Lecture Notes for Matematik F2

N. Emil J. Bjerrum-Bohr and Joachim Mathiesen

Version: June 9, 2020

Contents

1 Algebra of complex numbers

The set of complex number satisfy the same algebraic rules as ordinary real numbers: Commutative of addition and multiplication:

$$z_1 + z_2 = z_2 + z_1\,, \qquad z_1\, z_2 = z_2\, z_1\,.$$

Associative of addition and multiplication:

$$(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)\,, \qquad (z_1\, z_2)\, z_3 = z_1\,(z_2\, z_3)\,.$$

Distribution of multiplication over addition:

$$z_1\,(z_2 + z_3) \;\;=\;\; z_1\, z_2 + z_1\, z_3\,.$$

Example 1

We can represent complex numbers in the $z = x + i\,y$ notation where $i \equiv \sqrt{-1}$. Using this notation we can do algebra with complex numbers. We define $\mathrm{Re}(x + i\,y) = x$ and $\mathrm{Im}(x + i\,y) \equiv y$. For instance, if we define $z_1 \equiv 2 + 4\,i$, $z_2 \equiv 8 - 2\,i$, we have

$$\begin{aligned}
z_1 + z_2 &= (2 + 4\,i) + (8 - 2\,i) = (2 + 8) + (4 - 2)\,i = 10 + 2\,i\,,\\
z_1\, z_2 &= (2 + 4\,i)\,(8 - 2\,i) = (2)\,(8) + (4\,i)(-2\,i) + (2)\,(-2\,i) + (4\,i)\,(8)\\
&= 16 - 8\,i^2 - 4\,i + 32\,i = 16 + 8 + 28\,i = 24 + 28\,i\,,\\
\frac{z_1}{z_2} &= \frac{2 + 4\,i}{8 - 2\,i} = \frac{(2 + 4\,i)(8 + 2\,i)}{(8 - 2\,i)(8 + 2\,i)} = \frac{(2 + 4\,i)(8 + 2\,i)}{8^2 + 2^2} = \frac{8 + 36\,i}{68} = \frac{2}{17} + \frac{9\,i}{17}.
\end{aligned}$$

Example 2

We define the complex conjungate of a complex number $z = x + i\,y$ as $\bar{z} = x - i\,y$. One see that $z\,\bar{z} = x^2 + y^2$. We define |z| as the length of the complex number z (or $\bar{z}$). Clearly $|z| = |\bar{z}|$. We have for complex numbers z_1 and z_2 that

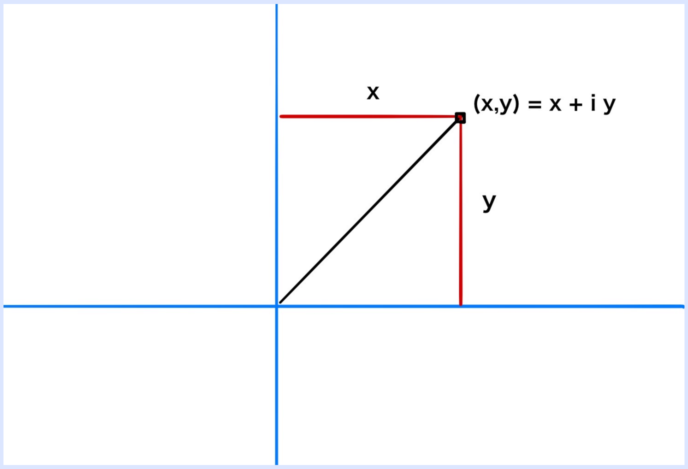
$$\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2\,, \quad \overline{z_1\, z_2} = \bar{z}_1\, \bar{z}_2\,,$$

as well as

$$\mathrm{Re} z = \frac{z + \bar{z}}{2}\,, \quad \mathrm{Im} z = \frac{z - \bar{z}}{2\,i}\,.$$

Example 3

We can visualize complex numbers $x + i\,y \in \mathbb{C}$ as cartesian coordinates $(x, y) \in \mathbb{R}^2$.

