Roughening of Fracture Surfaces: The Role of Plastic Deformation

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Post mortem analysis of fracture surfaces of ductile and brittle materials on the μm-mm and the nm scales, respectively, reveal self-affine cracks with anomalous scaling exponent $\zeta \approx 0.8$ in three dimensions and $\zeta \approx 0.65$ in two dimensions. Attempts to use elasticity theory to explain this result failed, yielding exponent $\zeta \approx 0.5$ up to logarithms. We show that when the cracks propagate via plastic void formations in front of the tip, followed by void coalescence, the void positions are positively correlated to yield exponents higher than 0.5.

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Quantitative studies of fracture surfaces reveal self-affine rough cracks with two scaling regimes: at small length scales (smaller than a typical crossover length ξ_c) the roughness exponents $\zeta \approx 0.5$, whereas at scales larger than ξ_c the roughness exponent is $\zeta \approx 0.8$. Such measurements were reported first for ductile materials (like metals) where ξ_c is of the order of 1 μm [1,2], and more recently for brittle materials like glass, but with a much smaller value of ξ_c, of about 1 nm [3]. Similar experiments conducted on two-dimensional samples reported rough cracks with large-scale exponents $\zeta \approx 0.65 \pm 0.04$ [4–6]. The exponent $\zeta \approx 0.5$ is characteristic of uncorrelated random surfaces, but higher exponents indicate correlated steps [7]; naturally the experimental discovery of such correlated, "anomalous" exponents attracted considerable interest with repeated attempts to derive them theoretically. A number of studies tried to map the problem to other models that are not derived from the field equations [8,9]. Up to now, however, attempts that were based on elasticity theory have failed to underpin the mechanism for correlated fracture steps. For realistic inplane loads, i.e., mode I or mode II fracture, these attempts invariably ended up with logarithmic roughening [10] or with the random surface scaling exponents $\zeta \approx 0.5$ [11].

In this Letter we present a quantitative model for self-affine fracture surfaces based on elasticity theory supplemented with considerations of plastic deformations; focusing on infinite two-dimensional materials we follow the qualitative picture presented recently in [12]; see Fig. 1. In this picture there exists a "process zone" in front of the crack tip in which plastic yield is accompanied by the evolution of damage cavities. A crucial aspect of this picture is the existence of a typical scale, ξ_c, which is roughly the distance between the crack tip and the first void, at the time of the nucleation of the latter. The voids are nucleated under the influence of the stress field $\sigma_{ij}(\boldsymbol{r})$ adjacent to the tip, but not *at* the tip, due to the existence of the plastic zone that cuts off the purely linear-elastic (unphysical) crack-tip singularities. The crack grows by coalescing the voids with the tip, creating a new stress field which induces the nucleation of new voids. In the picture of [12] the scale ξ_c is also identified with the typical size of the voids at coalescence. A consequence of this picture is that the roughening exponent $\zeta \approx 0.5$ corresponds to the surface structure of individual voids, whereas the large-scale anomalous exponent has to do with the correlation between the positions of different voids that coalesce to constitute the evolving crack. To dress this picture with quantitative content we need first to provide a theory for the scale ξ_c and, second, to demonstrate that the positions of consecutive voids are positively correlated. These are the main goals of this Letter.

A simple model for ξ_c can be developed by assuming the process zone to be properly described by the Huber–von Mises plasticity theory [14]. This theory focuses on the deviatoric stress $s_{ij} \equiv \sigma_{ij} - \frac{1}{3}\mathrm{Tr}\boldsymbol{\sigma}\delta_{ij}$ and on its invariants. The second invariant, $J_2 \equiv \frac{1}{2}s_{ij}s_{ij}$, corresponds to the distortional energy. The material yields as the distortional energy exceeds a material-dependent

