Exercises Week 9

Longitudinal vibrations of a long elastic rod

Consider the continuum limit equation obtained in Week 6 for the coupled springs, i.e. the Navier-Cauchy equation in one dimension,

$$\frac{1}{c^2}\frac{\partial^2 u(x,t)}{\partial t^2} - \frac{\partial^2 u(x,t)}{\partial x^2} = 0, \tag{1}$$

where u is the displacement and c is the speed of sound. We will now solve the equation for a rod of length L with the boundary conditions

$$\begin{cases} \sigma_{xx}(L,t) &= 0 \\ u(0,t) &= 0 \\ \frac{\partial u}{\partial t}\Big|_{t=0} &= 0 \\ u(x,0) &= \alpha x \end{cases} \tag{2}$$

1) Explain what these boundary conditions mean physically. Is the rod initially strained? How is it fixed in space?

We will try to find a solution by separation of variables, that is a solution on the form

$$u(x,t) = T(t)X(x). \tag{3}$$

2) By inserting the product form in Eq. (1), show that one obtains the two equations

$$\frac{X''}{X} = \frac{1}{c^2}\frac{T''}{T} = -\lambda^2 \tag{4}$$

with a separation constant λ.

3) Show next that the full solution that satisfies the boundary conditions assumes the form

$$u(x,t) = \sum_n A_n \cos(\lambda_n ct)\sin(\lambda_n x), \tag{5}$$

where

$$\lambda_n = \left(\frac{1}{2}+n\right)\frac{\pi}{L} \tag{6}$$

For $t = 0$, we obtain from the boundary condition, the following equation for the coefficients A_n

$$\alpha x = \sum_n A_n \sin(\lambda_n x), \tag{7}$$

4) Find the coefficients A_n (how well do you remember the Fourier-transformation?? ;) and plot the solution by truncating the sum Eq. (5) at a finite n. The coefficients should be

$$A_n = \frac{2\alpha(-1)^n L}{\pi^2(1/2+n)^2}$$

Radial vibration of a linear elastic sphere

Find the characteristic frequencies for the radial vibrations of a linear elastic sphere in vacuum. Note that for radial vibrations, the displacement field is rotation free and can therefore be written as the gradient of a scalar field $\varphi(r,t)$, i.e.

$$\mathbf{u}(r,t) = \nabla\varphi(r,t) = \partial_r\varphi(r,t)\mathbf{e}_r \tag{8}$$

You can further assume $\varphi(r,t)$ can be written on product form

$$\varphi(r,t) = f(r)e^{i\omega t} \tag{9}$$

Since the sphere is assumed to vibrate in vacuum there will be no stresses acting from the outside on the sphere, that is use the boundary condition

$$\sigma_{rr}(a) = 0, \tag{10}$$

where a is the radius of the sphere. Note that you could as well solve directly the radial-symmetric Nacvier-Cauchy equation without introducing the auxiliary scalar field φ.

Exercises from the book

12.8, 12.10, 24.1, 24.4, 24.6