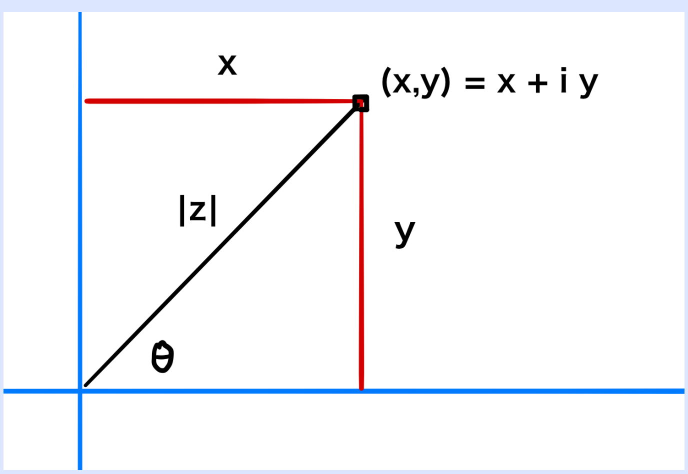


We find the length $|z|$ of a complex number from $|z| = \sqrt{x^2 + y^2}$. Addition of complex numbers are in this visualisation identical to addition of vectors.

Example 4

We can also visualize complex numbers $x + i\,y \in \mathbb{C}$ as polar coordinates $(\theta, |z|)$. Given that $x = |z|\,\cos(\theta)\,, y = |z|\sin(\theta)$ we can write $z = |z|\cos(\theta) + i|z|\sin(\theta) \equiv |z|e^{i\theta}$. We call θ the argument of z. For $-\pi < \theta < \pi$ we have $\theta = \arctan\left(\frac{x}{y}\right)$. Multiplication of complex numbers is easy in polar coordinates as for complex numbers z_1 and z_2 we have

$$z_1\,z_2 = |z_1|e^{i\theta_1}|z_2|e^{i\theta_2} = |z_1|\,|z_2|\,e^{i(\theta_1+\theta_2)}.$$



2 Complex functions

We can define a function on a complex domain in the following way

$$f(z) \quad \equiv \quad u(x,y) + i\,v(x,y)\,,$$

where $z \in \mathbb{C}$ and $u(x,y) \equiv \mathrm{Re} f(z)$ and $v(x,y) \equiv \mathrm{Im} f(z)$.

Example 5

For the function $f(z) = (2x+y^2)+i\,(-\frac{1}{x}+8y^3)$, we directly identify $u(x,y) = (2x+y^2)$ and $v(x,y) = (-\frac{1}{x} + 8y^3)$.

A function $f(z)$ is said to be differential in a point z if the limit

$$\lim_{h\to 0} \frac{f(z+h) - f(z)}{h}\,, \tag{1}$$

exists and is unique. We say, if, a function is differential in all points in a subdomain A of the complex plane, that it is an *analytical function* on A. To determine if a function is analytical in A we have to verify if the limit eq. (1) exists and is unique in all points.

Example 6

We will test if the function $f(z) = \mathrm{Re}(z)$ is an analytical function. We will check this directly by verifying that the limit in eq. (1) exists and is unique in all points $z \in \mathbb{C}$. If we set $h = \epsilon e^{i\theta}$ and take the limit $\epsilon \to 0$ (which implies that $h \to 0$),

$$\lim_{\epsilon\to 0} \frac{\mathrm{Re}(z+\epsilon e^{i\theta}) - \mathrm{Re}(z)}{\epsilon e^{i\theta}} = \lim_{\epsilon\to 0} \frac{\mathrm{Re}(z+\epsilon e^{i\theta} - z)}{\epsilon e^{i\theta}} = \lim_{\epsilon\to 0} \frac{\epsilon\cos(\theta)}{\epsilon\, e^{i\theta}} = e^{-i\theta}\cos\theta$$

We immediately see that the limit depends on θ and therefore that it is not unique.

Another way to check if a function is analytical is to verify if $f(z) = u(x,y)+iv(x,y)$ satisfies the Cauchy-Riemann relations everywhere on A

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y}, \\ \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x}. \end{aligned}$$

One can show that verifying the Cauchy-Riemann relations is equivalent to showing that the limit in eq. (1) exists and is unique.

