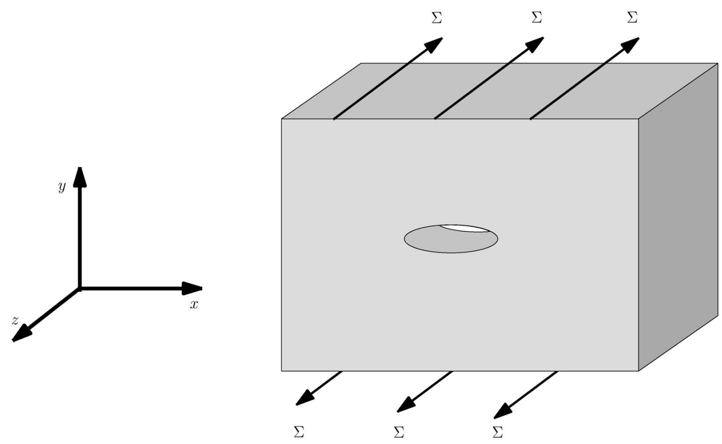
Exercises Week 8

Exercises from the book

9.1, 9.8

Fractures by anti-plane shear stress



We shall now consider the static stress around a small elliptical crack in an infinite elastic medium deformed by an anti-plane shear stress. The crack and the loading configuration is shown in the figure where at infinity a shear stress is applied, $\sigma_{zy} \to -\Sigma$ for $|y| \to \infty$. Note that the crack extends indefinitely in the z-direction. Under these conditions the only nonzero displacement is $u_z(x, y)$, where u_z does not depend on z.

1) Show that it then follows from the Navier-Cauchy equation that u_z is a harmonic function,
$$\nabla^2 u_z = 0$$

Any harmonic function can written as the real part of a complex analytic function $\psi(\omega)$,

$$u_z(\omega) = \text{Re}[\Psi(\omega)] = \frac{1}{2}(\ \Psi(\omega) + \overline{\Psi(\omega)}\ ) \tag{1}$$

where $\omega = x + iy$ and a bar over a function/variable refers to complex conjugation, $\bar{\omega} = x - iy$.

2) Show that the shear strain is given by the following expression in the complex variables
$$u_{zy} = \frac{i}{2}\left(\frac{\partial \Psi}{\partial \omega} - \frac{\partial \bar{\Psi}}{\partial \bar{\omega}}\right)$$

3) From Hooke's law, we know that $\sigma_{zy} = 2\mu u_{zy}$. In order to satisfy the boundary condition at infinity, show that, for $|y| \to \infty$, Ψ must have an asymptotic form

$$\Psi(\omega) = \frac{i\Sigma\omega}{\mu} .$$

Let us assume that the elliptical crack is described by a curve parametrized by a variable s,

$$\begin{pmatrix} x(s) \\ y(s) \end{pmatrix}$$

4) The surface of the crack cannot support a normal stress. Show that this boundary condition ($\sigma \cdot \mathbf{n}$ - where $\mathbf{n}$ is a surface normal) is equivalent to the following condition

$$\frac{\partial u_z}{\partial x}\frac{\partial y}{\partial s} - \frac{\partial u_z}{\partial y}\frac{\partial x}{\partial s} = 0 \tag{2}$$

5) Show Eq. (2) can be written in following form using the complex variables formulation in Eq. (1)

$$\frac{\partial \Psi}{\partial s} = \frac{\partial \bar{\Psi}}{\partial s}$$

We can then deduce that up to a constant the following equality most hold on the boundary of the crack $\Psi = \bar{\Psi}$. If now consider the crack to be circular and coincide with the unit circle (i.e. the crack has a unit radius), we can show that

$$\Psi(w) = \frac{i\Sigma}{\mu}(w - 1/w) \tag{3}$$

6) Argue why Eq.(3) has this form.

A function $w = \Phi(\xi)$ is a conformal mapping of a complex domain if and only if it is analytic and has a non-zero derivative. If u_z is harmonic, it follows that $u_z(\Phi(\xi))$ is harmonic too since the Laplace equation is conformally invariant

$$0 = \partial_\xi \partial_{\bar{\xi}}\, u_z[\Phi(\xi)] = |\Phi'(\xi)|^2 \partial_w \partial_{\bar{w}}\, u_z(w),$$

since $\Phi' \neq 0$.

7) Show that the mapping

$$z = f(w) = \frac{1}{2}\left(w + \frac{\alpha}{w}\right) \tag{4}$$

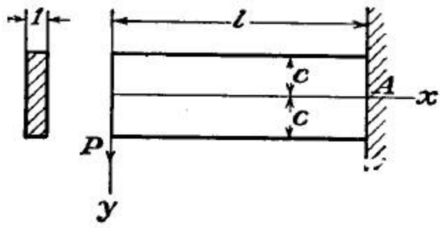
maps the unit circle to an ellipse.

8) For $\alpha = 1$, show that the circle is mapped by $f(z)$ to a slit extending from $x = -1$ to $x = 1$ (a thin crack).

If we now set $\Phi(z) = f^{-1}(z)$ we can find a solution for u_z around the crack.

10) Show that the boundary condition $\mathbf{n} \cdot \nabla u_z = 0$ is conformal invariant and use this together with the conformal invariance of the harmonic equation to find a general solution for u_z around the thin crack.
11) Show that the stress diverges as the reciprocal square root as we approach one of the tips of the crack.

Bending of a cantilever



Consider the bending of a cantilever with a narrow rectangular cross section of unit width (see figure). The upper and lower edges at $y = \pm c$ are free from load and shearing forces. At the end of the cantilever a resultant force P is distributed such that

$$-\int_{-c}^{c} \sigma_{xy} dy = P.$$

One way to find solutions to this type of problem is to guess on a polynomial form of the stress potential. Consider a stress potential on the form

$$\Phi(x, y) = b_2 xy + \frac{a_4}{12} x^4 + \frac{b_4}{6} x^3 y + \frac{c_4}{2} x^2 y^2 + \frac{d_4}{6} xy^3 + \frac{e_4}{12} y^4.$$

First find the conditions on the coefficients in order for the stress potential to satisfy the bi-Laplace equation and then afterwards the conditions on the coefficients that will satisfy the boundary conditions.