Complete Devil's Staircase, Fractal Dimension, and Universality of Mode-Locking Structure in the Circle Map

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It is shown numerically that the stability intervals for limit cycles of the circle map form a complete devil's staircase at the onset of chaos. The complementary set to the stability intervals is a Cantor set of fractal dimension $D = 0.87$. This exponent is found to be universal for a large class of functions.

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The transition to chaos in dynamical systems has been intensively studied by utilizing discrete maps. In particular, Feigenbaum[1] found that the bifurcation route to chaos, in which the transition is governed by one external parameter, is characterized by universal indices. Recently another route, namely through quasiperiodic behavior, has been studied by several authors.[2-4] This transition can be established by varying two frequencies and may be studied by means of the so-called circle map,

$$f(\theta) = \theta + \Omega - (K/2\pi)\sin(2\pi\theta). \tag{1}$$

The ratio between the frequencies is given by the winding number

$$W(K, \Omega) = \lim_{n \to \infty} n^{-1}[f^n(\theta) - \theta]. \tag{2}$$

In numerical studies Shenker[2] found that when the ratio W approaches the reciprocal Golden mean the mapping exhibits unusual scaling behavior at the critical point ($K = 1$) where chaos sets in. This transition has been elegantly treated by means of a renormalization-group technique by Feigenbaum, Kadanoff, and Shenker[3] and Rand *et al.*[4]

Our objective is quite different. We have studied the *global* mode-locking phenomenon at the critical point, $K = 1$. A similar mode locking has been observed experimentally in a Josephson junction when an external frequency is applied.[5] By numerical iterations of the mapping (1) at different values of Ω we have found evidence for mode locking of the mapping at every single rational value of W. Also, we find that the stability intervals for the different rational values fill up the whole Ω axis. The steps in the W vs Ω function thus form a *complete* devil's staircase, of a type which has been found in quite different contexts, such as the one-dimensional Ising model[6] and the Frenkel-Kontorowa model.[7] The complementary set (on the Ω axis) to a complete devil's staircase is a Cantor set of fractal dimension D smaller than or equal to 1. For the staircase of the circle map we have found $D = 0.87$ but this number seems to be universal for a large class of functions. The dimension D is therefore a bona fide critical index characterizing the transition to chaos.

The mapping (1) can be considered as the reduced (due to dissipation) one-dimensional map of a more general map of the plane onto itself.[3] The iteration of (1) from a given starting point θ_0 converges towards a limit cycle or an aperiodic trajectory. A limit cycle is characterized by a rational winding number $W = P/Q$, where Q is the period of the cylcle and P is the number of sweeps through the unit interval [0; 1] in a cycle when the mapping (1) is considered modulo 1. We denote the interval in Ω for which the iteration converges to a limit cycle P/Q as $\Delta\Omega(P/Q)$. As shown by Herman[8] for $0 < K < 1$ the iteration locks in to every single rational number in a finite interval $\Delta\Omega$. However, the total width of the steps $\sum \Delta\Omega(P/Q)$, where the summation is over all rational numbers, does not fill out the interval [0; 1] so that there is room of finite Lebesgue measure for solutions with irrational winding numbers.

At $K = 1$ (the critical point) the inverse function f^{-1} exists but is no longer differentiable at $\theta = 0$. We have investigated the situation for this value of K. Figure 1 shows the staircase formed by stability steps of width $\Delta\Omega$ larger than 0.0015. The inset is magnified 10 times and shows steps with $\Delta\Omega > 0.000\,15$.

The width of the first step ($P/Q = 0/1$) can be found analytically. The solution corresponds to a fixed point of (1), $f(\theta^*) = \theta^*$, which, with increasing value of Ω, becomes unstable when $f'(\theta^*) = 1$. This value of Ω is easily found to be $1/2\pi$, so that $\Delta\Omega(0/1) = [1/2\pi; 1/2\pi]$ (only [0; 1] is shown in Fig. 1). For general Q cycles of the

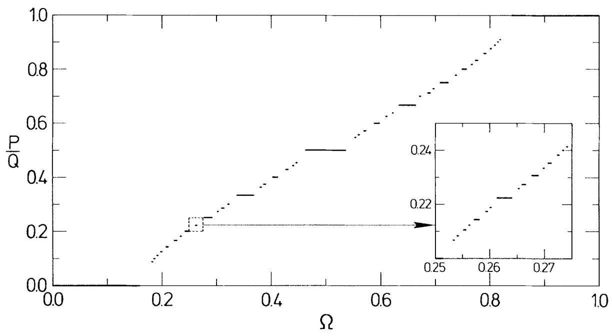

FIG. 1. The complete devil's staircase of the critical circle map. Stability intervals with $\Delta\Omega > 0.0015$ are shown. The inset is a magnification showing intervals with $\Delta\Omega > 0.00015$.

mapping, $\theta_1, \ldots, \theta_Q$, the stability interval is defined by the values of Q for which

$$T = \prod_{i=1}^{Q} f'(\theta_i) = \prod_{i=1}^{Q} [1 - \cos(2\pi\theta_i)] < 1. \qquad (3)$$

T is parabolalike within the stability interval. The relation (3) can be useful when searching for the endpoints where $T = 1$. We have found $\Delta\Omega(P/Q)$ with a precision of the order of 10^{-6} for all rational values in [0; 1] with $Q \leq 50$. Note the symmetry of the staircase around $\Omega = \frac{1}{2}$, $\Delta\Omega(P/Q) = \Delta\Omega(1 - P/Q)$.