Example 7

To test if a function is analytical it is often easier to check the Cauchy-Riemann relations. If $z = x + iy$, and we have $f(z) = \mathrm{Re}(z) = \mathrm{Re}(x + iy) = x$. We see that $u(x, y) = x$ and $v(x, y) = 0$ and therefore the first Cauchy-Riemann relation is not satisfied,

$$\frac{\partial u}{\partial x} = 1 \neq 0 = \frac{\partial v}{\partial y}.$$

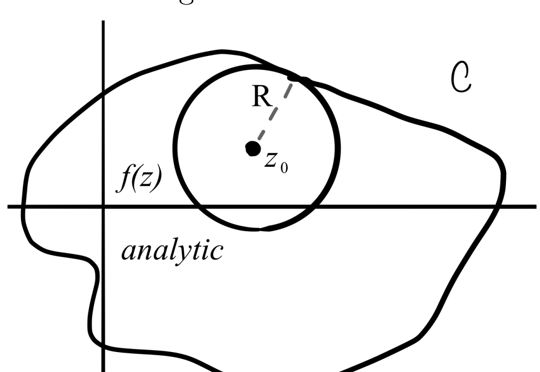
The function $f(z) = \mathrm{Re}(z) = \mathrm{Re}(x + iy) = x$ is therefore not analytical.

3 Complex power series

Complex functions that are analytic in a domain can be expanded in a power series. We can for such functions write

$$f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n ,$$

where $z_0 = 0$ for a series around the origin.



Considering the real positive series

$$|f(z)| = \sum_{n=0}^{\infty} |c_n|\,|z - z_0|^n\,,$$

we can check absolute convergence using standard tools for real power series such as the concept of the *radius of convergence*. The radius of convergence R is defined by

$$\frac{1}{R} = \lim_{n\to\infty} |c_n|^{\frac{1}{n}}$$

and the power series is absolute convergent for $|z| < R$.

Example 8

The complex exponential is an example of an elementary series with infinite radius of convergence

$$\exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}\,, \qquad 0 \le z < \infty\,.$$

Example 9

The trigonometric functions also have elementary series with infinite radius of convergence

$$\cos(z) = \sum_{n=0}^{\infty} \frac{(-1)^n\, z^{2n}}{2n!}\,, \qquad \sin(z) = \sum_{n=0}^{\infty} \frac{(-1)^n\, z^{2n+1}}{(2n+1)!}\,, \qquad 0 \le z < \infty\,.$$

Example 10

For the hypergeometric functions we have $\sin(iz) = i\sinh(z)$ and $\cos(iz) = \cosh(z)$ Thus it follows that

$$\cosh(z) = \sum_{n=0}^{\infty} \frac{z^{2n}}{2n!}\,, \qquad \sinh(z) = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!}\,, \qquad 0 \le z < \infty\,.$$

Example 10

An important example of a series with finite convergence is the geometric series

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n\,, \qquad 0 \leq z < 1\,.$$

A complex function can be differentiated any number of times inside its radius of convergence. We can use this to generate results for new power series from other known series.

Example 11

An example is the geometric series

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n\,, \qquad 0 \leq z < 1\,.$$

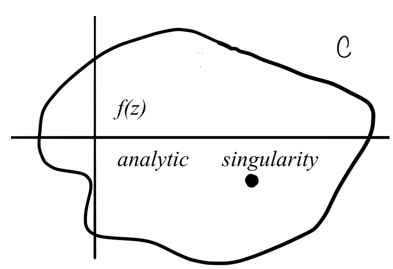
If we differentiate it we get

$$\frac{d}{dz}\frac{1}{1-z} = \frac{1}{(1-z)^2} = \sum_{n=1}^{\infty} n z^{n-1}\,, \qquad 0 \leq z < 1\,.$$

This way we can generate results for all power series of the type $\frac{1}{(1-z)^k}$, $0 \leq z < 1$.

4 Singularities and branch points

Singularities of complex functions are an important concept in complex analysis. At domains including singularities complex functions are non-analytic, *i.e.* not expandable in a power series.



The most common example we encounter is that of an *isolated singularity*, for instance, a pole. We talk of a pole of order ρ for $f(z)$ if the limit

$$\lim_{z\to z_j}\left[(z-z_j)^\rho f(z)\right],$$

is a finite and nonzero complex number. A *simple pole* is a pole of order $\rho = 1$.

Example 12

The function

$$f(z) = \frac{z^2-2z+1}{(z-1)(z+1)^3},$$

has a pole of order 3 in $z = -1$, since

$$f(z) = \lim_{z\to -1}(z+1)^3\frac{z^2-2z+1}{(z-1)(z+1)^3} = \lim_{z\to -1}\frac{(1-z)^2}{(1-z)} = 2\,.$$

If it is not possible to find a ρ so that the limit

$$\lim_{z\to z_j}\left[(z-z_j)^\rho f(z)\right]$$

yields a nonzero finite complex number, call the singularity at z_j an *essential singularity*.