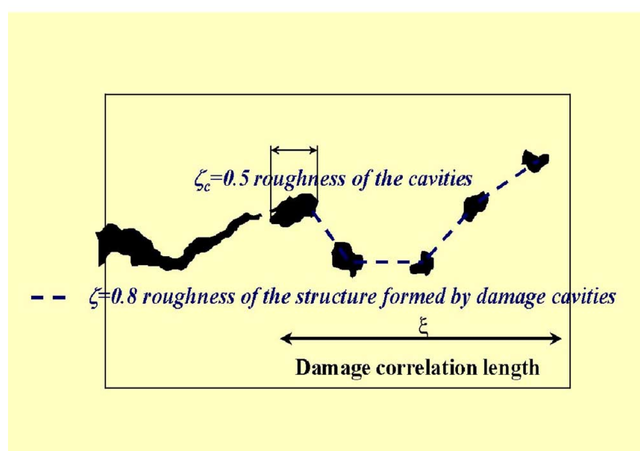

FIG. 1 (color online). The fracture scenario suggested in [12]. This scenario had been documented in detail in corrosive glass fracture, and also more recently in the fracture of paper [13]. (Figure courtesy of E. Bouchaud.)

threshold σ_Y^2. In two dimensions this yield condition reads [14]

$$J_2 = \frac{\sigma_1^2 - \sigma_1\sigma_2 + \sigma_2^2}{3} = \sigma_Y^2. \tag{1}$$

Here $\sigma_{1,2}$ are the principal stresses given by

$$\sigma_{1,2} = \frac{\sigma_{yy} + \sigma_{xx}}{2} \pm \sqrt{\frac{(\sigma_{yy} - \sigma_{xx})^2}{4} + \sigma_{xy}^2}. \tag{2}$$

In the purely linear-elastic solution the crack-tip region is where high stresses are concentrated (in fact, diverging near a sharp tip). Plasticity implies on the one hand that the tip is blunted, and on the other hand, that inside the plastic zone the Huber–von Mises criterion (1) is satisfied. The outer boundary of the plastic zone is called below the "yield curve" and in polar coordinates around the crack tip is denoted $R(\theta)$.

Whatever is the actual shape of the blunted tip its boundary cannot support normal components of the stress. Together with Eq. (1) this implies that on the crack interface

$$\sigma_1 = \sqrt{3}\sigma_Y, \qquad \sigma_2 = 0. \tag{3}$$

On the other hand, the linear-elastic solution, which is still valid outside the plastic zone, imposes the outer boundary conditions on the yield curve. Below we compute the outer stress field *exactly* for an arbitrarily shaped crack using the recently developed method of iterated conformal mappings [15]. For the present argument we take the outer stress field to conform with the universal linear-elastic stress field for mode I symmetry,

$$\sigma_{ij}(r,\theta) = \frac{K_I}{\sqrt{2\pi r}}\Sigma_{ij}^I(\theta). \tag{4}$$

For a crack of length L with σ^∞ being the tensile load at infinity, the stress intensity factor K_I is expected to scale like $K_I \sim \sigma^\infty\sqrt{L}$. Using this field we can find the yield curve $R(\theta)$. Typical yield curves for straight and curved cracks are shown in the insets of Figs. 3 and 6.

The typical scale ξ_c follows from the physics of the nucleation process. We assume that void nucleation occurs where the hydrostatic tension P, $P \equiv \frac{1}{2}\mathrm{Tr}\boldsymbol{\sigma}$, exceeds some threshold value P_c. Other assumptions on the nature of the nucleation process do not affect qualitatively our main result. The hydrostatic tension increases when we go away from the tip and reaches a maximum near the yield curve. To see this, note that on the crack surface $P = (\sqrt{3}/2)\sigma_Y$ [cf. Equation (3)]. On the yield curve we use Eq. (4) and the Huber–von Mises criterion together to solve the angular dependence of the hydrostatic tension in units of σ_Y. It attains a maximal value of $\sqrt{3}\sigma_Y$ and is considerably higher than $(\sqrt{3}/2)\sigma_Y$ for a wide range of angles. On the other hand, the linear-elastic solution (4) implies a monotonically decreasing P outside the yield curve. We thus expect P to *attain its maximum value near the yield curve*. This conclusion is fully supported by finite element method calculations; cf. [16]. Finally, since the nucleation occurs when P exceeds a threshold P_c, this threshold is between the limit values found above; i.e., $(\sqrt{3}/2)\sigma_Y < P_c < \sqrt{3}\sigma_Y$. The first void thus appears at a typical distance ξ_c as shown in Fig. 2. An immediate consequence of the above discussion is that ξ_c is related to the crack length via

$$\xi_c \sim \frac{K_I^2}{\sigma_Y^2} \sim \left(\frac{\sigma^\infty}{\sigma_Y}\right)^2 L. \tag{5}$$

It is worthwhile to put this prediction to experimental test.