To investigate the completeness of the staircase we consider the fraction of the interval $-1 < \Omega < 1$ which is not occupied by plateaus larger than a particular scale. As successive decreasing values of the scale we (arbitrarily) choose $r_{\bar{Q}} = \Delta\Omega(1/\bar{Q})$ for $6 \leq \bar{Q} \leq 29$. We now measure the total length, $S(r_{\bar{Q}})$, of the steps larger than $r_{\bar{Q}}$ (in general this involves plateaus from many $Q > \bar{Q}$). The remainder (holes between steps) has the total length $1 - S(r_{\bar{Q}})$ and we define the number $N(r_{\bar{Q}})$ as this length measured on the scale $r_{\bar{Q}}$, i.e., $N(r_{\bar{Q}}) = [1 - S(r_{\bar{Q}})]/r_{\bar{Q}}$. Figure 2, line a, shows a plot of $\log N(r_{\bar{Q}})$ vs $\log(1/r_{\bar{Q}})$. The points fall nicely on a straight line indicating a power-law behavior

$$N(r) \sim (1/r)^D. \qquad (4)$$

From Fig. 2, line a, one finds $D \sim 0.87$. The length $1 - S(r)$ is thus determined as $1 - S(r) \sim r^{1-D}$. This quantity vanishes with decreasing scale when $D < 1$ which shows that the staircase is complete; i.e., the space between the plateaus is of zero measure. Of course we cannot be sure that the power law holds for scales below r_{29}. However, this scale is extremely small ($r_{29} = 0.000\,077$) and the straight line fits the points

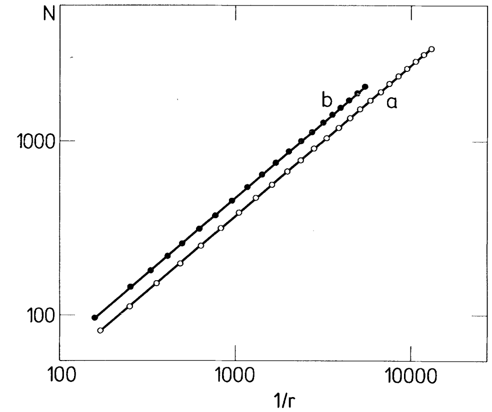

FIG. 2. Plot of $\log N(r)$ vs $\log(1/r)$ for (line a) the critical circle map (1), and (line b) the map (5) with $a = -0.8$. The slopes determined by linear regression are (a) 0.8669 and (b) 0.8652.

excellently with the R^2 factor of a standard regression analysis equal to 1.000.

The complementary set to the Ω intervals for which there is a plateau is thus a Cantor set. D is the *fractal dimension* of the Cantor set.[9] We believe that the fractal dimension is a useful way of characterizing the scaling properties of the staircase at the critical point: D is a characteristic index for the mode-locking structure at the transition to chaos.

A large periodicity (high Q) corresponds to high iterates of the function (1), and in such cases one would expect that only local properties of the mapping are relevant. At $K = 1$ the cubic singularity at $\theta = 0$ of the inverse mapping is the essential property, and one might speculate that the dimension D is universal for mappings with such a singularity. To investigate if this is the case we have studied a class of cubic critical mappings

$$f(\theta) = \theta + \Omega - (K/2\pi)[\sin(2\pi\theta) + a\sin^3(2\pi\theta)], \qquad (5)$$

with $K = 1$, which are monotonic for $-\frac{4}{3} \leq a \leq \frac{1}{6}$. Indeed the mode-locking structures of the class (5) also seem to form complete devil's staircases, and more importantly, *the fractal dimension appears to be $D \sim 0.87$!* We therefore conjecture that this number is universal for maps where the inverse has a cubic singularity. Figure 2, line b, shows a plot of $\log N$ vs $\log(1/r)$ for the mapping (5) with $a = -0.8$. The slopes of the two lines are identical, namely $D = 0.87$. Further numerical studies of this universal behavior are in progress. When K exceeds 1 the stability intervals may overlap and so the convergence of the iteration depends on the starting point.

In a real experiment, the instability of the system with two coupled frequencies is usually not described by a simple mapping of the form (1) [or (5)]. However, by varying one frequency (corresponding to variation of Ω) one should indeed observe mode locking in a series of finite intervals. For the Josephson junction steps down to $Q = 32$ have been observed[5] and it would be possible to check if the scaling of the locking structure is characterized by the universal number $D \sim 0.87$ at the critical point. Above this point an overlap in the stability intervals is observed.[5] This overlap gives rise to hysteresis effects, which might correspond to the case $K > 1$ for the mappings (5). A Rayleigh-Bénard experiment perturbed by up-down oscillations in the gravitation field[10] should also be a good candidate for observation of the staircase behavior in a dynamical system.

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