Example 13

The function $\exp\left(\frac{1}{z}\right)$ has an essential singularity at $z = 0$ since the $\lim_{z\to 0}\left|\left[(z)^\rho \exp\left(\frac{1}{z}\right)\right]\right|$ is infinite for any ρ.

We have a *removable singularity* if we have a pole that we can uniquely cancel.

Example 14

The function

$$f(z) = \frac{z^2-2z+1}{(z-1)(z+1)^3}$$

has a *removable singularity* at $z = 1$, since $z^2-2z+1 = (z-1)^2$, so we can uniquely cancel the pole at $(z-1)^{-1}$ in the denominator with the zero at $(z-1)$ in the numerator. Thus we can define $f(z) = \dfrac{1-z}{(1+z)^3}$ as the function with the singularity at $z = 1$ uniquely removed.

Finally we can also talk about *singularities at infinity*. We can determine the behaviour of $f(z)$ at infinity, by considering if $f\left(\dfrac{1}{z}\right)$ has a pole for $z = 0$.

Example 15

The function $f(z) = z$ has a pole at infinity, since $f\left(\dfrac{1}{\tilde{z}}\right) = \dfrac{1}{\tilde{z}}$ has a pole at $\tilde{z} = 0$.

An important concept in complex analysis is multivariate functions. While many many complex functions are single valued functions, for instance polynomials, there are also a number of complex functions that only are single valued in certain domains. Given a complex number in polar coordinates, $z = re^{i\theta}$, we can define a multivalued valued complex logarithm function

$$\mathrm{Ln}(z) = \ln(r) + i(\theta + 2\pi k)\,,$$

where k is an integer. We next define the principal valued complex logarithm ($k = 0$)

$$\ln(z) = \ln(r) + i\theta, \quad -\pi < \theta < \pi\,.$$

Example 16

Given the complex number $z = 1 + i$, we see that in polar coordinates $z = \sqrt{2}e^{i\pi/4}$ thus

$$\ln(z) = \frac{\ln(2)}{2} + i\,\frac{\pi}{4}\,.$$

while

$$\mathrm{Ln}(z) = \frac{\ln(2)}{2} + i\left(\frac{\pi}{4} + 2\pi k\right).$$

We can also define $z^a \equiv e^{\mathrm{Ln}(z)a}$.

Example 17

If $z = i$ we have $z^2 = e^{2\mathrm{Ln}(i)} = e^{2(\ln(1)+i\frac{\pi}{2}+2i\pi k)} = e^{i\pi}e^{i4\pi k}$. We see that $z^2 = -1$ for any integer k as expected.

Example 18

If $z = i$ we have $z^{\frac{1}{2}} = e^{\frac{1}{2}\mathrm{Ln}(i)} = e^{\frac{1}{2}(\ln(1)+i\frac{\pi}{2}+2i\pi k)} = e^{i(\frac{\pi}{4}+\pi k)}$. We see that we have two values, namely $z^{\frac{1}{2}} = e^{i\frac{\pi}{4}}$ and $z^{\frac{1}{2}} = e^{i\frac{3\pi}{4}}$, for $k = 0$ and $k = 1$. More generally we can define the nth root of a complex number in the following way. Given a complex number $z = re^{i\theta}$ its n roots are given by

$$z_k^{\frac{1}{n}} = e^{\frac{1}{n}\ln(r)+i\,\frac{\theta+2\pi k}{n}} = r^{\frac{1}{n}}e^{i\frac{\theta+2\pi k}{n}}$$

for $k = 0, 1, \ldots, n-1$.

Working with multivariate functions in the complex plane requires a certain amount of care since we need to find domains for which they can be considered single-valued. In such domains, we can apply many of the concepts we know from analytic functions.

Example 19

For the complex root of $z = re^{i\theta}$ we have

$$z^{\frac{1}{n}} = r^{\frac{1}{n}}e^{i\frac{\theta+2\pi k}{n}}.$$

For a contour that encloses the origin, we do not return to original value for the function after one circuit around the contour. On the other hand, if we do not enclose the origin, we return to the original value after a circuit around the contour.

We call $z = 0$ a *branch point* for the function $z^{\frac{1}{n}}$.

