Week 8

Fractures by Anti-Plane Shear Stress

1. The Navier-Cauchy equation is,

$$\mathbf{f} + \mu\nabla^2\mathbf{u} + (\lambda+\mu)\nabla(\nabla.\mathbf{u}) = 0 \tag{1}$$

where $\mathbf{f}$ is a volumetric body force and $\mathbf{u}$ is the displacement vector. Since we are given that the only non zero displacement is

$$\mathbf{u} = u_z(x,y)\hat{z}$$

we immediately see that $\nabla.\mathbf{u} = 0$. We are also told that there are no body forces and therefore, $\mathbf{f} = 0$. Therefore, one can easily see that the Navier-Cauchy equation reduces to

$$\nabla^2\mathbf{u} = \frac{\partial^2 u_z(x,y)}{\partial z^2} = 0 \tag{2}$$

2. The following relations will be useful,

$$\frac{\partial w}{\partial y} = i \tag{3}$$

$$\frac{\partial \bar{w}}{\partial y} = -i \tag{4}$$

We see that,

$$u_{zy} = \partial_y u_z \tag{5}$$

$$= \frac{1}{2}\left(\frac{\partial\psi}{\partial y} + \frac{\partial\bar{\psi}}{\partial y}\right) \tag{6}$$

$$= \frac{1}{2}\left(\frac{\partial\psi}{\partial w}\frac{\partial w}{\partial y} + \frac{\partial\bar{\psi}}{\partial \bar{w}}\frac{\partial \bar{w}}{\partial y}\right) \tag{7}$$

$$= \frac{i}{4}\left(\frac{\partial\psi}{\partial w} - \frac{\partial\bar{\psi}}{\partial \bar{w}}\right) \tag{8}$$

where we have used (3) and (4) in going from (5) to (8).

3. We now integrate (8) with respect to w.

$$\int u_{zy}dw = \frac{i}{4}\left(\int\frac{\partial\psi}{\partial w}dw - \int\frac{\partial\bar{\psi}}{\partial \bar{w}}dw\right) \tag{9}$$

$$\int \frac{\sigma_{zy}}{2\mu}dw = \frac{i}{4}\left(\int\frac{\partial\psi}{\partial w}dw - \int\frac{\partial\bar{\psi}}{\partial w}\frac{\partial w}{\partial \bar{w}}dw\right) \tag{10}$$

$$\int \frac{\sigma_{zy}}{2\mu}dw = \frac{i}{4}\psi(w) \tag{11}$$

$$\frac{-\Sigma w}{i\mu} = \psi(w) \tag{12}$$

$$\psi(w) = \frac{i\Sigma w}{\mu} \tag{13}$$

In (10), we have used the fact that $\frac{\partial w}{\partial \bar{w}} = 0$. In going from (14) to (15), we have taken the asymptotic limit of the stress tensor component σ_{zy} at infinity while performing the integral over w.

4. In the parametrized form, the unit tangent vector can be represented as,

$$\hat{\mathbf{t}} = \frac{\partial\mathbf{r}}{\partial s} = \begin{bmatrix} \frac{\partial x}{\partial s} \\ \frac{\partial y}{\partial s} \end{bmatrix} \tag{14}$$

The unit normal should be orthogonal to this tangent vector and we can easily see that

$$\hat{\mathbf{n}} = \begin{bmatrix} -\frac{\partial y}{\partial s} \\ \frac{\partial x}{\partial s} \end{bmatrix} \tag{15}$$

satisifies $\hat{\mathbf{t}}.\hat{\mathbf{n}} = 0$. The no jump boundary condition is thus given by,

$$\begin{aligned} \sigma.\hat{\mathbf{n}} &= 0 \quad (16)\\ \sigma_{zx}n_x + \sigma_{zy}n_y &= 0 \quad (17)\\ \frac{\partial u_z}{\partial x}\frac{\partial y}{\partial s} - \frac{\partial u_z}{\partial y}\frac{\partial x}{\partial s} &= 0 \quad (18) \end{aligned}$$

