Answer Exercises Week 12

Radial vibration of a linear elastic sphere

Find the characteristic frequencies for the radial vibrations of a linear elastic sphere in vacuum.

Answer

We need to solve the Navier-Cauchy equation for a radial-symmetric problem with the boundary condition

$$\sigma_{rr}(a) = 0,$$

where a is the radius of the sphere.

First we note that for a radial-symmetric problem we have that $\nabla \times \mathbf{u} = 0$ and that we only have longitudinal vibrations of the sphere,

$$\frac{\partial^2 \mathbf{u}_L}{\partial t^2} = c_L^2 \nabla^2 \mathbf{u}_L = c_L^2 \nabla\nabla \cdot \mathbf{u}_L.$$

Here we have used that $\nabla^2 \mathbf{u}_L = \nabla\nabla \cdot \mathbf{u}_L - \nabla \times \nabla \times \mathbf{u}_L = \nabla\nabla \cdot \mathbf{u}_L$. There are two more or less similar ways to solve this equation, the first one is to solve directly the equation.

Direct approach

If we insert a solution on the form $u_r = g(r)e^{i\omega t}$, we end with the following equation for the radial component

$$-k^2 g(r) = g''(r) + \frac{2}{r} g'(r) - \frac{2}{r^2} g(r),$$

where $k = \omega/c_L$. If you look up in your quantum mechanics books or something similar, you will see that this is the Bessel equation and that the solution, which is consistent with the boundary conditions, is the first Bessel function j_1,

$$u_r(r, t) = A j_1(kr) e^{i\omega t} = A \left(\frac{\sin kr}{r^2} - \frac{\cos kr}{r} \right) e^{i\omega t},$$

where A is a constant to be matched by the boundary conditions.

Alternative approach

We can alternatively use that the displacement field is rotation free and therefore can be written as the gradient of a scalar field $\mathbf{u}_L = \nabla\varphi$. Inserting this in the wave equation we get

$$\frac{\partial^2 \nabla\varphi}{\partial t^2} = c_L^2 \nabla\nabla^2\varphi$$

which is equivalent to

$$\nabla\left(\frac{\partial^2\varphi}{\partial t^2} - c_L^2\nabla^2\varphi\right) = 0.$$

Integrating once we get

$$\frac{\partial^2\varphi}{\partial t^2} = c_L^2\nabla^2\varphi.$$

Similar to the displacement field, the scalar field will also be radial-symmetric, we therefore seek a solution to the problem on the form

$$\varphi = f(r)e^{i\omega t}$$

If we insert this in the equation above (using the spherical expression of the Laplace operator), we get an equation for f

$$-k^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right)$$

We can rewrite the right hand side, such that equation assumes the form

$$-k^2 f = \frac{1}{r}\frac{\partial^2}{\partial r^2}(rf) \tag{1}$$

If we introduce an auxiliary function $\psi = rf$ the equation reads

$$\psi'' = -k^2\psi \tag{2}$$

which has a solution $\psi = A\sin kr$. We disregard the cosine solution because we want $\psi(r=0)=0$. It then follows that

$$f(r) = A\frac{\sin kr}{r}. \tag{3}$$

This expression should be familiar to people with Bessel functions fresh in mind. We now determine possible values of k from the boundary condition $\sigma_{rr}(a) = 0$.

We first use that

$$\sigma_{rr}(r) = 2\mu u_{rr} + \lambda(u_{rr} + 2u_{tt}) \tag{4}$$

which then becomes (with $u_{rr} = \partial_r^2 \varphi$ and $u_{tt} = (\partial_r \varphi)/r$)

$$\sigma_{rr}(a) = \rho c_L^2 f''(a) + 2\rho(c_L^2 - 2c_T^2) f'(a)/a \tag{5}$$

where we have omitted the time factor – please verify if this expression is correct. Putting this to zero we have the following equation for ka

$$0 = c_L^2 \left(\frac{2 \sin ka}{a^3} - \frac{2ka \cos ka}{a^3} - \frac{k^2 a^2 \sin ka}{a^3} \right) + 2(c_L^2 - 2c_T^2) \left(\frac{ka \cos ka}{a^3} - \frac{\sin ka}{a^3} \right) \tag{6}$$

which can be written on the form

$$\tan ka = \frac{ka}{1 - \left(\frac{kac_L}{2c_T} \right)^2}. \tag{7}$$

From this equation we can numerically or graphically solve for the characteristic values of ka and corresponding frequencies.