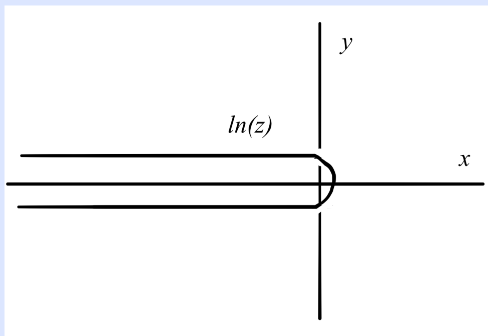
Example 20

For the complex logarithm $\mathrm{Ln}(z)$, we see that we have a branch point at $z = 0$, where we never return to the original value after any number of circuits around the origin.

To work with single-valued complex functions, we can define *branch cuts* which are lines in the complex plane that one must not cross. By defining such branch cuts we avoid making a full circuit around a function's branch points.

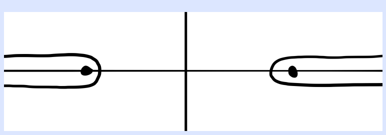
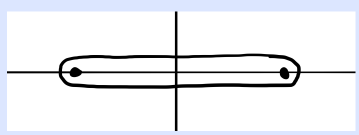
Example 21

The function $\mathrm{Ln}(z)$ can be defined as a single-valued function by making a branch cut along the negative real axis. That way we define the single-valued complex logarithm $\ln(z)$.



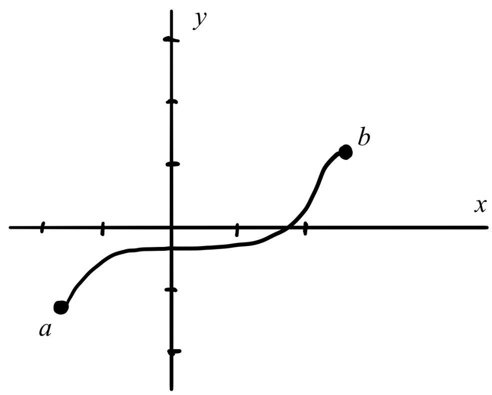
Example 22

For the function $(a^2 - z^2)^{\frac{1}{2}}$ where a is real, we see that $(a-z)^{\frac{1}{2}}(a+z)^{\frac{1}{2}}$, and that we have branch points at $z = a$ and $z = -a$. We will have a branch cuts from $[a; \infty[$ and $]-\infty; -a]$ or from $[-a; a]$, since for any circuits we make on these contours we have a single-valued function.

5 Integration in the complex plane

Integration with regards to a real variable is well known from calculus. In this section, we will discuss integration concerning complex variables.



The complex integral over a complex function $f(z) = u(x, y) + iv(x, y)$ along the curve $\mathcal{C}$ (parametrised by coordinates $(x(t), y(t))$ and where $a \leq t \leq b$) is defined as follows

$$\oint_{\mathcal{C}} f(z)\, dz \;\; \equiv \;\; \oint_{\mathcal{C}} \Big(u(x(t), y(t)) + iv(x(t), y(t))\Big)\,(dx + i\, dy) =$$

$$\int_a^b \Big(u(x(t), y(t))\frac{dx(t)}{dt} - v(x(t), y(t))\frac{dy(t)}{dt}\Big) dt + i\int_a^b \Big(u(x(t), y(t))\frac{dy(t)}{dt} + v(x(t), y(t))\frac{dx(t)}{dt}\Big) dt$$

Example 23

We can integrate the function $f(z) = \frac{1}{z}$ along the unit circle. We parametrize the unit circle using $\exp(it) = \cos(t) + i\,\sin(t) = x(t) + i\,y(t)\,,\ 0 \leq t \leq 2\pi$. We have

$$x(t) = \cos(t)\,, \quad y(t) = \sin(t)\,, \quad \frac{dx(t)}{dt} = -\sin(t)\,, \quad \frac{dy(t)}{dt} = \cos(t)\,,$$

and

$$\begin{aligned}\oint_{\mathcal{C}} f(z)\, dz &= \oint_{\mathcal{C}} \Big(\frac{1}{x(t) + i\,y(t)}\Big)\, dt = \oint_{\mathcal{C}} \Big(\frac{x(t) - i\,y(t)}{x(t)^2 + y(t)^2}\Big)\, dt = \oint_{\mathcal{C}} \Big(x(t) - i\,y(t)\Big)\, dt \\ &= \int_0^{2\pi} \Big(\cos(t)(-\sin(t)) - (-\sin(t))\cos(t)\Big) dt \\ &+ i\int_0^{2\pi} \Big(\cos(t)\cos(t) + (-\sin(t)(-\sin(t))\Big) dt = 0 + i\int_0^{2\pi} dt = 2\pi i\,.\end{aligned}$$