where we have used the relations

$$\sigma_{zx} = 2\mu u_{zx} = 2\mu\partial_x u_z$$

$$\sigma_{zy} = 2\mu u_{zy} = 2\mu\partial_y u_z$$

5. Let us take the first term in (18). We rewrite the strain component as follows,

$$\begin{aligned} u_{zx} &= \frac{1}{2}\frac{\partial}{\partial x}\left[\psi(w) + \bar{\psi}(w)\right] \quad (19)\\ &= \frac{1}{2}\left[\frac{\partial\psi}{\partial w}\frac{\partial w}{\partial x} + \frac{\partial\bar{\psi}}{\partial\bar{w}}\frac{\partial\bar{w}}{\partial x}\right] \quad (20)\\ &= \frac{1}{2}\left[\frac{\partial\psi}{\partial w} + \frac{\partial\bar{\psi}}{\partial\bar{w}}\right] \quad (21)\\ &= \frac{1}{2}\left[\frac{\partial\psi}{\partial y}\frac{\partial y}{\partial w} + \frac{\partial\bar{\psi}}{\partial y}\frac{\partial y}{\partial\bar{w}}\right] \quad (22)\\ &= \frac{1}{2}\left[\frac{\partial\psi}{\partial y}\frac{-i}{2} + \frac{\partial\bar{\psi}}{\partial y}\frac{i}{2}\right] \quad (23)\\ &= \frac{i}{4}\left[-\frac{\partial\psi}{\partial y} + \frac{\partial\bar{\psi}}{\partial y}\right] \quad (24) \end{aligned}$$

Similarly we obtain,

$$u_{zy} = \frac{i}{4}\left[-\frac{\partial\psi}{\partial x} + \frac{\partial\bar{\psi}}{\partial x}\right] \quad (25)$$

Substituting (24) and (25) into (18), we find that

$$\frac{\partial\psi}{\partial s} = \frac{\partial\bar{\psi}}{\partial s} \quad (26)$$

6. Since it is fair to assume that $\psi = \bar{\psi}$, let us now consider a form of the analytic function Ψ as given below

$$\begin{aligned} \psi &= \frac{i\Sigma}{\mu}\left(w - \frac{1}{w}\right) \quad (27)\\ &= \frac{i\Sigma}{\mu}\left(w - \frac{w\bar{w}}{w}\right) \quad (28)\\ &= \frac{i\Sigma}{\mu}(w - \bar{w}) \quad (29) \end{aligned}$$

The complex conjugate of this function gives

$$\begin{aligned} \bar{\psi} &= \frac{-i\Sigma}{\mu}(\bar{w} - w) \quad (30)\\ &= \frac{i\Sigma}{\mu}(w - \bar{w}) \quad (31) \end{aligned}$$

This form of Ψ also satisfies the asymptotic limit trivially and so it is justified to assume this particular form.

7. The complex variable $w = re^{i\theta}$ in polar coordinates. In this case, we consider a unit circle in the complex plane and set $r = 1$. The mapping is then given by

$$\begin{aligned} z = f(w) &= \frac{1}{2}\left(w + \frac{\alpha}{w}\right) && (32) \\ &= \frac{1}{2}\left(e^{i\theta} + \alpha e^{-i\theta}\right) && (33) \\ &= \frac{1}{2}\left[(\cos\theta + i\sin\theta) + (\alpha\cos\theta - i\alpha\sin\theta)\right] && (34) \\ &= \frac{1}{2}\left[\cos\theta(\alpha+1) - i\sin\theta(\alpha-1)\right] && (35) \end{aligned}$$

(35) is the equation of an ellipse with major axis $2a = (\alpha+1)$ and minor axis $2b = (\alpha-1)$

8. When $\alpha = 1$, we find that (35) reduces to,

$$z = f(w) = 1.\cos\theta \tag{36}$$

When $\theta = 0$ and $\theta = \pi$,

$$z = 1.\cos 0 = 1$$

$$z = 1.\cos\pi = -1$$

When $\theta = \frac{\pi}{2}$ and $\theta = \frac{3\pi}{2}$,

$$z = 1.\cos\frac{\pi}{2} = 1.\cos\frac{3\pi}{2} = 0$$

9. We are asked to show that the boundary condition given by $\boldsymbol{n}.\nabla u_z = 0$ is conformal invariant between the w and the z domains under the conformal mapping given by

$$z = f(w) = \frac{1}{2}\left(w + \frac{1}{w}\right)$$

which has the inverse mapping $w = \Phi(z)$. This means that we mush show,

$$(\boldsymbol{n}.\nabla u_z)_w = 0 \rightarrow (\boldsymbol{n}.\nabla u_z)_z = 0 \tag{37}$$

modulo some analytic non-zero function which in this case is some function of the mapping itself. We know $\boldsymbol{a}.\boldsymbol{b} = \mathcal{R}(\bar{\boldsymbol{a}}\boldsymbol{b})$ where $\boldsymbol{a}$ and $\boldsymbol{b}$ are two complex vectors In the w domain, we have

$$\begin{aligned} \hat{\boldsymbol{n}} &\rightarrow w && (38) \\ \nabla &\rightarrow \partial_{\bar{w}} = \partial_x + i\partial_y && (39) \end{aligned}$$

