma_{xx}}{E} + \beta$$

②
$$\frac{\partial \sigma_{xx}}{\partial x} - K u_x = 0$$

③
$$\frac{\partial^2 \sigma_{xx}}{\partial x^2} - K u_{xx} = 0$$

Use ①

$$\frac{\partial^2 u_{xx}}{\partial x^2} - \frac{K}{E} u_{xx} = 0$$

Solution

$$u_{xx} = A \cosh \sqrt{\frac{k}{E}}\, x + B \sinh \sqrt{\frac{k}{E}}\, x$$

Use boundary condition $\sigma_{xx}\left(\pm \frac{L}{2}\right) = 0$

$$u_{xx}\left(\pm\frac{L}{2}\right) = \beta$$

$$u_{xx} = \beta \frac{\cosh\sqrt{\frac{k}{E}}\,x}{\cosh\sqrt{\frac{k}{E}}\,\frac{L}{2}}$$

Maximum stress occurs at $x = 0$

$$|\sigma(x=0)| = E\left|\left(u_{xx}(x=0) - \beta\right)\right|$$

$$= E|\beta|\left(1 - \frac{1}{\cosh\sqrt{\frac{k}{E}}\,\frac{L}{2}}\right)$$

(4) Maximum stress linear in β and maximum stable L can be found from the condition

$$\sigma_{max}(L) = \sigma_Y$$