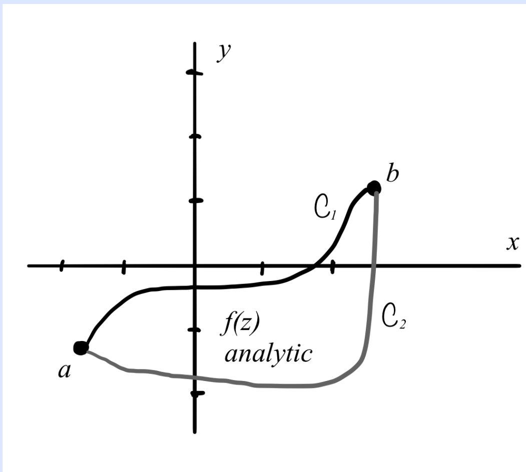
6 Cauchy's theorem and integral formula

Cauchy's theorem is an important result. It states the following, given an analytic funtion $f(z)$ with continuous derivative $f^{(1)}(z)$ everywhere on and inside a closed contour $\mathcal{C}$, then it follows that

$$\oint_{\mathcal{C}} f(z)dz = 0\,.$$

Example 24

We use Cauchy's theorem in many different contexts in complex analysis. One application is if we have a region that inclosed by a contour $\mathcal{C}$ for which the theorem is satisfied.



Then we know for $\mathcal{C} = \mathcal{C}_1 + \mathcal{C}_2$ that

$$\oint_{\mathcal{C}} f(z)dz = \oint_{\mathcal{C}_1} f(z)dz + \oint_{\mathcal{C}_2} f(z)dz = 0\,.$$

If for instance $\mathcal{C}_1$ is an integration path we are interested in deducing (could be along the real axis) we now know that we can compute it from computing the path along $\mathcal{C}_2$. We will for example use Cauchy's theorem when we consider multivariate integrands.

One of the most important formulas of complex analysis and complex integration is Cauchy's integral formula. It states the following. If $f(z)$ is an analytic function everywhere within and on the boundary of a closed contour $\mathcal{C}$ and z_0 is a point within $\mathcal{C}$, then

$$f(z_0) = \frac{1}{2\pi i}\oint_{\mathcal{C}} \frac{f(z)}{z - z_0}dz\,,$$

as well as it follows that

$$f^n(z_0) = \frac{n!}{2\pi i} \oint_{\mathcal{C}} \frac{f(z)}{(z-z_0)^{n+1}} dz \, ,$$

where $f^n(z_0)$ is the n-th derivative of $f(z)$ evaluated at z_0.

7 Laurent series

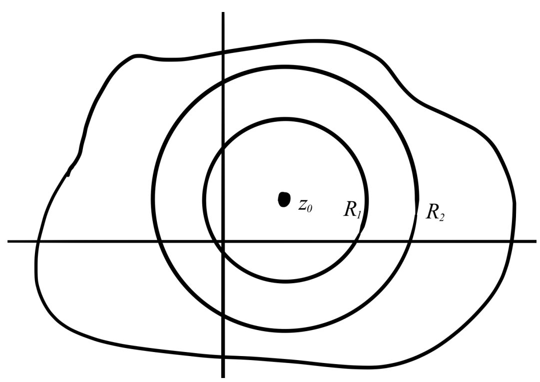
An important extension of the Taylor series is called a *Laurent series.* If a function is analytic we can expand it in a Taylor series, however, if the function has a pole of order ρ at z_0 then the function is not analytic and we cannot expand it in a Taylor series. However, we can consider the Laurent extension. If we consider a function $f(z)$ with a pole of order ρ at the point z_0, then we can consider the function $g(z) = f(z)(z-z_0)^\rho$ and expand it in a Taylor series. We thus have

$$g(z) = \sum_{n=0}^{\infty} c_n (z-z_0)^n \, ,$$

as well as

$$f(z) = \sum_{n=-\rho}^{\infty} c_n (z-z_0)^n \, ,$$

The power series expansion for $f(z)$ around z_0 is called a Laurent series expansion. We call the terms in the series for $n \geq 0$ for the *analytic part of the series* and the terms for $n < 0$ for the *principal part of the series.* If the pole a z_0 is an essential singularity, ρ become ∞.



The convergence of a Laurent series is defined by the radii of convergence of the analytic and the principal part of the series.