Since $w = \mathrm{e}^{i\theta}$, $|w| = 1$ and, if we consider the curve describing the crack, then θ serves as the parameter describing the curve. We know that $\hat{\boldsymbol{n}} = -i\hat{\boldsymbol{t}}$, and the unit tangent vector is defined to be

$$\begin{aligned} \hat{\boldsymbol{t}} &= \frac{\frac{df}{d\theta}}{\left|\frac{df}{d\theta}\right|} && (40) \\ &= \frac{\frac{df}{dw}\frac{dw}{d\theta}}{\left|\frac{df}{dw}\right|} && (41) \\ &= \frac{iw\frac{df}{dw}}{\left|\frac{df}{dw}\right|} && (42) \end{aligned}$$

and the unit normal is then given by,

$$\hat{\boldsymbol{n}}_z = \frac{w\frac{dz}{dw}}{\left|\frac{df}{dw}\right|} \tag{43}$$

and

$$\partial_{\bar{w}} \to \partial_{\bar{z}}$$

Putting it all together, we find

$$\begin{aligned}
\bar{\boldsymbol{n}}_z.\nabla u_z &= \frac{\bar{w}\frac{d\bar{z}}{d\bar{w}}}{\left|\frac{df}{dw}\right|}\partial_{\bar{z}}u_z \qquad (44)\\
&= \frac{\bar{w}\frac{d\bar{z}}{d\bar{w}}}{\left|\frac{df}{dw}\right|}\frac{d\bar{w}}{d\bar{z}}\partial_{\bar{w}}u_z \qquad (45)\\
&= \frac{1}{\left|\frac{df}{dw}\right|}\bar{w}\partial_{\bar{w}}u_z \qquad (46)
\end{aligned}$$

The factor $\left|\frac{df}{dw}\right|$ is non-zero by definition and the rest of it given by $\bar{w}\partial_{\bar{w}}u_z$ is just $(\boldsymbol{n}.\nabla u_z)_w = 0$ in the w domain which is zero. Therefore, the conformal invariance preserves the boundary condition.

10. We know that the stress is given by

$$\begin{aligned}
\sigma_{zy} &= 2\mu u_{zy} \qquad (47)\\
&= \frac{\mu i}{2}\left(\frac{\partial \psi}{\partial w} - \frac{\partial \bar{\psi}}{\partial \bar{w}}\right) \qquad (48)
\end{aligned}$$

From the mapping relation, we obtain,

$$w = z \pm \sqrt{z^2 - 1} \qquad (49)$$

and since we are given that

$$\psi(w) = \frac{i\Sigma}{\mu}\left(w - \frac{1}{w}\right) \qquad (50)$$

we can obtain by using (49) in (50),

$$\psi(z) = \frac{i\Sigma}{\mu}\left(z \pm \sqrt{z^2-1} - \frac{1}{z \pm \sqrt{z^2-1}}\right) \qquad (51)$$

If we use the above in the stress relation (48), we obtain

$$\sigma_{zy} = \frac{\mu i}{2}\left(\frac{\partial \psi}{\partial z}\frac{\partial z}{\partial w} - \frac{\partial \bar{\psi}}{\partial \bar{z}}\frac{\partial \bar{z}}{\partial \bar{w}}\right) \qquad (52)$$

The divergence is most easily seen by simply focussing attention on one of the derivatives contained in (52), in this case

$$\frac{\partial \psi}{\partial z} = \frac{i\Sigma}{\mu}\left[1 \pm \frac{z}{\sqrt{z^2-1}} + \frac{1}{\sqrt{z^2-1}(z \pm \sqrt{z^2-1})}\right] \qquad (53)$$

In the case of a thin crack, we know that $y \simeq 0$ whereas $-1 \leq x \leq 1$, and we see that as we approach the edges, the second term in (53) diverges.