Naturally, the precise location of the nucleating void experiences a high degree of stochasticity due to material inhomogeneities. In our model below we assume that the nucleation can occur randomly anywhere in the region in which $P > P_c$ with a probability proportional to $P - P_c$.

The simplest possible crack propagation model is obtained if we assume that a void is nucleated and then the crack coalesces with the void before a new void is nucleated. In experiments it appears that several voids may nucleate before the coalescence occurs [12,13], but we demonstrate that already a one void model induces positive correlations between consecutive void nucleations, leading eventually to an anomalous roughness exponent larger than 0.5. Clearly, even this simple model requires strong tools to compute the stress field around an arbitrarily shaped crack, to determine at each stage of growth the location of the yield curve and nucleating randomly the next void according to the probability

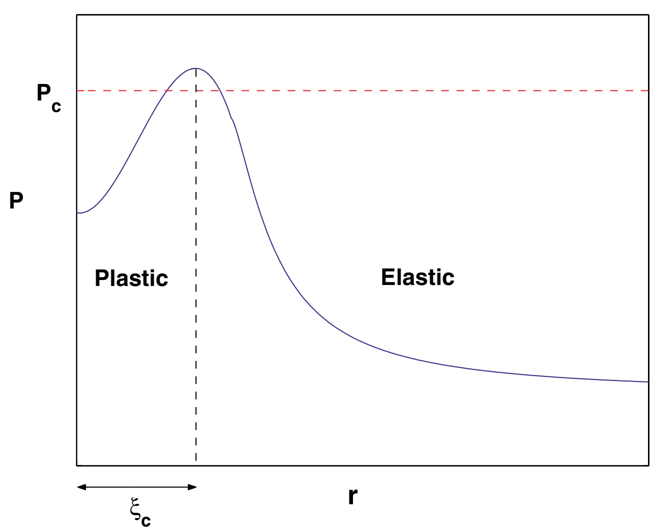

FIG. 2 (color online). A forward direction profile of the hydrostatic tension P in units of σ_Y. On the crack $P = \sqrt{3}/2$, and it attains a maximum of $\sqrt{3}$ on the yield curve. The threshold line indicates a value of P_c such that $\sqrt{3}/2 < P_c < \sqrt{3}$. The typical length ξ_c is shown. Other directions exhibit qualitatively similar profiles.

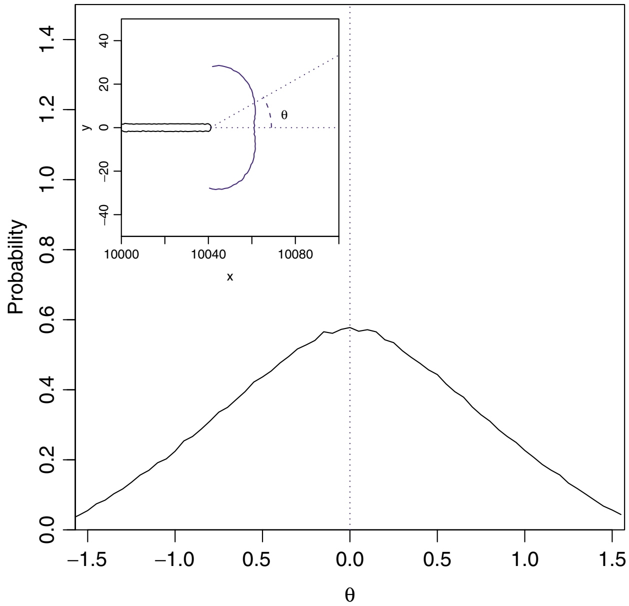

FIG. 3 (color online). The yield curve (inset) and the probability distribution function on it for a long straight crack. The distribution is symmetric and wide enough to allow for deviations from the forward direction.