Example 25

The geometric series has Taylor power series expansion around $z_0 = 0$

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \,, \qquad 0 \le z < 1 \,.$$

we can define a Laurent series expansion for the geometric series around $z_0 = \infty$ by considering

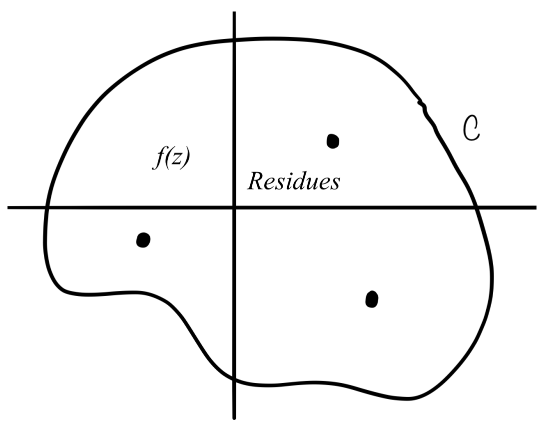
$$\frac{-1}{z(1-\frac{1}{z})} = -\frac{1}{z}\sum_{n=0}^{\infty} \frac{1}{z^n} = \sum_{n=-1}^{\infty} \frac{-1}{z^n} \,, \qquad z > 1 \,.$$

8 Residue theorem

Given a function $f(z)$ with poles inside a contour $\mathcal{C}$. The residue theorem states that

$$\oint_{\mathcal{C}} f(z) dz = 2\pi i \sum_j \mathrm{Res}_j \,,$$

where Res_j denotes the residue of the jth inside the contour.



If the pole is simple (of order one) we can compute the residue from

$$\mathrm{Res}_j = \lim_{z \to z_j} \left[(z - z_j) f(z) \right] .$$

For a general pole (of order ρ) one has to compute the limit

$$\mathrm{Res}_j = \lim_{z\to z_j}\left[\frac{1}{(\rho-1)!}\frac{d^{\rho-1}}{dz^{\rho-1}}(z-z_j)^\rho f(z)\right].$$

One often has to use l'Hôspital's rule in the computations of residues.

Example 26

We compute the residue of $f(z) = \dfrac{1}{1-z}$ at $z=1$. We have

$$\mathrm{Res}_j = \lim_{z\to 1}\left[(z-1)\frac{1}{1-z}\right] = \lim_{z\to 1}\left[-1\right] = -1\,.$$

Thus we have for a contour $\mathcal{C}$ around $z=1$

$$\oint_{\mathcal{C}} \frac{1}{1-z}dz = 2\pi i(-1) = -2\pi i\,.$$

Example 27

We compute the residue of $f(z) = \dfrac{e^z}{(1-z^2)^2}$ at $z=1$. We have $(1-z^2)^2 = (1-z)^2(1+z)^2$ so

$$\begin{aligned}\mathrm{Res}_{z=1} &= \lim_{z\to 1}\left[\frac{1}{(2-1)!}\frac{d}{dz}(z-1)^2\frac{e^z}{(1-z)^2(1+z)^2}\right]\\ &= \lim_{z\to 1}\left[\frac{d}{dz}\frac{e^z}{(1+z)^2}\right] = \lim_{z\to 1}\left[\frac{e^z}{(z+1)^2}-\frac{2e^z}{(z+1)^3}\right] = \frac{1}{4}(e-e) = 0\,.\end{aligned}$$

We compute the residue of $f(z) = \dfrac{e^z}{(1-z^2)^2}$ at $z=-1$. We have $(1-z^2)^2 = (1-z)^2(1+z)^2$ so

$$\begin{aligned}\mathrm{Res}_{z=-1} &= \lim_{z\to -1}\left[\frac{1}{(2-1)!}\frac{d}{dz}(z+1)^2\frac{e^z}{(1-z)^2(1+z)^2}\right]\\ &= \lim_{z\to -1}\left[\frac{d}{dz}\frac{e^z}{(1-z)^2}\right] = \lim_{z\to -1}\left[\frac{e^z}{(1-z)^2}+\frac{2e^z}{(1-z)^3}\right] = \frac{1}{2e}\,.\end{aligned}$$

Thus we have for a contour $\mathcal{C}$ around $z = 1$ and $z = -1$

$$\oint_{\mathcal{C}} \frac{e^z}{(1-z^2)^2} dz = 2\pi i (0 + \frac{1}{2e}) = \frac{i\pi}{e}\,.$$

9 Integration with a multivalued integrand

Integration with a multivalued integrand is a very important feature of complex analysis. Defining a complex contour in such a way that a real integration can be performed is a very important application of complex analysis. We will here consider integrations of functions that need to be defined using branch cuts.