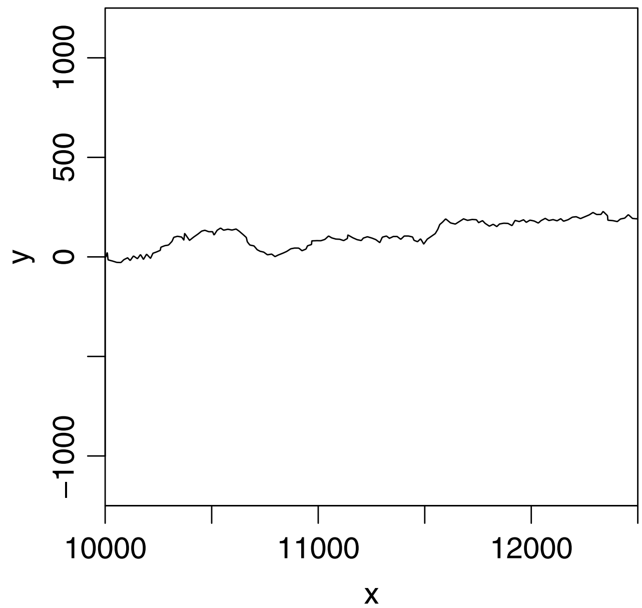

FIG. 4. A crack that was generated using our model.

distribution discussed above. In a recent work we have developed precisely the necessary tool in the form of the method of iterated conformal mappings [15].

In the method of iterated conformal mappings one starts with a crack for which the conformal map from the exterior of the unit circle to the exterior of the crack is known. (Below we start with a long crack, in the form of a mathematical branch cut of length 1000 in units of ξ_c.) We can then grow the crack by little steps in desired directions, computing at all times the conformal map from the exterior of the unit circle to the exterior of the resulting crack. Having the conformal map makes the *exact* calculation of the stress field (for arbitrary loads at infinity) straightforward in principle and highly affordable in practice. The details of the method and its machine implementations are described in full detail in [15]. We should stress here that the method naturally grows cracks with finite curvature tips, and each step adds on a small addition to the tip, also of a finite size that is controlled in the algorithm.

Having the stress field around the crack we can readily find the yield curve, and the physical region in its vicinity where a void can be nucleated (naturally, the width of this region depends on the critical value P_c which is a parameter of the algorithm, as σ_Y is). Choosing with probability $\propto P - P_c$ the position of the next void, we use this site as a pointer that directs the crack tip. Figure 3 shows a typical yield curve and the corresponding probability distribution function ($\propto P - P_c$) on this curve for a straight crack. The distribution is symmetric and wide enough to allow for deviations from the forward direction.

We then use the method of iterated conformal mappings to make a growth step to coalesce the tip with the void. Naturally the step sizes are of the order of ξ_c. Thus the radius of curvature at the tip is also of the order of ξ_c. We note that this model forsakes the details of the void structure and all the length scales below ξ_c. This is clearly acceptable as long as we are mainly interested in the scaling properties on scales larger than ξ_c.

In Fig. 4 we present a typical crack that had been grown using this method. The positive correlation between successive void nucleation and coalescence events is obvious

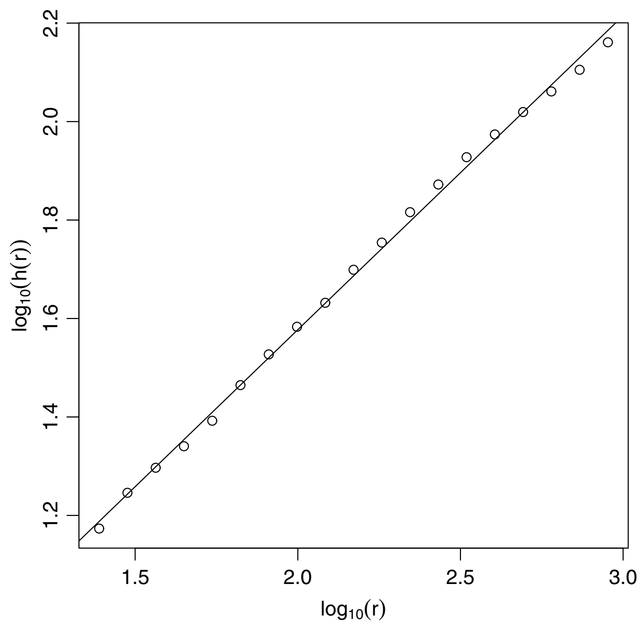

FIG. 5. Calculation of the anomalous roughening exponent.