Example 28

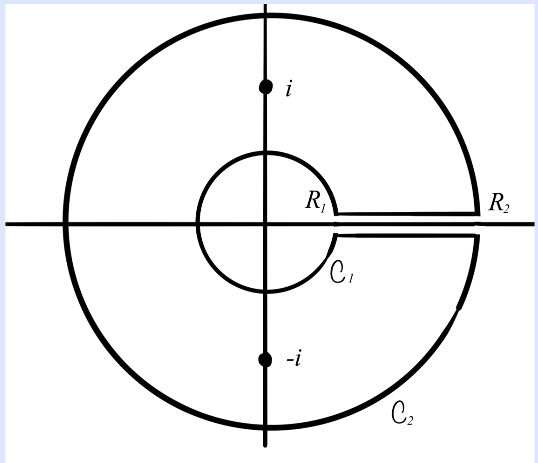
Let us consider the real integration (for $0 < a < 1$)

$$\int_0^\infty \frac{x^a}{1+x^2} dx\,.$$

We will compute this integral by a contour integration in the complex plane. We will thus consider

$$\oint_{\mathcal{C}} \frac{z^a}{1+z^2} dz\,.$$

We see that we have $(1+z^2) = -(i-z)(i+z)$ thus we have poles for $z = i$ and $z = -i$. We now consider the following contour that enclose both poles.



By the residue theorem, we have

$$\oint_{\mathcal{C}} \frac{z^a}{1+z^2}dz = 2\pi i\big(\mathrm{Res}(z=i) + \mathrm{Res}(z=-i)\big)\,.$$

We have

$$\mathrm{Res}(z=i) = \lim_{z\to i}\left[(z-i)\frac{z^a}{1+z^2}\right] = \lim_{z\to i}\left[\frac{z^a}{z+i}\right] = -\frac{1}{2}ie^{\frac{i\pi a}{2}}\,.$$

and

$$\mathrm{Res}(z=-i) = \lim_{z\to -i}\left[(z+i)\frac{z^a}{1+z^2}\right] = \lim_{z\to i}\left[\frac{z^a}{z-i}\right] = \frac{1}{2}ie^{\frac{3}{2}i\pi a}\,.$$

Now we split the integration up along the contour according to

$$\oint_{\mathcal{C}} \frac{z^a}{1+z^2}dz = \int_{R_1}^{R_2} \frac{x^a}{1+x^2}dx + \oint_{\mathcal{C}_1} \frac{z^a}{1+z^2}dz + \int_{R_2}^{R_1} \frac{(e^{2\pi i}x)^a}{1+x^2}dx + \oint_{\mathcal{C}_2} \frac{z^a}{1+z^2}dz\,.$$

If we take $R_1 \to 0$ and $R_2 \to \infty$ we have that the integrals $\oint_{\mathcal{C}_1} \frac{z^a}{1+z^2}dz \to 0$ and $\oint_{\mathcal{C}_2} \frac{z^a}{1+z^2}dz \to 0$ (since $0 < a < 1$) and that

$$\begin{aligned}\oint_{\mathcal{C}} \frac{z^a}{1+z^2}dz &= \int_0^\infty \frac{x^a}{1+x^2}dx - e^{2a\pi i}\int_0^\infty \frac{x^a}{1+x^2} \\ &= (1-e^{2a\pi i})\int_0^\infty \frac{x^a}{1+x^2}dx = \pi\Big(e^{\frac{i\pi a}{2}} - e^{3\pi a}\Big)\,.\end{aligned}$$

So we can write

$$\int_0^\infty \frac{x^a}{1+x^2}dx = \pi(1-e^{2a\pi i})^{-1}(e^{\frac{i\pi a}{2}} - e^{\frac{3}{2}i\pi a}) = \pi\frac{e^{\frac{i\pi a}{2}} - e^{\frac{3i\pi a}{2}}}{1-e^{2i\pi a}} = \frac{\pi e^{\frac{i\pi a}{2}}}{1+e^{i\pi a}}$$
$$= \frac{\pi}{e^{-\frac{i\pi a}{2}} + e^{\frac{i\pi a}{2}}} = \frac{\pi}{2}\sec\left(\frac{\pi a}{2}\right).$$