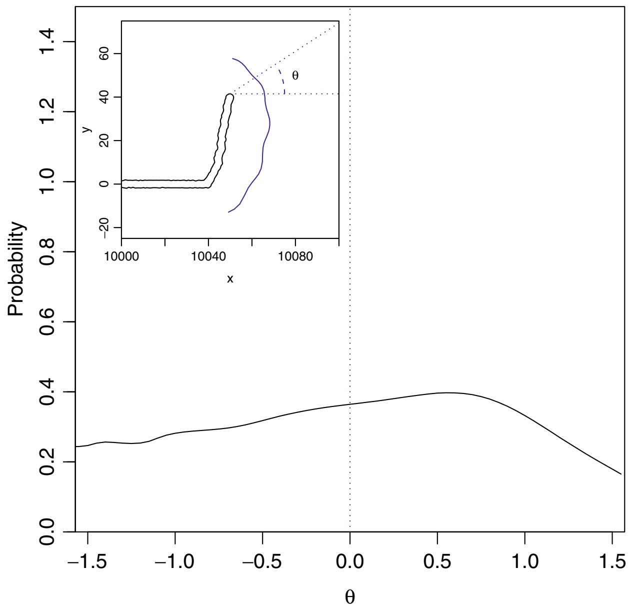

FIG. 6 (color online). The yield curve (inset) and the probability distribution function on it for a straight crack followed by an upturn. The angle θ is measured relative to the horizontal direction. It is clear that the distribution is skewed in favor of positive angles.

even to the naked eye. Once the crack steps upward, there is a high probability to continue upward, and vice versa. This is precisely the property that we were after. A quantitative measurement of this tendency is the roughening exponent, which we compute as follows. Measuring the height fluctuations $y(x)$ in the graph of the crack, one defines $h(r)$ according to

$$h(r) \equiv \langle \max\{y(\tilde{x})\}_{x<\tilde{x}<x+r} - \min\{y(\tilde{x})\}_{x<\tilde{x}<x+r}\rangle_x. \quad (6)$$

For self-affine graphs the scaling exponent ζ is defined via the scaling relation

$$h(r) \sim r^{\zeta}. \quad (7)$$

In Fig. 5 we present a log-log plot of $h(r)$ vs r, with a best power-law fit of $\zeta = 0.64 \pm 0.04$. Indeed as anticipated from the visual observation of Fig. 4 the exponent is higher than 0.5.

We note that our measured scaling exponent is very close to the exponents observed in two-dimensional experiments. (Of course we cannot expect a two-dimensional theory to agree with three-dimensional experiments—the scaling exponents are, as always, strongly dimension dependent). The question of universality of this exponent should await, however, the inclusion of more voids; this may very well lead to more correlated surfaces with higher exponents. We are more interested in the physcial aspects of our model and the main points of the model are worth reiterating. First, we have a new typical scale, ξ_c, which is crucial. Growing directly at the tip of the crack results in a very strong preference for the forward direction, meaning that a step up will most likely be followed by a step down, and vice versa, as shown in [11]. The introduction of the physics of the plastic zone results in creating a finite distance away from the tip to realize the next growth step. Second, a growth in the upward (or downward) direction is affecting the next stress field such as to bias the next growth step to be correlated with the last one. To see this clearly we present in Fig. 6 the yield curve and the corresponding probability distribution function ($\propto P - P_c$) on this curve for a long straight crack followed by an upward turn. It is clear that the distribution is skewed in favor of positive angles with respect to the forward direction. When the crack grows further this tendency becomes more pronounced. This is the essence of the positive correlation mechanism.

To improve our model further one needs to solve exactly for the stress field around a crack and a single void ahead. This will allow the introduction of two voids in the physically required places. Such an improved model, which is presently under construction, calls for mapping conformally doubly connected regions; The results of this model will be presented in a forthcoming paper.

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